Envelopes of commutative rings

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Abstract

Given a significative class $\mathcal{F}$ of commutative rings, we study the precise conditions under which a commutative ring R has an $\mathcal{F}$-envelope. A full answer is obtained when $\mathcal{F}$ is the class of fields, semisimple commutative rings or integral domains. When $\mathcal{F}$ is the class of Noetherian rings, we give a full answer when the Krull dimension of R is zero and when the envelope is required to be epimorphic. The general problem is reduced to identifying the class of non-Noetherian rings having a monomorphic Noetherian envelope, which we conjecture is the empty class.

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1   Introduction

The classical concepts of injective envelope and projective cover of a module led to the introduction of envelopes and covers with respect to an arbitrary class of objects in a given category. These more general concepts were introduced by Enochs ([7] and, under the names of left and right minimal approximations, by Auslander's school ([4], [3]). Lets recall their definition. Given an arbitrary category $\mathcal{C}$, a morphism $f : X \longrightarrow Y$ in it is called left **minimal** if every endomorphism $v : Y \longrightarrow Y$ such that $vf = f$ is necessarily an isomorphism. Dually one defines the concept of right minimal morphism. If $\mathcal{F}$ is a given class of objects in $\mathcal{C}$, a $\mathcal{F}$-**preenvelope** of an object X is a morphism $f : X \longrightarrow F$, with $F \in \mathcal{F}$, satisfying the property that any morphism $X \longrightarrow F'$ to an object of $\mathcal{F}$ factors through f. When, in addition, the morphism f is left minimal that preenvelope is called a $\mathcal{F}$-**envelope**. The concepts of $\mathcal{F}$-precover and $\mathcal{F}$-cover are defined dually. Since in this paper we shall deal only with preenvelopes and envelopes, a left minimal morphism will be called simply minimal. The study of envelopes and covers is generally rather fruitful when the class $\mathcal{F}$ is a significative one, i.e., a class of objects having nice properties from different points of views (homological, arithmetical, etc).

In particular, the concepts have proved very useful in module categories and, more generally, in the context of arbitrary additive categories. For example, a long standing open question asked wether every module had a flat cover. The question was answered affirmatively by Bican, El Bashir and Enochs ([5]) for modules over an associative ring with unit and it has turned out to be very useful in the study of adjoints in the homotopy category of an abelian category, in particular in the homotopy category of a module category or that of quasicoherent sheaves on a scheme (see, e.g. [14] and [13]).

It seems however that, apart from the additive 'world', the concepts have been somehow neglected. In this paper we consider initially the situation when $\mathcal{C} = Rings$ is the category of rings (always associative with unit in this paper) and $\mathcal{F}$ is a significative class of commutative rings. If $CRings$ denotes the category of commutative rings, then the forgetful functor $j :$ $CRings \longrightarrow Rings$ has a left adjoint which associates to any ring R its quotient R_{com} by the ideal generated by all differences $ab - ba$, with $a, b \in R$. As a consequence if $f : R \longrightarrow F$ is a morphism, with $F \in \mathcal{F}$, it is uniquely factored in the form

$$R \overset{pr}{\twoheadrightarrow} R_{com} \overset{\bar{f}}{\longrightarrow} F$$

and one readily sees that f is a $\mathcal{F}$-(pre)envelope if, and only if, so is $\bar{f}$. That allows us to restrict to the world of commutative rings all through the paper. So in the sequel, unless otherwise specified, the term 'ring' will mean 'commutative ring'.

Our initial motivation for the paper was of geometric nature. Algebraic schemes have the nicest properties when they enjoy some sort of Noetherian condition. Therefore it is natural to try to approximate any given scheme by a Noetherian one and, as usual in Algebraic Geometry, the first step should be to understand the affine case. Given the duality between the categories of affine schemes and rings [9], our initial task was to understand envelopes and cover in $CRings$ with respect to the class of Noetherian rings. But once arrived at this step, it was harmless to try an analogous study with respect to other significative classes of rings (e.g. fields, semisimple rings or domains).

The content of our paper is devoted to the study of envelopes of rings with respect to those significative classes. The organization of the papers goes as follows. The results of section 2 are

summarized in the following table, where $\mathcal{F}$ is a class of commutative rings:

$\mathcal{F}$	rings R having a $\mathcal{F}$-envelope	$\mathcal{F}$-envelope
fields	R local and $K - dim(R) = 0$	$R \xrightarrow{pr} R/\mathbf{m}$, $\mathbf{m}$ maximal
semisimple rings	$Spec(R)$ finite	$R \xrightarrow{can} \prod_{\mathbf{p} \in Spec(R)} k(\mathbf{p})$
integral domains	$Nil(R)$ is a prime ideal	$R \xrightarrow{pr} R_{red} = R/Nil(R)$

In the subsequent sections we study the more complicated case of **Noetherian envelopes**, i.e., envelopes of rings in the class of Noetherian rings. In section 3 we prove that if R is a ring having a Noetherian preenvelope then R satisfies ACC on radical ideals and $Spec(R)$ is a Noetherian topological space with the Zariski topology (Proposition 3.1). In section 4 we prove that a ring of zero Krull dimension has a Noetherian (pre)envelope if, and only it, it is a finite direct product of local rings which are Artinian modulo the infinite radical (Theorem 4.7). In section 5 we show that a ring R has an epimorphic Noetherian envelope if, and only if, it has a nil ideal I such that R/I is Noetherian and $\mathbf{p}I_{\mathbf{p}} = I_{\mathbf{p}}$, for all $\mathbf{p} \in Spec(R)$ (Theorem 5.2). After this last result, the identification of those rings having a Noetherian envelope reduces to identify those having a monomorphic Noetherian envelope. We then tackle in the final section the problem of the existence of a non-Noetherian ring with a monomorphic Noetherian envelope. The existence of such a ring would lead to the existence of a 'minimal' local one (Proposition 6.1) of which the trivial extension $\mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ would be the prototype. We prove that this ring does not have an Noetherian envelope (Theorem 6.3) and conjecture that there does not exist any non-Noetherian ring having a monomorphic Noetherian envelope.

The notation and terminology on commutative rings followed in the paper is standard. The reader is referred to any of the classical textbooks [2], [10] and [11] for all undefined notions. For the little bit of Category theory that we need, the reader is referred to [12].

2   Envelopes of rings in some significative classes

In this section we will have a class $\mathcal{F}$ of (always commutative) rings, made precise at each step, and we shall identify those rings which have a $\mathcal{F}$-(pre)envelope. A trivial but useful fact will be used all through, namely, that if $f : R \longrightarrow F$ is a $\mathcal{F}$-(pre)envelope, then the inclusion $Im(f) \hookrightarrow F$ is also a $\mathcal{F}$-(pre)envelope.

Our first choice of $\mathcal{F}$ is the class of fields or the class of semisimple rings. For the study of envelopes in these classes, the following well-known result will be used. We include a short proof for completeness.

