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SPACELIKE BIHARMONIC GENERAL HELICES WITH TIMELIKE NORMAL IN THE LORENTZIAN GROUP OF RIGID MOTIONS $\mathbb{E}(2)$

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Abstract

In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(2)$. We characterize the spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions $\mathbb{E}(2)$.

key words. Biharmonic curve, harmonic curve, rigid motions.

AMS(MOS) subject classifications.

1 Introduction

The theory of biharmonic functions is an old and rich subject. Biharmonic functions have been studied since 1862 by Maxwell and Airy to describe a mathematical model of elasticity. The theory of polyharmonic functions was developed later on, for example, by Almansi, Levi-Civita and Nicolescu.

Firstly, harmonic maps are given as follows:

Harmonic maps $f : (M, g) \longrightarrow (N, h)$ between Riemannian manifolds are the critical points of the energy

$$E(f) = \frac{1}{2} \int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f) = \text{trace}\nabla df. \tag{1.2}$$

Secondly, biharmonic maps are given as follows:

As suggested by Eells and Sampson in [4], we can define the bienergy of a map f by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [5], showing that the Euler–Lagrange equation associated to E_2 is

$$\begin{aligned} \tau_2(f) &= -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df \\ &= 0, \end{aligned} \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic.

In this paper, we study spacelike biharmonic general helices in the Lorentzian group of rigid motions $\mathbb{E}(2)$. We characterize the spacelike biharmonic general helices in terms of their curvature and torsion in the Lorentzian group of rigid motions $\mathbb{E}(2)$.

2 The Group of Rigid Motions $\mathbb{E}(2)$

Let $E(2)$ be the group of rigid motions of Euclidean 2-space. This consists of all matrices of the form

$$\begin{pmatrix} \cos x & -\sin x & y \\ \sin x & \cos x & z \\ 0 & 0 & 1 \end{pmatrix}.$$

Topologically, $E(2)$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$ under the map

$$E(2) \longrightarrow \mathbb{S}^1 \times \mathbb{R}^2 : \begin{pmatrix} \cos[x] & -\sin[x] & y \\ \sin[x] & \cos[x] & z \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow ([x], y, z),$$

where $[x]$ means x modulo $2\pi z$. It's Lie algebra has a basis consisting of

$$\mathbf{e}_1 = \frac{\partial}{\partial x},\ \mathbf{e}_2 = \cos x\frac{\partial}{\partial y} + \sin x\frac{\partial}{\partial z},\ \mathbf{e}_3 = -\sin x\frac{\partial}{\partial y} + \cos x\frac{\partial}{\partial z}, \tag{2.1}$$

[9] and coframe

$$\theta^1 = dx,\ \theta^2 = \cos x dy + \sin x dz,\ \theta^3 = -\sin x dy + \cos x dz.$$

It is easy to check that the metric g is given by

$$g = \left(\theta^1\right)^2 + \left(\theta^2\right)^2 - \left(\theta^3\right)^2. \tag{2.2}$$

The bracket relations are

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \ \ [\mathbf{e}_2, \mathbf{e}_3] = 0, \ \ [\mathbf{e}_3, \mathbf{e}_1] = \mathbf{e}_2.$$

Proposition 2.1. *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric g, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & 0 & 0 \\ -\mathbf{e}_3 & 0 & -\mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.3}$$

where the (i, j)-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)W, Z).$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices i, j, k and l take the values $1, 2$ and 3.

$$R_{121} = -\mathbf{e}_2, \ \ R_{131} = -\mathbf{e}_3, \quad R_{232} = \mathbf{e}_3 \tag{2.4}$$

and

$$R_{1212} = 1, \ \ R_{1313} = -1, \ \ R_{2323} = 1. \tag{2.5}$$

3 Spacelike Biharmonic General Helices with Timelike Normal in the Lorentzian Group of Rigid Motions $\mathbb{E}(2)$

Let $\gamma : I \longrightarrow \mathbb{E}(2)$ be a non geodesic spacelike curve with timelike normal in the group of rigid motions $\mathbb{E}(2)$ parametrized by arc length. Let $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ be the Frenet frame fields tangent to the group of rigid motions $\mathbb{E}(2)$. along γ defined as follows:

