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Comments On Differentiable Over Function of Split Quaternions

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Abstract

The theory of mathematical analysis over split quaternions is formulated in a closest possible analogy to the usual theory of analytic functions of a complex variable. After reviewing split quaternion algebra via an isomorphic 4×4 matrix representation, a different definition is given to partial derivatives involving split quaternions. This takes care of the ambiguity involved in the no commutative properties of split quaternions. A closely analogous condition for analyticity of functions of a split quaternion variable is found. The analogy with complex variables is illustrated for both the derivative.

key words. Split quaternion, function of split quaternion, analyticity of a split quaternion.

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1 Algebra Of Split Quaternions

By 'calculus of hyper complex variables' we mean the extension of mathematical calculus as used in the theory of complex variables to that of hyper complex variables such as split quaternions.

The algebra of hyper complex number systems [2, 3] may be obtained by relaxing a requirement ofGrassman algebra. In a Grassman algebra operators unticommute under a product wedge operation so that $e_i \wedge e_j + e_j \wedge e_i = 0$.

The most natural assumption is then that $e_i \wedge e_i = 0$ when $i = j$. In a hyper complex system one uses $e_i \wedge e_i = \pm 1$ instead.

A split quaternions q is an expression of the form

$$q = t + ix + iy + kz$$

Where t, x, y and z are real numbers, and i, j, k are split quaternion units which satisfy the non-commutative multiplication rules

$$i^2 = -1, j^2 = k^2 = 1,$$

$$ij = -ji = k, jk = -kj = -i, ki = -ik = j \tag{1}$$

Let us denote the algebra of split quaternions by Q and its natural basis by $\{1, i, j, k\}$. An element of Q is called a split quaternion [3, 4].
Here we simply put the operators next to each other to indicate product without explicitly writing out the wedge operation. The rules for this product operation on different operators are the same as those of the cross-product between the unit vectors i, j and k of a right-hand coordinate system. The product of two split quaternions q_1 and q_2 is

$$\begin{aligned} q_1 q_2 &= (t_1 + ix_1 + jy_1 + kz_1)(t_2 + ix_2 + iy_2 + kz_2) \\ &= (t_1 t_2 - x_1 x_2 + y_1 y_2 + z_1 z_2) \\ &+ i(t_1 x_2 + t_2 x_1 - y_1 z_2 + y_2 z_1) \\ &+ j(t_1 y_2 + t_2 y_1 - x_1 z_2 + x_2 z_1) \\ &+ k(t_1 z_2 + t_2 z_1 - x_2 y_1 + x_1 y_2) \end{aligned}$$

Which is certainly not commutative ($q_1 q_2 \neq q_2 q_1$).
A particularly revealing form for split quaternions can be obtained by considering that ordinary complex variables can be represented as 2×2 matrices for $z = x + iy$, written as

$$\begin{bmatrix} x & y \\ -y & x \end{bmatrix} \tag{2}$$

The norm $x^2 + y^2 = \|z\|^2$ is also simply the determinant of the matrix.
The norm of a product is the product of the norms, through the property that the determinant of a product is the product of the determinants. In the theory of complex variables the product happens to be commutative. This difference in commutative properties must be taken into account in order to successfully develop mathematical analysis over split quaternions. If q is a split quaternion, then in complex numbers space we can write:

$$\begin{aligned} q &= t + ix + jy + kz \\ &= t + ix + jy - jiz \\ &= (t + ix) + j(y - iz) \end{aligned}$$

Let $a = t + ix$ and $b = y - iz$ are two complex numbers, then $q = a + jb$. We introduce the transformation representing multiplication in Q. Let q be a split quaternion, then $T_q : Q \to Q$ is defined as follows:

$$T_q(q') = q'q \qquad , \qquad q' \in Q$$

Now the rules for composition written above are automatically embodied by representing a split quaternion with the matrix

$$T_q \equiv \begin{bmatrix} t & x & y & z \\ -x & t & z & -y \\ y & z & t & x \\ z & -y & -x & t \end{bmatrix} \tag{3}$$

Isomorphic to split quaternion algebra. A two-by-two representation with complex numbers is also well known [1, 4, 5], this representation is:

$$T_q = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

Similar to the representation given for complex numbers, this matrix gives the unit matrix when $q = 1 + 0i + 0j + 0k$. The order for matrix multiplication is the same for split quaternion multiplication.