Lemma 2.1 *Let* $\mathbf{p}$ *and* $\mathbf{q}$ *two prime ideals of* R *and* $u_\mathbf{p} : R \longrightarrow k(\mathbf{p})$ *and* $u_\mathbf{q} : R \longrightarrow k(\mathbf{q})$ *the canonical ring homomorphisms to the respective residue fields. If* $h : k(\mathbf{p}) \longrightarrow k(\mathbf{q})$ *is a field homomorphism such that* $hu_\mathbf{p} = u_\mathbf{q}$, *then* $\mathbf{p} = \mathbf{q}$ *and* $h = 1_{k(\mathbf{p})}$ *is the identity map.*

Proof. We have $u_\mathbf{q}(\mathbf{p}) = (hu_\mathbf{p})(\mathbf{p}) = 0$. That means that the ideal $\mathbf{p}R_\mathbf{q}$ of $R_\mathbf{q}$ is mapped onto zero by the canonical projection $R_\mathbf{q} \twoheadrightarrow k(\mathbf{q})$. In case $\mathbf{p} \not\subseteq \mathbf{q}$, that leads to contradiction for $\mathbf{p}R_\mathbf{q} = R_\mathbf{q}$. So we can assume $\mathbf{p} \subseteq \mathbf{q}$. If this inclusion is strict, then we choose an element $s \in \mathbf{q} \setminus \mathbf{p}$ and get that $h(s+\mathbf{p}) = h(u_\mathbf{p}(s)) = u_\mathbf{q}(s) = s+\mathbf{q} = 0$. Since every field homomoprhism is injective, we conclude that $s + \mathbf{p} = 0$ in $k(\mathbf{p})$, which is false because $0 \neq s + \mathbf{p} \in R/\mathbf{p} \subset k(\mathbf{p})$.

We then necessarily have $\mathbf{p} = \mathbf{q}$. If we denote by $i_\mathbf{p} : R/\mathbf{p} \hookrightarrow k(\mathbf{p})$ the inclusion, then we get that $hi_\mathbf{p} = i_\mathbf{p}$ and, by the universal property of localization with respect to multiplicative sets, we conclude that $h = 1_{k(\mathbf{p})}$. $\blacksquare$

Theorem 2.2 *Let* $\mathcal{F}$ *be the class of fields, let* $\mathcal{S}$ *be the class of semisimple commutative rings and let* R *be any given commutative ring. The following assertions hold:*

1. *R has a $\mathcal{F}$-(pre)envelope if, and only if, R is local and $K - dim(R) = 0$. In that case, the projection $R \twoheadrightarrow R/\mathbf{m}$ is the $\mathcal{F}$-envelope, where $\mathbf{m}$ is the maximal ideal.*

2. *R has a $\mathcal{S}$-(pre)envelope if, and only if, $Spec(R)$ is finite. In that case, the canonical map $R \longrightarrow \prod_{\mathbf{p} \in Spec(R)} k(\mathbf{p})$ is the $\mathcal{S}$-envelope.*

Proof. 1) Suppose that $f : R \longrightarrow F$ is a $\mathcal{F}$-preenvelope. If $\mathbf{m}$ is any maximal ideal of R, then the canonical projection $p : R \twoheadrightarrow R/\mathbf{m}$ factors through f, so that we have a field homomorphism $h : F \longrightarrow R/\mathbf{m}$ such that $hf = p$. Then $f(\mathbf{m}) \subseteq Ker(h) = 0$, so that $\mathbf{m} \subseteq Ker(f)$ and hence $\mathbf{m} = Ker(f)$. It follows that R is local with $Ker(f)$ as unique maximal ideal. Let then $\bar{f} : R/Ker(f) \longrightarrow F$ be the field homomorphism such that $\bar{f}p = f$ and, using the $\mathcal{F}$-preenveloping condition of f, choose a field homomorphism $g : F \longrightarrow R/Ker(f)$ such that $gf = p$. Then we have that $g\bar{f}p = gf = p$, so that $g\bar{f} = 1$ and hence $\bar{f}$ and g are isomorphisms. Since there is a canonical ring homomorphism $R \longrightarrow k(\mathbf{p})$ for every $\mathbf{p} \in Spec(R)$, Lemma 2.1 implies that $Spec(R) = \{Ker(f)\}$ and, hence, that $K - dim(R) = 0$.

Conversely, let R be a local ring with maximal ideal $\mathbf{m}$ such that $K - dim(R) = 0$. Any ring homomorphism $f : R \longrightarrow F$, which F field, has a prime ideal as kernel. Then $Ker(f) = \mathbf{m}$ and f factors through the projection $p : R \twoheadrightarrow R/\mathbf{m}$. This projection is then the $\mathcal{F}$-envelope of R.

2) It is well-known that a commutative ring is semisimple if, and only if, it is a finite direct product of fields. Given a ring homomorphism $f : R \longrightarrow S$, with S semisimple, it follows that

f is a $\mathcal{S}$-preenvelope if, and only if, every ring homomorphism $g : R \longrightarrow K$, with K a field, factors through f. If we fix a decomposition $S = K_1 \times ... \times K_r$, where the K_i are fields, any ring homomorphism $h : S \longrightarrow K$ to a field vanishes on all but one of the canonical idempotents $e_i = (0, ..., \overset{i}{1}, ...0)$, so that h can be represented by a matrix map

$$h = \begin{pmatrix} 0 & ... & 0 & h' & 0 & ... & 0 \end{pmatrix} : K_1 \times ... \times K_r \longrightarrow K,$$

where $h' : K_i \longrightarrow K$ is a field homomorphism. We shall frequently use these facts.

Suppose that $f = \begin{pmatrix} f_1 \\ . \\ . \\ f_r \end{pmatrix} : R \longrightarrow K_1 \times ... \times K_r$ is a $\mathcal{S}$-preenvelope (each $f_i : R \longrightarrow K_i$ being a ring homomorphism). For every $j \in \{1, ..., r\}$, we put $\mathbf{p}_j := Ker(f_j)$, which is a prime ideal of R. By the universal property of localization, there is a unique field homomorphism $g_j : k(\mathbf{p}_j) \longrightarrow K_j$ such that $g_j u_{\mathbf{p}_j} = f_j$. Let now $\mathbf{p} \in Spec(R)$ be any prime ideal. The canonical map $u_{\mathbf{p}} : R \longrightarrow k(\mathbf{p})$ factors through f and, by the last paragraph we get an index $i \in \{1, ..., r\}$ together with a morphism $h' : K_i \longrightarrow k(\mathbf{p})$ such that $h'f_i = u_{\mathbf{p}}$. But then $h'g_i u_{\mathbf{p}_i} = u_{\mathbf{p}}$ and Lemma 2.1 tells us that $\mathbf{p} = \mathbf{p}_i$. That proves that $Spec(R) = \{\mathbf{p}_1, ..., \mathbf{p}_r\}$.

Conversely, suppose that $Spec(R) = \{\mathbf{p}_1, ..., \mathbf{p}_r\}$ is finite. If $g : R \longrightarrow K$ is a ring homomorphism, with K a field, then $\mathbf{q} := Ker(g)$ is a prime ideal and g factors through $u_{\mathbf{q}} : R \longrightarrow k(\mathbf{q})$ and, hence, also through the canonical map $f : R \longrightarrow \prod_{1 \leq i \leq r} k(\mathbf{p}_i)$. So f becomes a $\mathcal{S}$-preenvelope. It only remains to check that it is actually an envelope. Indeed, if $\varphi : \prod_{1 \leq i \leq r} k(\mathbf{p}_i) \longrightarrow \prod_{1 \leq i \leq r} k(\mathbf{p}_i)$ is a ring homomorphism such that $\varphi f = f$ then, bearing in mind that $f = \begin{pmatrix} u_{\mathbf{p}_1} \\ . \\ . \\ u_{\mathbf{p}_r} \end{pmatrix}$, Lemma 2.1 tells us that $\varphi = 1_{\prod k(\mathbf{p}_i)}$ is the identity map. $\blacksquare$

Example 2.3 *A Noetherian ring of zero Krull dimension is a typical example of ring having a semisimple envelope. An example with nonzero Krull dimension is given by a discrete valuation domain D (e.g. the power series algebra $K[[X]]$ over the field K).*

We end this section with the characterization of rings which have preenvelopes in the class of integral domains. Recall that a ring R is reduced if $Nil(R) = 0$. We define the reduced ring associated to a ring R as $R_{red} = R/Nil(R)$.