$\mathbf{t}$ is the unit vector field γ' tangent to γ, $\mathbf{n}$ is the unit vector field in the direction of $\nabla_{\mathbf{t}}\mathbf{t}$ (normal to γ) and $\mathbf{b}$ is chosen so that $\{\mathbf{t}, \mathbf{n}, \mathbf{b}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{t}(s)}\mathbf{t}(s) &= \kappa(s)\mathbf{n}(s), \\ \nabla_{\mathbf{t}(s)}\mathbf{n}(s) &= \kappa(s)\mathbf{t}(s) + \tau(s)\mathbf{b}(s), \\ \nabla_{\mathbf{t}(s)}\mathbf{b}(s) &= \tau(s)\mathbf{n}(s), \end{aligned} \tag{3.1}$$

where $\kappa(s) = |\tau(\gamma)| = |\nabla_{\mathbf{t}(s)}\mathbf{t}(s)|$ is the curvature of γ, $\tau(s)$ is its torsion and

$$\begin{aligned} g(\mathbf{t}(s), \mathbf{t}(s)) &= 1, \; g(\mathbf{n}(s), \mathbf{n}(s)) = -1, \; g(\mathbf{b}(s), \mathbf{b}(s)) = 1, \\ g(\mathbf{t}(s), \mathbf{n}(s)) &= g(\mathbf{t}(s), \mathbf{b}(s)) = g(\mathbf{n}(s), \mathbf{b}(s)) = 0. \end{aligned} \tag{3.2}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{t}(s) &= t_1(s)\mathbf{e}_1 + t_2(s)\mathbf{e}_2 + t_3(s)\mathbf{e}_3, \\ \mathbf{n}(s) &= n_1(s)\mathbf{e}_1 + n_2(s)\mathbf{e}_2 + n_3(s)\mathbf{e}_3, \\ \mathbf{b}(s) &= \mathbf{t}(s) \times \mathbf{n}(s) = b_1(s)\mathbf{e}_1 + b_2(s)\mathbf{e}_2 + b_3(s)\mathbf{e}_3. \end{aligned} \tag{3.3}$$

Theorem 3.1. $\gamma : I \longrightarrow \mathbb{E}(2)$ *is a non geodesic spacelike biharmonic curve with timelike normal in the Lorentzian group of rigid motions* $\mathbb{E}(2)$ *if and only if*

$$\begin{aligned} \kappa(s) &= \text{constant} \neq 0, \\ \kappa^2(s) + \tau^2(s) &= 1 - 2b_1^2(s), \\ \tau'(s) &= 2n_1(s)b_1(s). \end{aligned} \tag{3.4}$$

Proof. Using (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_{\mathbf{t}}^3\mathbf{t}(s) + \kappa(s)R(\mathbf{t}(s), \mathbf{n}(s))\mathbf{t}(s) \\ &= (3\kappa_1'(s)\kappa(s))\mathbf{t}(s) + (\kappa''(s) + \kappa^3(s) + \kappa(s)\tau^2(s))\mathbf{n}(s) \\ &\quad + (2\tau(s)\kappa'(s) + \kappa(s)\tau'(s))\mathbf{b}(s) + \kappa(s)R(\mathbf{t}(s), \mathbf{n}(s))\mathbf{t}(s). \end{aligned}$$

By (1.1), we see that γ is a unit speed spacelike biharmonic curve with timelike normal if and only if

$$\begin{aligned} \kappa(s)\kappa'(s) &= 0, \\ \kappa''(s)+\kappa^3(s)+\kappa(s)\tau^2(s) &= -\kappa(s)R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)), \\ 2\tau(s)\kappa'(s)+\tau'(s)\kappa(s) &= -\kappa(s)R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)). \end{aligned} \tag{3.5}$$

Since $\kappa \neq 0$ by the assumption that is non-geodesic

$$\begin{aligned} \kappa(s) &= \text{constant} \neq 0, \\ \kappa^2(s)+\tau^2(s) &= -R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)), \\ \tau'(s) &= -R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)). \end{aligned} \tag{3.6}$$

A direct computation using (2.5), yields

$$\begin{aligned} R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{n}(s)) &= -1+2b_1^2(s), \\ R(\mathbf{t}(s),\mathbf{n}(s),\mathbf{t}(s),\mathbf{b}(s)) &= -2n_1(s)\,b_1(s). \end{aligned} \tag{3.7}$$

These, together with (3.6), complete the proof of the theorem.