This notation is very convenient since it immediately reveals when an object is a split quaternion. The norm is

$$\|q\|^2 = q\bar{q} = t^2 + x^2 - y^2 - z^2 \tag{4}$$

The determinant of the matrix is simply related to the norm, since it equals $(t^2 + x^2 - y^2 - z^2)^2$, therefore $\det T_q = \|q\|^4$.

2 Mathematical Analysis with Split Quaternions

Analogously to defining the derivative $\frac{df}{dz}$ of a function of a complex variable, one seeks to define $f'(q) = \frac{df}{dq}$ in a similar way, where f and q are split quaternions. Also one seeks a condition similar to that of analytic functions of a complex variable in order to define uniquely this derivative, independently of direction.

DEFINITION. Given a function

$$f = f_1 + if_2 + jf_3 + kf_4$$

Of q

$$f'(q) = \frac{df}{dq} = \lim_{\Delta q \to 0} [f(q + \Delta q) - f(q)](\Delta q)^{-1} \tag{5}$$

Comment 1. Notice that the order is important; the above definition is generally different from

$$\lim_{\Delta q\to 0}(\Delta q)^{-1}[f(q+\Delta q)-f(q)]$$

Comment 2. When this derivative is evaluated in the different directions

$$\frac{\partial f}{\partial t}=(\frac{df}{dq})_t=\lim_{\Delta t\to 0}[f(q(t+\Delta t,x,y,z))-f(q(t,x,y,z))](\Delta t)^{-1}$$
$$\frac{\partial f}{\partial x}=(\frac{df}{dq})_x=\lim_{\Delta x\to 0}[f(q(t,x+\Delta x,y,z))-f(q(t,x,y,z))](i\Delta t)^{-1}$$
$$\frac{\partial f}{\partial y}=(\frac{df}{dq})_y=\lim_{\Delta y\to 0}[f(q(t,x,y+\Delta y,z))-f(q(t,x,y,z))](j\Delta t)^{-1}$$
$$\frac{\partial f}{\partial z}=(\frac{df}{dq})_z=\lim_{\Delta z\to 0}[f(q(t,x,y,z+\Delta z))-f(q(t,x,y,z))](k\Delta t)^{-1}$$

Comment 3. $f'(q)$ Can be defined if such derivatives taken along different directions areequal.

Comment 4. Each of this is still asplit quaternion. To indicate a definite component we write, for example, $\frac{\partial f_i}{\partial x}$ Where $i=1,2,3,4$.

Before we go further to illustrate the use of this definition we turn back once more to thecase of a complex variable. One may use the ordinary partial derivative since no non-commutative complications are present.

$\frac{df}{dz}$ can be written as the matrix

$$\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x}\\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y}\end{bmatrix}$$

indeed, because of the Cauchy-Riemann conditions

$$\frac{\partial f_1}{\partial x}=\frac{\partial f_2}{\partial y}\quad and\quad \frac{\partial f_2}{\partial x}=-\frac{\partial f_1}{\partial y}$$

The form of the matrix is that of one representing a complex number. The derivative is itself a complex variable and the structure of the matrix is a transparent way of visualizing the Cauchy-Riemann conditions. In this example the Jacobian is equal to

$$(\frac{\partial f_1}{\partial x})^2+(\frac{\partial f_2}{\partial x})^2$$

Similarly one expends $\frac{df}{dq}$ to be represented by a matrix whose form corresponds to that of a split quaternion.

THEOREM. The condition that $\frac{df}{dq}$ be independent of direction along which it is evaluated implies that the 4×4 matrix

$$\begin{bmatrix} \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} & \frac{\partial f_4}{\partial t} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} & \frac{\partial f_4}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} & \frac{\partial f_4}{\partial y} \\ \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} & \frac{\partial f_3}{\partial z} & \frac{\partial f_4}{\partial z} \end{bmatrix} \tag{6}$$

reduces to split quaternion form.