Proposition 2.4 *Let R be a commutative ring and $\mathcal{D}$ the class of integral domains. The following conditions are equivalent:*

1. R has a $\mathcal{D}$-(pre)envelope;

2. $Nil\,(R)$ is a prime ideal of R.

In that case, the projection $p : R \twoheadrightarrow R_{red}$ is the $\mathcal{D}$-envelope.

Proof. 2) $\Longrightarrow$ 1) Every ring homomorphism $f : R \longrightarrow D$, with $D \in \mathcal{D}$, vanishes on $Nil(R)$. That proves that f factors through $p : R \twoheadrightarrow R_{red}$, so that this latter map is a $\mathcal{D}$-envelope.

1) $\Longrightarrow$ 2) Let $f : R \longrightarrow D$ be a $\mathcal{D}$-preenvelope. Then $f(Nil(R)) = 0$ and the induced map $\bar{f} : R_{red} \longrightarrow D$ is also a $\mathcal{D}$-preenvelope. Replacing R by R_{red} if necessary, we can and shall assume that R is reduced and will have to prove that then R is an integral domain.

Indeed $\mathbf{q} := Ker(f)$ is a prime ideal and the projection $\pi_{\mathbf{q}} : R \twoheadrightarrow R/\mathbf{q}$ is also a $\mathcal{D}$-preenvelope. It is actually a $\mathcal{D}$-envelope since it is surjective. But then, for every $\mathbf{p} \in Spec(R)$, the projection $\pi_{\mathbf{p}} : R \twoheadrightarrow R/\mathbf{p}$ factors through $\pi_{\mathbf{q}}$. This implies that $\mathbf{q} \subseteq \mathbf{p}$, for every $\mathbf{p} \in Spec(R)$, and hence that $\mathbf{q} \subset \bigcap_{\mathbf{p} \in Spec(R)} \mathbf{p} = Nil(R) = 0$ (cf. [10][Corollary I.4.5]). Therefore $0 = \mathbf{q}$ is a prime ideal, so that R is an integral domain. ∎

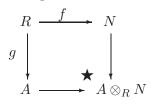# 3 Rings with a Noetherian preenvelope

All through this section we fix a ring R having a Noetherian preenvelope $f : R \longrightarrow N$. An ideal I of R will be called **restricted** if $I = f^{-1}(f(I)N)$ or, equivalently, if $I = f^{-1}(J)$ for some ideal J of N. The following result gathers some useful properties of the rings having a Noetherian preenvelope.

Proposition 3.1 *Let $f : R \longrightarrow N$ be a Noetherian preenvelope. The following assertions hold:*

1. *$Ker(f)$ is contained in $Nil(R)$*

2. *Every radical ideal of R is restricted*

3. *R satisfies ACC on restricted ideals*

4. *$Spec(R)$ is a Noetherian topological space with the Zariski topoloy. In particular, if I is an ideal of R there are only finitely many prime ideals minimal over I*

Proof. 1) If follows from the fact that, for every $\mathbf{p} \in Spec(R)$, the canonical map $u_{\mathbf{p}} : R \longrightarrow k(\mathbf{p})$ factors through f, and hence $Ker(f) \subseteq Ker(u_{\mathbf{p}}) = \mathbf{p}$.

2) Since the assignment $J \rightsquigarrow f^{-1}(J)$ preserves intersections, it will be enough to prove that every prime ideal of R is restricted. If $g : R \longrightarrow A$ is any ring homomorphism and we consider the R-module structures on A and N given by restriction of scalars via g and f, respectively, then $A \otimes_R N$ becomes an R-algebra which fits in the following pushout in $CRings$:

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & N \\ {\scriptstyle g}\downarrow & & \downarrow \\ & \bigstar & \\ A & \longrightarrow & A \otimes_R N \end{array}$$

In case $A \otimes_R N$ is Noetherian, the universal property of pushouts tells us that the botton map $A \longrightarrow A \otimes_R N$ is a Noetherian preenvelope of A. We shall frequently use this fact in the paper.

For our purposes in this proof, we take $A = R/\mathbf{p}$ and $g = \pi_{\mathbf{p}} : R \twoheadrightarrow R/\mathbf{p}$, the projection, for any fixed $\mathbf{p} \in Spec(R)$. Then the map $\bar{f} : R/\mathbf{p} \longrightarrow N/f(\mathbf{p})N \cong (R/\mathbf{p}) \otimes_R N$ is a Noetherian preenvelope and, in particular, the inclusion $i_{\mathbf{p}} : R/\mathbf{p} \hookrightarrow k(\mathbf{p})$ factors through it. This implies that $f^{-1}(f(\mathbf{p})N)/\mathbf{p} = Ker(\bar{f}) \subseteq Ker(i_{\mathbf{p}}) = 0$ and, hence, that $\mathbf{p}$ is a restricted ideal.

3) Clear since N is Noetherian and every ascending chain $I_0 \subseteq I_1 \subseteq ...$ of restricted ideals of R is the preimage of the chain $f(I_0)N \subseteq f(I_1)N \subseteq ...$ of ideals of N.

4) There is an order-reversing bijection between (Zariski-)closed subsets of $Spec(R)$ and radical ideals of R. Therefore $Spec(R)$ is Noetherian if, and only if, R has ACC on radical ideals (cf. [10][Chapter I, section 2]). But this latter property is satisfied due to assertions 2) and 3). Finally, if I is any ideal of R then the prime ideals of R which are minimal over I are precisely those corresponding to the irreducible components of the closed subset $\mathcal{V}(I) = \{\mathbf{p} \in Spec(R) : I \subseteq \mathbf{p}\}$, which is a Noetherian topological space since so is $Spec(R)$. Therefore those prime ideals are a finite number (cf. [10][Proposition I.2.14]). ∎

4 The case of Krull dimension zero

In this section we shall identify the rings of zero Krull dimension having a Noetherian (pre)envelope.

Lemma 4.1 *Let $R = R_1 \times ... \times R_n$ be a ring decomposed into a finite product of nonzero rings. The following assertions are equivalent:*

1. *R has a Noetherian (pre)envelope*

2. Each R_i has a Noetherian (pre)envelope.

In such a case, if $f_i : R_i \longrightarrow N_i$ is a Noetherian (pre)envelope for each $i = 1, ..., n$, then the diagonal map $f = diag(f_1, ..., f_n) : R_1 \times ... \times R_n \longrightarrow N_1 \times ... \times N_n$ is a Noetherian (pre)envelope.

Proof. We shall prove the equivalence of 1) and 2) for the case of preenvelopes, leaving the minimality of morphisms for the end.

1) $\Longrightarrow$ 2) Fix a Noetherian preenvelope $f : R = R_1 \times \cdots \times R_n \longrightarrow N$. If now $e_i = \left(0, \ldots, \overset{i}{1}, 0, \ldots, 0 \right)$ $(i = 1, \ldots, n)$, then we have that $N_i =: f(e_i) N$ is a Noetherian ring, for every $i = 1, \ldots, n$. Clearly $f(R_i) \subseteq N_i$ and we have an induced ring homomorphism $f_i = f \mid_{R_i} : R_i \longrightarrow N_i$, for every $i = 1, \ldots, n.$. We clearly have $N \cong N_1 \times ... \times N_r$ and f can be identified with the diagonal map

$$diag(f_1, ..., f_n) : R_1 \times ... \times R_n \longrightarrow N_1 \times ... \times N_n.$$

Let now $h : R_i \longrightarrow N'$ be a ring homomorphism, with N' Noetherian. Then the matrix map

$$\left(0 \quad .. \quad 0 \quad \overset{i}{h} \quad 0 \quad .. \quad 0 \right) : R_1 \times ... \times R_n \longrightarrow N'$$

is also a ring homomorphism, which must factor through $f \equiv diag(f_1, ..., f_n)$. That implies that h factors through f_i, so that f_i is a Noetherian preenvelope for each $i = 1, ..., n$.