If we write this curve in the another parametric representation $\gamma = \gamma(\theta)$, where $\theta = \int_0^s \kappa(s)\,ds$. We have new Frenet equations as follows:

$$\begin{aligned} \nabla_{\mathbf{t}(\theta)}\mathbf{t}(\theta) &= \mathbf{n}(\theta), \\ \nabla_{\mathbf{t}(\theta)}\mathbf{n}(\theta) &= \mathbf{t}(\theta)+f(\theta)\,\mathbf{b}(\theta), \\ \nabla_{\mathbf{t}(\theta)}\mathbf{b}(\theta) &= f(\theta)\,\mathbf{n}(\theta), \end{aligned} \tag{3.8}$$

where $f(\theta) = \frac{\tau(\theta)}{\kappa(\theta)}$.

If we write $\{\mathbf{t}(\theta), \mathbf{n}(\theta), \mathbf{b}(\theta)\}$ with respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ as following:

$$\begin{aligned} \mathbf{t}(\theta) &= t_1(\theta)\,\mathbf{e}_1+t_2(\theta)\,\mathbf{e}_2+t_3(\theta)\,\mathbf{e}_3, \\ \mathbf{n}(\theta) &= n_1(\theta)\,\mathbf{e}_1+n_2(\theta)\,\mathbf{e}_2+n_3(\theta)\,\mathbf{e}_3, \\ \mathbf{b}(\theta) &= \mathbf{t}(\theta)\times\mathbf{n}(\theta) = b_1(\theta)\,\mathbf{e}_1+b_2(\theta)\,\mathbf{e}_2+b_3(\theta)\,\mathbf{e}_3. \end{aligned} \tag{3.9}$$

Theorem 3.2. *Let $\gamma : I \longrightarrow \mathbb{E}(2)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(2)$. Then, the parametric equations of γ are*

$$\begin{aligned} x(\theta) &= \cos\wp\theta + a_1, \\ y(\theta) &= \tfrac{\sin\wp}{\cos^2\wp+\Xi_1^2}\left((\cos\wp - \Xi_1)\sin\left[\cos\wp\theta + a_1\right]\cosh\left[\Xi_1\theta + \Xi_2\right]\right. \\ &\quad \left. + (\cos\wp + \Xi_1)\cos\left[\cos\wp\theta + a_1\right]\sinh\left[\Xi_1\theta + \Xi_2\right]\right) + a_2, \\ z(\theta) &= \tfrac{\sin\wp}{\cos^2\wp+\Xi_1^2}\left((\Xi_1 - \cos\wp)\cos\left[\cos\wp\theta + a_1\right]\cosh\left[\Xi_1\theta + \Xi_2\right]\right. \\ &\quad \left. + (\cos\wp + \Xi_1)\sin\left[\cos\wp\theta + a_1\right]\sinh\left[\Xi_1\theta + \Xi_2\right]\right) + a_3, \end{aligned} \tag{3.10}$$

where a_1, a_2, a_3, Ξ_1, Ξ_2 *are constants of integration and* $\wp$ *is constant angle.*

Proof. Suppose that γ is a non geodesic spacelike biharmonic curve. Substituting the first equation of the Frenet equations (3.8) in the second equation of (3.8), we obtain

$$\mathbf{b}(\theta) = \frac{1}{f(\theta)}\left[\nabla^2_{\mathbf{t}(s)}\mathbf{t}(\theta) - \mathbf{t}(\theta)\right]. \tag{3.11}$$

Using the last equation of (3.8), we obtain

$$\nabla^3_{\mathbf{t}(s)}\mathbf{t}(\theta) - \left(1 + f^2(\theta)\right)\nabla_{\mathbf{t}(s)}\mathbf{t}(\theta) = 0. \tag{3.12}$$

Since the curve $\gamma(\theta)$ is a spacelike general helix, i.e. the tangent vector $\mathbf{t}(\theta)$ makes a constant angle $\wp$, with the constant spacelike vector called the axis of the general helix. So, without loss of generality, we take the axis of a general helix as being parallel to the spacelike vector $\mathbf{e}_1$. Then, using first equation of (3.9), we get

$$t_1(\theta) = g(\mathbf{t}(\theta), \mathbf{e}_1) = \cos\wp. \tag{3.13}$$

On other hand, the tangent vector $\mathbf{T}(\theta)$ is a unit spacelike vector, so the following condition is satisfied:

$$t_2^2(\theta) - t_3^2(\theta) = 1 - \cos^2\wp. \tag{3.14}$$

The general solution of (3.14) can be written in the following form:

$$\begin{aligned} t_2(\theta) &= \sin\wp\cosh\sigma(\theta), \\ t_3(\theta) &= \sin\wp\sinh\sigma(\theta), \end{aligned} \tag{3.15}$$

where σ is an arbitrary function of θ.