PROOF. Along the t, x, y directions one has

$$\frac{\partial f}{\partial t} = \frac{df \partial q}{dq \partial t} = f'(q) \Rightarrow f'(q) = \frac{\partial f_1}{\partial t} + i\frac{\partial f_2}{\partial t} + j\frac{\partial f_3}{\partial t} + k\frac{\partial f_4}{\partial t} \tag{7}$$

$$\frac{\partial f}{\partial x} = \frac{df \partial q}{dq \partial x} = f'(q)i \Rightarrow f'(q) = -\frac{\partial f}{\partial x}i = \frac{\partial f_2}{\partial x} - i\frac{\partial f_1}{\partial x} - j\frac{\partial f_4}{\partial x} + k\frac{\partial f_3}{\partial x} \tag{8}$$

$$\frac{\partial f}{\partial y} = \frac{df \partial q}{dq \partial y} = f'(q)j \Rightarrow f'(q) = \frac{\partial f}{\partial y}j = \frac{\partial f_3}{\partial y} + i\frac{\partial f_4}{\partial y} + j\frac{\partial f_1}{\partial y} + k\frac{\partial f_2}{\partial y} \tag{9}$$

$$\frac{\partial f}{\partial z} = \frac{df \partial q}{dq \partial z} = f'(q)k \Rightarrow f'(q) = \frac{\partial f}{\partial z}k = \frac{\partial f_4}{\partial z} - i\frac{\partial f_3}{\partial z} - j\frac{\partial f_2}{\partial x} + k\frac{\partial f_1}{\partial z} \tag{10}$$

Since these four expressions must be equated, they must be equal by term. This gives rise to 24 possible equalities of which only 12 are really independent. These conditions are:

$$\frac{\partial f_1}{\partial t} = \frac{\partial f_2}{\partial x} = \frac{\partial f_3}{\partial y} = \frac{\partial f_4}{\partial z}$$
$$\frac{\partial f_2}{\partial t} = -\frac{\partial f_1}{\partial x} = \frac{\partial f_4}{\partial y} = -\frac{\partial f_3}{\partial z}$$
$$\frac{\partial f_3}{\partial t} = -\frac{\partial f_4}{\partial x} = \frac{\partial f_1}{\partial y} = -\frac{\partial f_2}{\partial z}$$
$$\frac{\partial f_4}{\partial t} = \frac{\partial f_3}{\partial x} = \frac{\partial f_2}{\partial y} = \frac{\partial f_1}{\partial z}$$

The matrix then has only four independent elements and can be written in split quaternion form.

From 3 and 4 we can write:

$$T_{f'} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} & \frac{\partial f_4}{\partial t} \\ -\frac{\partial f_2}{\partial t} & \frac{\partial f_1}{\partial t} & \frac{\partial f_4}{\partial t} & -\frac{\partial f_3}{\partial t} \\ \frac{\partial f_3}{\partial t} & \frac{\partial f_4}{\partial t} & \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} \\ \frac{\partial f_4}{\partial t} & -\frac{\partial f_3}{\partial t} & -\frac{\partial f_2}{\partial t} & \frac{\partial f_1}{\partial t} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial f_1}{\partial t} & \frac{\partial f_2}{\partial t} & \frac{\partial f_3}{\partial t} & \frac{\partial f_4}{\partial t} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial x} & \frac{\partial f_4}{\partial x} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_2}{\partial y} & \frac{\partial f_3}{\partial y} & \frac{\partial f_4}{\partial y} \\ \frac{\partial f_1}{\partial z} & \frac{\partial f_2}{\partial z} & \frac{\partial f_3}{\partial z} & \frac{\partial f_4}{\partial z} \end{bmatrix}$$

The determinant or Jacobian can obviously be written as

$$\left[\left(\frac{\partial f_1}{\partial t}\right)^2 + \left(\frac{\partial f_2}{\partial t}\right)^2 - \left(\frac{\partial f_3}{\partial t}\right)^2 - \left(\frac{\partial f_4}{\partial t}\right)^2\right]^2$$

References

[1] BOCHERS, S., and Martin, W. T., 1948, Several Complex Variables (Princeton: Princeton University Press).

[2] BORDEN, R. S., 1983, a Course in Advanced Calculus (Amsterdam: North Holland).

[3] JONES, G. A., and SINGERMAN, D., 1987, Complex Functions (Cambridge: Cambridge University Press).

[4] KULA, L., and YAYLI, Y., Split Quaternions and Rotations in Semi Euclidean Space . J. Korean Math. Soc. 44 (2007), No. 6, pp. 1313-1327.

[5] TERRAS, A., 1985, Harmonic Analysis on Symmetric Spaces and Applications II (Berlin: Springer-Verlag).

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