2) $\Longrightarrow$ 1) Suppose that $f_i : R_i \longrightarrow N_i$ is a Noetherian preenvelope for $i = 1, \ldots, n$ and let us put

$$f := diag(f_1, ..., f_n) : R_1 \times ... \times R_n \longrightarrow N_1 \times ... \times N_n.$$

Given any ring homomorphism $g : R = R_1 \times \cdots \times R_n \longrightarrow N'$, where N' is a Noetherian ring, put $N_i' = N'g(e_i)$ for $i = 1, ..., n$. Then we have an isomorphism $N' \cong N_1' \times ... \times N_n'$ and the N_i' are also Noetherian rings, some of them possibly zero. Viewing that isomorphism as an identification, we can think of g as a diagonal matrix map

$$g \equiv diag(g_1, ..., g_n) : R_1 \times ... \times R_n \longrightarrow N_1' \times ... \times N_n' \cong N',$$

where each g_i is a ring homomorphism. Then g_i factors through f_i, for every $i = 1, ..., n$, and so g factors through f. Therefore f is a Noetherian preenvelope.

We come now to the minimality of morphisms. We can consider a Noetherian preenvelope given by a diagonal map $f = diag(f_1, ..., f_n) : R_1 \times ... \times R_n \longrightarrow N_1 \times ... \times N_n$, where each f_i is a Noetherian preenvelope. If f is a minimal morphism in $CRings$ one readily sees that each f_i is also minimal. Conversely, suppose that each f_i is minimal and consider any ring homomorphism $\varphi : N_1 \times ... \times N_n \longrightarrow N_1 \times ... \times N_n$ such that $\varphi f = f$. We can identify φ with a matrix (φ_{ij}), where $\varphi_{ij} : N_j \longrightarrow N_i$ is a a map preserving addition and multiplication (but not necessarily the unit) for all $i, j \in \{1, ..., n\}$. Viewing the equality $\varphi f = f$ as a matricial equality, we get:

$$\varphi_{ij} f_j = 0, \text{ for } i \neq j$$
$$\varphi_{ii} f_i = f_i.$$

The first equality for $i \neq j$ gives that $\varphi_{ij}(1) = (\varphi_{ij} f_j)(1) = 0$, and then $\varphi_{ij} = 0$ since φ_{ij} preserves multiplication. Therefore $\varphi = diag(\varphi_{11}, ..., \varphi_{nn})$ and the minimality of the morphisms f_i gives that each φ_{ii} is a ring isomorphism. It follows that φ is an isomorphism. ∎

Lemma 4.2 *If R has a Noetherian preenvelope and $K - dim(R) = 0$ then R is finite direct product of local rings with zero Krull dimension.*

Proof. If $K - dim(R) = 0$ and R has a Noetherian preenvelope then, since all its prime ideals are both maximal and minimal, Proposition 3.1 tells us that there are only finitely many of them. Then R is a semilocal ring and, since $Nil(R)$ is a nil ideal, idempotents lift modulo $Nil(R)$ (cf. [15][Proposition VIII.4.2]). The result then follows immediatly since $R_{red} = R/Nil(R)$ is a finite direct product of fields (cf. [10], Proposition I.1.5). ∎

The last two lemmas reduce our problem to the case of a local ring. We start by considering the case in which R has a monomorphic Noetherian preenvelope.

Lemma 4.3 *Let R be a local ring with maximal ideal $\mathbf{m}$ and K-dim $(R) = 0$. If R has a monomorphic Noetherian preenvelope then $\mathbf{m}$ is nilpotent. In particular, the following conditions are equivalent:*

1. *R is a Noetherian ring;*

2. *R is an Artinian ring;*

3. *$\mathbf{m}/\mathbf{m}^2$ is a finite dimensional $R/\mathbf{m}$-vector space.*

Proof. Let $j : R \longrightarrow N$ be a Noetherian monomorphic preenvelope of R. Since $\mathbf{m} = Nil(R)$ is nil it follows that $\mathbf{m}N$ is a nil ideal in a Noetherian ring. Then it is nilpotent, and so there is a $n > 0$ such that $\mathbf{m}^n \subset (\mathbf{m}N)^n = 0$.

For the second part, note that when $K\text{-}dim\,(R) = 0$ then R is Artinian if and only if R is Noetherian (see [2, Theorem 8.5]). Assume that $\mathbf{m}/\mathbf{m}^2$ is finitely generated. If $\{x_1, ..., x_r\}$ is a finite set of generators of $\mathbf{m}$ modulo $\mathbf{m}^2$, then the products $x_{\sigma(1)} \cdot ... \cdot x_{\sigma(m)}$, with σ varying in the set of maps $\{1, ..., m\} \longrightarrow \{1, ..., r\}$, generate $\mathbf{m}^m/\mathbf{m}^{m+1}$ both as an R-module and as a $R/\mathbf{m}$-vector space. In particular, each $\mathbf{m}^m/\mathbf{m}^{m+1}$ $(m = 0, 1, ...)$ is an R-module of finite length. Since there is a $n > 0$ such that $\mathbf{m}^n = 0$ we conclude that R has finite length as R module, i.e., R is Artinian. ■

Lemma 4.4 *Let A be a non-Noetherian commutative ring and $\mathbf{a}$ be an ideal of A such that the projection $p : A \twoheadrightarrow A/\mathbf{a}$ is a Noetherian envelope. Then $\mathbf{a}$ does not have a simple quotient.*

Proof. Any simple quotient of $\mathbf{a}$ is isomorphic to $\mathbf{a}/\mathbf{a}'$, for some ideal $\mathbf{a}'$ such that $\mathbf{a}' \subsetneq \mathbf{a}$. Now the canonical exact sequence

$$0 \to \mathbf{a}/\mathbf{a}' \hookrightarrow A/\mathbf{a}' \twoheadrightarrow A/\mathbf{a} \to 0$$

has the property that its outer nonzero terms are Noetherian A-modules. Then $A/\mathbf{a}'$ is Noetherian, both as an A-module and as a ring. But then the projection $q : A \twoheadrightarrow A/\mathbf{a}'$ factors through the Notherian envelope p, which implies that $\mathbf{a} = Ker(p) \subseteq Ker(q) = \mathbf{a}'$. This contradicts our choice of $\mathbf{a}'$. ■

Lemma 4.5 *Let R be a local ring with zero Krull dimension having a monomoprhic Noetherian preenvelope. Then R is Artinian.*

Proof. By Lemma 4.3 it is enough to prove that R is Noetherian. Suppose then that there exists a non-Noetherian local R such that $K - dim(R) = 0$ and R has a monomorphic Noetherian preenvelope. Fix such a preenvelope $j : R \longrightarrow N$ and view it as an inclusion. The set of restricted ideals I of R such that R/I is not Noetherian has a maximal element, say J (see Proposition 3.1). Note that the induced map $\bar{j} : R/J \longrightarrow N/JN$ is also a monomorphic Noetherian preenvelope. Then, replacing R by R/J if necessary, we can and shall assume that R/I is Noetherian, for every restricted ideal $I \neq 0$.