So, substituting the components $t_1(\theta)$, $t_2(\theta)$ and $t_3(\theta)$ in the first equation of (3.9), we have the following equation

$$\mathbf{t} = \cos\wp\mathbf{e}_1 + \sin\wp\cosh\sigma(\theta)\,\mathbf{e}_2 + \sin\wp\sinh\sigma(\theta)\,\mathbf{e}_3. \tag{3.16}$$

If we substitute (3.5) in (3.12), we have

$$\sigma'(\theta)\,\sigma''(\theta) = 0. \tag{3.17}$$

The general solution of (3.17) is

$$\sigma(\theta) = \Xi_1\theta + \Xi_2, \tag{3.18}$$

where Ξ_1, Ξ_2 are constants of integration.

Thus (3.16) and (3.18), imply

$$\mathbf{t} = \cos\wp\mathbf{e}_1 + \sin\wp\cosh[\Xi_1\theta + \Xi_2]\,\mathbf{e}_2 + \sin\wp\sinh[\Xi_1\theta + \Xi_2]\,\mathbf{e}_3. \tag{3.19}$$

Using (2.1) in (3.19), we obtain

$$\begin{aligned} \mathbf{t} = \; & (\cos\wp, \cos[\cos\wp\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ & -\sin[\cos\wp\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2], \\ & \sin[\cos\wp\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ & +\cos[\cos\wp\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2]), \end{aligned} \tag{3.20}$$

where a_1 is constant of integration.

Also, we have

$$\begin{aligned} \frac{dx}{d\theta} &= \cos\wp, \\ \frac{dy}{d\theta} &= \cos[\cos\wp\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ &\quad -\sin[\cos\wp\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2], \\ \frac{dz}{d\theta} &= \sin[\cos\wp\theta + a_1]\sin\wp\cosh[\Xi_1\theta + \Xi_2] \\ &\quad +\cos[\cos\wp\theta + a_1]\sin\wp\sinh[\Xi_1\theta + \Xi_2]. \end{aligned} \tag{3.21}$$

If we take the integral (3.21), we get (3.10). Thus, the proof is completed.

Theorem 3.3. *Let $\gamma : I \longrightarrow \mathbb{E}(2)$ is a non geodesic spacelike biharmonic general helix with timelike normal in the Lorentzian group of rigid motions $\mathbb{E}(2)$. Then, the parametric equations of*

γ *are*

$$\begin{aligned} x^1(s) &= \cos\wp\kappa s + a_1, \\ x^2(s) &= \frac{\sin\wp}{\cos^2\wp+\Xi_1^2}((\cos\wp - \Xi_1)\sin[\cos\wp\kappa s + a_1]\cosh[\Xi_1\kappa s + \Xi_2] \\ &\quad + (\cos\wp + \Xi_1)\cos[\cos\wp\kappa s + a_1]\sinh[\Xi_1\kappa s + \Xi_2]) + a_2, \\ x^3(s) &= \frac{\sin\wp}{\cos^2\wp+\Xi_1^2}((\Xi_1 - \cos\wp)\cos[\cos\wp\kappa s + a_1]\cosh[\Xi_1\kappa s + \Xi_2] \\ &\quad + (\cos\wp + \Xi_1)\sin[\cos\wp\kappa s + a_1]\sinh[\Xi_1\kappa s + \Xi_2]) + a_3, \end{aligned} \tag{3.22}$$

where a_1, a_2, a_3 *are constants of integration.*

Proof. From first equation of (3.4) and the definition of θ, we have

$$\theta = \kappa s. \tag{3.23}$$

So, substituting (3.23) in the system (3.10), we have (3.22) and the assertion is proved.

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