By the proof of assertion 2 in Proposition 3.1, we know that the induced map $R/\mathbf{m}^2 \longrightarrow N/\mathbf{m}^2N$ is a Noetherian preenvelope and it factors in the form

$$R/\mathbf{m}^2 \xrightarrow{p} R/I \rightarrowtail N/\mathbf{m}^2 N,$$

where $I := \mathbf{m}^2 N \cap R$. If $\mathbf{m}^2 \neq 0$ then $p : R/\mathbf{m}^2 \twoheadrightarrow R/I$ would be a Noetherian envelope because I is nonzero and restricted. Since, by Lemma 4.3, $R/\mathbf{m}^2$ is not a Noetherian ring it follows that $\mathbf{m}^2 \subsetneq I$. But then we contradict Lemma 4.4 for $I/\mathbf{m}^2$ is a semisimple $R/\mathbf{m}^2$-module (it is annihilated by $\mathbf{m}/\mathbf{m}^2$) and, hence, it always has simple quotients.

We then have that $\mathbf{m}^2 = 0$. Consider the set

$$\mathcal{I} = \{\mathbf{a} \text{ ideal of } R : 0 \neq \mathbf{a} \subseteq \mathbf{m} \text{ and } \mathbf{a} \text{ not restricted}\}.$$

In case $\mathcal{I} \neq \emptyset$, we pick up any $\mathbf{a} \in \mathcal{I}$. Then $\mathbf{a}N \cap R/\mathbf{a}$ is a nonzero semisimple R-module since it is a subfactor of the $R/\mathbf{m}$-vector space $\mathbf{m}$. We can then find an intermediate ideal $\mathbf{a} \subseteq J \subsetneq \mathbf{a}N \cap R =: J'$ such that J'/J is a simple module. Notice that $\mathbf{a}N = JN = J'N$. Now the induced map $\tilde{j} : R/J \longrightarrow N/JN$ is a Noetherian preenvelope and an argument already used in the previous paragraph shows that the projection $R/J \twoheadrightarrow R/J'$ is a Noetherian envelope. This contradicts Lemma 4.4 for J'/J is simple. Therefore we get $\mathcal{I} = \emptyset$. Since $\mathbf{m}$ is a semisimple R-module, we can take a minimal ideal $I_0 \subset \mathbf{m}$, which is then necessarily restricted. Then I_0 and R/I_0 are both Noetherian R-modules, from which we get that R is a Noetherian ring and, hence, a contradiction. ∎

Definition 4.6 *Let R be a local ring with maximal ideal $\mathbf{m}$ such that $K - dim(R) = 0$. We shall call it* **Artinian modulo the infinite radical** *in case $R/\bigcap_{n>0} \mathbf{m}^n$ is Artinian. Equivalently, if there is an integer $n > 0$ such that $\mathbf{m}^n = \mathbf{m}^{n+1}$ and $R/\mathbf{m}^n$ is Artinian.*

We are now ready to prove the main result of this section.

Theorem 4.7 *Let R be a commutative ring such that K-dim $(R) = 0$. The following assertions are equivalent*

1. *R has a Noetherian (pre)envelope*

2. *R is isomorphic to a finite product $R_1 \times ... \times R_r$, where the R_i are local rings which are Artinian modulo the infinite radical.*

In that situation, if $\mathbf{m}_i$ is the maximal ideal of R_i and $p_i : R_i \twoheadrightarrow R_i/\bigcap_{n>0} \mathbf{m}_i^n$ is the projection, for each $i = 1, ..., r$, then the 'diagonal' map $diag(p_1, ..., p_n) : R_1 \times ... \times R_r \longrightarrow R_1/\bigcap_{n>0} \mathbf{m}_1^n \times ... \times R_r/\bigcap_{n>0} \mathbf{m}_r^n$ is the Noetherian envelope.

Proof. Using Lemmas 4.2 and 4.1, the proof reduces to the case when R is local, something that we assume in the sequel.

1) $\Longrightarrow$ 2) Let $f : R \longrightarrow N$ be the Noetherian preenvelope. Then we have a factorization

$$f : R \overset{p}{\twoheadrightarrow} R/Ker\,(f) \overset{\overline{f}}{\rightarrowtail} N,$$

where $\overline{f}$ is a Noetherian (monomorphic) preenvelope. It follows from Lemma 4.5 that $R/Ker\,(f)$ is artinian. Putting $I = Ker\,(f)$, we then get that the projection $R \overset{p}{\longrightarrow} R/I$ is the Noetherian envelope. The case $I = 0$ is trivial, for then R is Artinian. Suppose that $I \neq 0$. In case $I^2 \neq I$, we get a contradiction with Lemma 4.4 for I/I^2 is a nonzero module over the Artinian ring R/I and, hence, it always has simple quotients. Therefore we have $I = I^2$ in that case, which implies that $I = I^n$, for all $n > 0$, and hence that $I \subseteq \bigcap_{n>0} \mathbf{m}^n$. But, since R/I is Artinian, we have that $\mathbf{m}^n \subseteq I$ for $n >> 0$. It follows that there exists a $k > 0$ such that $I = \mathbf{m}^n$, for all $n \geq k$. Then R is Artinian modulo the infinite radical.

2) $\Longrightarrow$ 1) Let R be local and Artinian modulo the infinite radical and let n be the smallest of the positive integers k such that $\mathbf{m}^k = \mathbf{m}^{k+1}$. Then $R/\mathbf{m}^n = R/\bigcap_{n>0} \mathbf{m}^n$ is Artinian.

We shall prove that if $h : R \longrightarrow N$ is any ring homomorphism, with N Noetherian, then $h(\mathbf{m}^n) = 0$, from which it will follow that the projection $p : R \twoheadrightarrow R/\mathbf{m}^n = R/\bigcap_{k>0} \mathbf{m}^k$ is the Noetherian envelope. Since $\mathbf{m} = Nil(R)$ is a nil ideal of R it follows that $h(\mathbf{m})N$ is a nil ideal of the Noetherian ring N, and thus nilpotent. But the equality $\mathbf{m}^n = \mathbf{m}^k$ implies that $(h(\mathbf{m})N)^n = (h(\mathbf{m})N)^k$, for all $k \geq n$. It then follows that $h(\mathbf{m}^n)N = (h(\mathbf{m})N)^n = 0$ and, hence, that $h(\mathbf{m}^n) = 0$.

The final statement is a direct consequence of the above paragraphs and of Lemma 4.1. $\blacksquare$

Example 4.8 *Let a_n $(n = 1, 2, ...)$ by the n-th term of the Fibonacci sequence $1, 1, 2, 3, 5, ...$ and within the power series algebra $K[[X_1, X_2, ...]]$ over the field K, consider the ideal I generated by the following relations:*

$$X_n = X_{n+1}X_{n+2}$$
$$X_n^{a_n+1} = 0,$$

for all positive integers n. Then $R = K[[X_1, X_2, ...]]/I$ is a local ring of zero Krull dimension which is Artinian modulo the infinite radical, but is not Artinian.

Proof. Since $\mathbf{m} = (X_1, X_2, ...)$ is the only maximal ideal of $K[[X_1, X_2, ...]]$ it follows that $\bar{\mathbf{m}} = \mathbf{m}/I$ is the only maximal ideal of R, so that R is local. On the other hand, the second set of

relations tells us that $\bar{\mathbf{m}}$ is a nil ideal for all generators $x_n := X_n + I$ of $\bar{\mathbf{m}}$ are nilpotent elements. In particular, we have $K - dim(R) = 0$.

On the other hand, the first set of relations tells us that $x_n \in \bar{\mathbf{m}}^2$, for all $n > 0$, so that $\bar{\mathbf{m}} = \bar{\mathbf{m}}^2$. Then $R/\bigcap_{n>0} \bar{\mathbf{m}}^n$ is isomorphic to the field $R/\bar{\mathbf{m}} \cong K$, so that R is Artinian modulo the infinite radical. In order to see that R is not Artinian it will be enough to check that the ascending chain

$$Rx_1 \subseteq Rx_2 \subseteq ... \subseteq Rx_n \subseteq ...$$

is not stationary. Indeed if $Rx_n = Rx_{n+1}$ then there exists $a \in R$ such that $x_{n+1} = ax_n$, and hence $x_{n+1} = ax_{n+1}x_{n+2}$. This gives $x_{n+1}(1 - ax_{n+2}) = 0$. But $1 - ax_{n+2}$ is invertible in R since every power series of the form $1 - X_{n+2}f$ is invertible in $K[[X_1, X_2, ...]]$. It follows that $x_{n+1} = 0$. So the equality $Rx_n = Rx_{n+1} = ...$ implies that $x_{n+k} = 0$, for all $k > 0$. By the first set of relations, this in turn implies that $x_i = 0$ for all $i > 0$.

The proof will be finished if we prove that $x_1 \neq 0$. But if $x_1 = 0$ then in the successive sustitutions using the first set of relations, we shall attain a power x_n^t, with $t > a_n$, for some $n > 0$. The successive sustitutions give $x_1 = x_2x_3 = x_3^2x_4 = x_4^3x_5^2 = x_5^5x_6^3 = ...$ The n-th expression is of the form $x_n^{a_n} x_{n+1}^{a_{n-1}}$, for all $n > 0$, convening that $a_0 = 0$. Therefore none of them is zero. That ends the proof. ■

5   Rings with an epimorphic Noetherian envelope

We start the section with the following lemma.

Lemma 5.1 *Let A be a local Noetherian ring with maximal ideal $\mathbf{m}$. Let M be a (not necessarily finitely generated) A-module such that $Supp(M) = \{\mathbf{m}\}$. If there exists a finite subset $\{x_1, x_2, \dots, x_r\} \subset M$ such that $ann_A(M) = ann_A(x_1, \dots, x_r)$, then $\mathbf{m}M \neq M$.*

Proof. If $Supp(M) = \{\mathbf{m}\}$ then $Supp(Ax) = \{\mathbf{m}\}$, for every $x \in M \setminus \{0\}$. Fixing such an x, we have that $\sqrt{ann_A(x)} = \mathbf{m}$ (cf. [11, Theorem 6.6, pg. 40]). Then there exist an integer $n > 0$ such that $\mathbf{m}^n x = 0$. It follows that $M = \bigcup_{n>0} ann_M(\mathbf{m}^n)$.

If now $\{x_1, x_2, \dots, x_r\} \subset M$ is a finite subset such that $ann_A(M) = ann_A(x_1, \dots, x_r)$, then there exists a large enough $n > 0$ such that $x_i \in ann_M(\mathbf{m}^n)$ for $i = 1, \dots, r$. Then $\mathbf{m}^n \subseteq ann_A(x_1, \dots, x_r) = ann_A(M)$, so that $\mathbf{m}^n M = 0$. If we had $\mathbf{m}M = M$ it would follow that $M = 0$, which is impossible since $Supp(M) \neq \varnothing$. ■

We are now ready to prove the main result of this section. Given any module M, we denote by $M_{\mathbf{p}}$ the localization at the prime ideal $\mathbf{p}$.

Theorem 5.2 *Let R be a ring and I an ideal of R such that R/I is Noetherian. The following assertions are equivalent:*

1. *The projection $p : R \longrightarrow R/I$ is a Noetherian envelope;*

2. *I is a nil ideal and $\mathbf{p}I_{\mathbf{p}} = I_{\mathbf{p}}$, for all $\mathbf{p} \in Spec\,(R)$.*

Proof. 1) $\Longrightarrow$ 2) By Proposition 3.1, we have that $I = Ker\,(p) \subseteq Nil\,(R)$ and so I is a nil ideal.

Let know $\mathbf{p}$ be a prime ideal of R. Then, by the proof of assertion 2 in Proposition 3.1, we see that the canonical projection $\pi = p_{\mathbf{p}} : R_{\mathbf{p}} \longrightarrow R_{\mathbf{p}}/I_{\mathbf{p}}$ is also a Noetherian envelope. So, replacing R and I by $R_{\mathbf{p}}$ and $I_{\mathbf{p}}$ respectively, we can assume that R is local (with maximal ideal $\mathbf{m}$) and have to prove that $\mathbf{m}I = I$. Indeed, if $\mathbf{m}I \neq I$ then $I/\mathbf{m}I$ (and hence I) has simple quotients, which contradicts Lemma 4.4.

2) $\Longrightarrow$ 1) We have to prove that if $f : R \longrightarrow N$ is a ring homomorphism, with N Noetherian, then $f\,(I) = 0$. Suppose that is not true and fix an f such that $f\,(I) \neq 0$. Note that $f\,(Nil\,(R))\,N$ is a nil ideal of the Noetherian ring N. It follows that $f\,(Nil\,(R))\,N$ is nilpotent and, as a consequence, that $f\,(Nil\,(R)\,I)\,N = f\,(Nil\,(R))\,Nf\,(I)\,N \neq f\,(I)\,N$ for otherwise we would get $f\,(I)\,N = 0$ and hence $f\,(I) = 0$, against the assumption.

We next consider the composition

$$I \xrightarrow{\;f\;} f\,(I)\,N \longrightarrow f\,(I)\,N/f\,(Nil\,(R)\,I)\,N.$$

Its kernel is $I \cap f^{-1}\,(f\,(Nil\,(R)\,I)\,N)$ and we get a monomorphism of R-modules

$$\overline{f} : M =: I/\left[I \cap f^{-1}\,(f\,(Nil\,(R)\,I)\,N)\right] \longrightarrow f\,(I)\,N/f\,(Nil\,(R)\,I)\,N.$$

Note that $M \neq 0$ for otherwise we would have $f\,(I) \subseteq f\,(Nil\,(R)\,I)\,N$ and hence $f\,(I)\,N = f\,(Nil\,(R)\,I)\,N$, that we have seen that is impossible.

On the other hand, since I is a nil ideal, we have that $I \subseteq Nil\,(R)$ and hence $R_{red} \cong (R/I)_{red}$ is a Noetherian ring. We shall view this latter isomorphism as an identification and put $A = R_{red}$ in the sequel. Since we clearly have $Nil\,(R)\,M = 0$ it follows that M is an A-module in the canonical way. Moreover $Im\,(\overline{f})$ generates the finitely generated N-module

$f(I) N/f(Nil(R) I) N$, which allows us to choose a finite subset $\{x_1, x_2, \ldots, x_r\} \subset M$ such that $\{\overline{f}(x_1), \ldots, \overline{f}(x_r)\}$ generates $f(I) N/f(Nil(R) I) N$ (as an N-module). We shall derive from that that $ann_A(M) = ann_A(x_1, \ldots, x_r)$. Indeed if $\overline{r} = r + Nil(R)$ is an element of A such that $rx_i = 0$ for $i = 1, \ldots, r$, then $f(r) \overline{f}(x_i) = 0$ for all $i = 1, \ldots, r$. It follows that $f(r) f(I) N \subseteq f(Nil(R) I) N$, so that $f(rI) \subseteq f(Nil(R) I) N$ and hence $rI \subseteq f^{-1}(f(Nil(R) I) N)$. This implies that $\overline{r}M = rM = 0$.

We claim now that

$$Supp_A(M) = \mathcal{V}(ann_A(M)) = \{\mathbf{q} \in Spec(A) : ann_A(M) \subseteq \mathbf{q}\}.$$

Indeed if $\mathbf{q} \in Spec(A)$ and $ann_A(x_1, \ldots, x_r) = ann_A(M) \nsubseteq \mathbf{q}$, then we can find an element $s \in A \backslash \mathbf{q}$ such that $sx_i = 0$, for $i = 1, \ldots, r$, and hence such that $sM = 0$. It follows that $M_{\mathbf{q}} = 0$ and so $\mathbf{q} \notin Supp(M)$. That proves that $Supp(M) \subseteq \mathcal{V}(ann_A(M))$. On the other hand, if $\mathbf{q} \notin Supp(M)$ then, for every $i = 1, \ldots, r$, we can find an element $s_i \in A \backslash \mathbf{q}$ such that $s_i x_i = 0$. Then $s = s_1 \cdots \cdots s_r$ belongs to $ann_A(x_1, \ldots, x_r) = ann_A(M)$, so that $ann_A(M) \subsetneq \mathbf{q}$ and hence $\mathbf{q} \notin \mathcal{V}(ann_A(M))$.

The equality $Supp_A(M) = \mathcal{V}(ann_A(M))$ and the fact that we are assuming $M \neq 0$ (and hence $ann_A(M) \neq A$) imply that we can pick up a prime ideal $\mathbf{q}$ of A which is minimal among those containing $ann_A(M)$, and thereby minimal in $Supp_A(M)$. We localize at $\mathbf{q}$ and obtain a module $M_{\mathbf{q}}$ over the local ring $A_{\mathbf{q}}$ such that $Supp_{A_{\mathbf{q}}}(M_{\mathbf{q}}) = \{\mathbf{q}A_{\mathbf{q}}\}$. Moreover the finite subset $\{x_1, \ldots, x_r\} \subseteq M_{\mathbf{q}}$ satisfies that $ann_{A_{\mathbf{q}}}(x_1, \ldots, x_r) = ann_{A_{\mathbf{q}}}(M_{\mathbf{q}})$. Indeed if $a/s \in A_{\mathbf{q}}$ satisfies that $ax_i/s = 0$ in $M_{\mathbf{q}}$, for all $i = 1, \ldots, r$, then we can find an element $t \in A \backslash \mathbf{q}$ such that $tax_i = 0$, for $i = 1, \ldots, r$. Since $ann_A(M) = ann_A(x_1, \ldots, x_r)$ it follows that $ta \in ann_A(M)$ and, hence, that $a/s = ta/ts \in ann_{\acute{A}_{\mathbf{q}}}(M_{\mathbf{q}})$. Now we can apply Lemma 0.4 to the local Noetherian ring $A_{\mathbf{q}}$ and the $A_{\mathbf{q}}$-module $M_{\mathbf{q}}$. We conclude that $\mathbf{q}M_{\mathbf{q}} \neq M_{\mathbf{q}}$.

We take now the prime ideal $\mathbf{p}$ of R such that $\mathbf{p}/Nil(R) = \mathbf{q}$. From the equality $\mathbf{p}I_{\mathbf{p}} = I_{\mathbf{p}}$ it follows that $\mathbf{p}M_{\mathbf{p}} = M_{\mathbf{p}}$. We will derive that $M_{\mathbf{q}} = \mathbf{q}M_{\mathbf{q}}$, thus getting a contradiction and ending the proof. Let $x \in M$ be any element. Since $x \in \mathbf{p}M_{\mathbf{p}}$, we have an equality $x = \sum_{1 \leq j \leq m} p_j y_j/s_j$, where $p_j \in \mathbf{p}$, $y_j \in M$ and $s_j \in R \backslash \mathbf{p}$. Multiplying by $s = s_1 \cdots s_m$, we see that $sx \in \mathbf{p}M$, which is equivalent to sayt that $\overline{s}x \in \mathbf{q}M$, where $\overline{s} = s + Nil(R) \in A \backslash \{\mathbf{q}\}$. Then we have that $x = \overline{s}x/\overline{s} \in \mathbf{q}M_{\mathbf{q}}$, for every $x \in M$, which implies that $M_{\mathbf{q}} = \mathbf{q}M_{\mathbf{q}}$ as desired. $\blacksquare$

Recall that the **trivial extension** of a ring A by the A-module N, denoted by $R = A \ltimes N$, has as underlying additive abelian group $A \oplus N$ and the multiplication is defined by the rule $(a, m) \cdot (b, n) = (ab, an + bm)$.

Example 5.3 *1. Let A be a Noetherian integral domain, X be a finitely generated A-module and D a torsion divisible A-module. Put $N = X \oplus D$ and take $R = A \ltimes N$. The ideal $I = 0 \ltimes D = \{(a,n) \in R : a = 0 \text{ and } n \in D\}$ satisfies condition 2 in the above theorem and that R/I is Noetherian. Therefore $p : R \longrightarrow R/I \cong A \ltimes X$ is the Noetherian envelope.*

2. If in the above example we do not assume D to be torsion then, with the same choice of I, the projection $p : R \longrightarrow R/I$ is not a Noetherian envelope.

Proof.

1. $Nil(R) = 0 \ltimes N$ is contained in any prime ideal of R, which implies that any such prime ideal is of the form $\widehat{\mathbf{p}} = \mathbf{p} \ltimes N$, where $\mathbf{p} \in Spec\,(A)$. Now one check the following equalities:

   (a) $R_{\widehat{\mathbf{p}}} = A_{\mathbf{p}} \ltimes N_{\mathbf{p}} = A_{\mathbf{p}} \ltimes (X_{\mathbf{p}} \oplus D_{\mathbf{p}})$

   (b) $I_{\widehat{\mathbf{p}}} = 0 \ltimes (0 \oplus D_{\mathbf{p}})$

   (c) $\widehat{\mathbf{p}}I_{\widehat{\mathbf{p}}} = 0 \ltimes (0 \oplus \mathbf{p}D_{\mathbf{p}})$.

   The divisibility of D gives that $D = \mathbf{p}D$, for every $\mathbf{p} \in Spec\,(A) \setminus \{0\}$, while we have that D_0 (=localization at $\mathbf{p} = \mathbf{0}$) is zero due to the fact that D is a torsion A-module. That proves that $\widehat{\mathbf{p}}I_{\widehat{\mathbf{p}}} = I_{\widehat{\mathbf{p}}}$, for all $\widehat{\mathbf{p}} \in Spec\,(R)$.

   2) The argument of the above paragraph shows that if D is not torsion (and hence $D_0 \neq 0$) then I does not satifies condition 2 of the Theorem. ∎

6   Does there exist a non-Noetherian ring with a monomorphic Noetherian envelope?

If $f : R \longrightarrow N$ is a Noetherian (pre)envelope of the ring R, then the inclusion $R' := Im(f) \hookrightarrow N$ is a monomorphic Noetherian (pre)envelope. In order to identify the rings having a Noetherian envelope, one needs to identify those having a monomorphic Noetherian envelope. That makes pertinent the question in the title of this section, which we address from now on. We start with the following result:

Proposition 6.1 *Suppose that there exists a non-Noetherian ring having a monomorphic Noetherian preenvelope. Then there is a non-Noetherian local ring R (with maximal ideal $\mathbf{m}$) having a monomorphic Noetherian preenvelope $j : R \hookrightarrow N$ satisfying the following properties:*

1. R has finite Krull dimension $K - dim(R) = d > 0$, and every ring of Krull dimension $< d$ having a monomorphic Noetherian preenvelope is Noetherian

2. R_{red} is a Noetherian ring

3. If $0 \neq I \subseteq Nil(R)$ is an ideal such that R/I is non-Noetherian, then $D := \frac{IN \cap R}{I}$ is an R-module such that $Supp(D) = \{\mathbf{m}\}$ and $\mathbf{m}D = D$.

Proof. If A is a non-Noetherian ring with a monomorphic Noetherian preenvelope $i : A \rightarrowtail N'$, then the set of restricted ideals $I' \subseteq A$ such that A/I' is non-Noetherian has a maximal element, say, J. Then $R = A/J$ is a non-Noetherian ring having a monomorphic Noetherian preenvelope $j = \bar{i} : R = A/J \rightarrowtail N'/N'J =: N$ with the property that, a for a nonzero ideal I of R, the following three assertions are equivalent:

   i) R/I is Noetherian

   ii) $I = IN \cap R$

   iii) R/I has a monomorphic Noetherian preenvelope

We claim that there is a maximal ideal $\mathbf{m}$ of R such that $R_{\mathbf{m}}$ is not Noetherian. Bearing in mind that $j_{\mathbf{m}} : R_{\mathbf{m}} \longrightarrow N_{\mathbf{m}}$ is also a (monomorphic) Noetherian preenvelope (see the proof of Proposition 3.1(2)), it will follow that, after replacing R by an appropriate factor of $R_{\mathbf{m}}$, one gets a non-Noetherian local ring with a monomorphic Noetherian preenvelope satisfying that the properties i)-iii) above are also equivalent for it.

Suppose our claim is false, so that $R_{\mathbf{m}}$ is Noetherian for all $\mathbf{m} \in Max(R)$. Let then $I_0 \subsetneq I_1 \subsetneq ...$ be a strictly increasing chain of ideals in R. We can assume that $0 \neq I_0 =: I$. Then R/I cannot be a Noetherian ring. Since the induced map $\bar{j} : R/I \longrightarrow N/NI$ is a Noetherian preenvelope it follows that $\bar{j}$ is not injective, and hence $0 \neq \hat{I}/I$, where $\hat{I} := R \cap NI$. But $N\hat{I} = NI$ and the induced map $R/\hat{I} \longrightarrow N/N\hat{I}$ is a monomorphic Noetherian preenvelope. Our assumptions on R imply that $R/\hat{I}$ is Noetherian, so that the canonical projection $p : R/I \twoheadrightarrow R/\hat{I}$ is a Noetherian envelope. According to Theorem 5.2, we have that $\mathbf{m}(\frac{\hat{I}}{I})_{\mathbf{m}} = (\frac{\hat{I}}{I})_{\mathbf{m}}$ (*) for every $\mathbf{m} \in Max(R)$. The fact that $R_{\mathbf{m}}$ is Noetherian implies that $(\frac{\hat{I}}{I})_{\mathbf{m}} = \frac{\hat{I}_{\mathbf{m}}}{I_{\mathbf{m}}}$ is a finitely generated $R_{\mathbf{m}}$-module. Then, using Nakayama's lemma, from the equality (*) we get that $(\frac{\hat{I}}{I})_{\mathbf{m}} = 0$, for all $\mathbf{m} \in Max(R)$. It follows that $\hat{I}/I = 0$, and we then get a contradiction.

So, from now on in this proof, we assume that R is a local non-Noetherian ring having a monomorphic Noetherian preenvelope, for which condition i)-iii) are equivalent. By Proposition

3.1, every $\mathbf{p} \in Spec(R)$ is restricted and therefore $R/\mathbf{p}$ is Noetherian for all $\mathbf{p} \in Spec(R)$. Since there are only finitely many minimal elements in $Spec(R)$ (cf. Proposition 3.1) we conclude that R_{red} is a Noetherian ring and, hence, that $K - dim(R) <\propto$. It implies, in particular, that one could have chosen our initial ring A with minimal finite Krull dimension. Having done so, this final local ring R is has also minimal finite Krull dimension among the non-Noetherian rings having a monomorphic Noetherian preenvelope. In particular, we have that $R_{\mathbf{p}}$ is Noetherian, for every $\mathbf{p} \in Spec(R) \setminus \{\mathbf{m}\}$.

Finally, if $0 \neq I \subset Nil(R)$ is an ideal such that R/I is not Noetherian (i.e. $I \subsetneq \hat{I} := R \cap NI$), the third paragraph of this proof shows that the canonical projection $p : R/I \twoheadrightarrow R/\hat{I}$ is a Noetherian envelope. Then Theorem 5.2 says that $D = \hat{I}/I$ has the property that $\mathbf{p}D_{\mathbf{p}} = D_{\mathbf{p}}$, for all $\mathbf{p} \in Spec(R)$. But then $D_{\mathbf{p}} = (\frac{\hat{I}}{I})_{\mathbf{p}} = 0$, for every non-maximal $\mathbf{p} \in Spec(R)$, because $R_{\mathbf{p}}$ is a Noetherian ring. It follows that $Supp(D) = \{\mathbf{m}\}$ and that $\mathbf{m}D = D$. $\blacksquare$

Example 6.2 *Let $\mathbf{Z}_{(p)}$ denote the localization of $\mathbf{Z}$ at the prime ideal $(p) = p\mathbf{Z}$ and consider the trivial extension $R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q}$. Then R is a non-Noetherian local ring and, in case of having a Noetherian preenvelope, this would be monomorphic and conditions 1)-3) of the above proposition would hold.*

Proof. Since $0 \times M$ is an ideal of R for each $\mathbf{Z}_{(p)}$-submodule M of $\mathbf{Q}$ it follows that R is not Noetherian. The prime ideals of R are $p\mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ and $0 \rtimes \mathbf{Q} = Nil(R)$, so that R is local with maximal ideal $\mathbf{m} := p\mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ and $K - dim(R) = 1$. In particular, condition 1) Proposition 6.1 is satisfied (see Theorem 4.7). Since R is a subring of the Noetherian ring $\mathbf{Q} \rtimes \mathbf{Q} \cong \mathbf{Q}[x]/(x^2)$, any Noetherian preenvelope $j : R \longrightarrow N$ that might exist would be necessarily monomorphic. On the other hand $R_{red} \cong \mathbf{Z}_{(p)}$ is a Noetherian ring.

Finally, suppose that $j : R \longrightarrow N$ is a Noetherian preenvelope, which we view as an inclusion, and let $0 \neq I \subseteq Nil(R)$ be an ideal of R such that R/I is non-Noetherian. We have that $I = 0 \rtimes A$ and $R/I \cong \mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/A)$, for some $\mathbf{Z}_{(p)}$-submodule $0 \neq A \subsetneq \mathbf{Q}$. Note that $\hat{I} = R \cap NI$ consists of nilpotent elements, so that $\hat{I} = 0 \rtimes B$, for some $\mathbf{Z}_{(p)}$-submodule $A \subseteq B \subseteq \mathbf{Q}$. We need to prove that $B = \mathbf{Q}$, and then condition 3) of Proposition 6.1 will be automatically satisfied.

Indeed, on one side we have that the induced map $\tilde{j} : R/\hat{I} \longrightarrow N/NI$ is a monomorphic Noetherian preenvelope. But in case $B \subsetneq \mathbf{Q}$, we have $R/\hat{I} \cong \mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/B)$ and Example 5.3 says that the canonical projection $\pi : R/\hat{I} \cong \mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/B) \twoheadrightarrow \mathbf{Z}_{(p)}$ is the Noetherian envelope. This is absurd for then we would have $0 \neq 0 \rtimes (\mathbf{Q}/B) = Ker(\pi) \subseteq Ker(\tilde{j}) = 0$. $\blacksquare$

The last proposition and example propose the ring $R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ as an obvious candidate to be a 'minimal' non-Noetherian ring having a monomorphic Noetherian preenvelope. We have the following result.

Theorem 6.3 *The ring $\mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ does not have a Noetherian envelope.*

The proof of this theorem will cover the rest of the paper and is based on a few lemmas.We proceed by reduction to absurd and, <u>in the sequel</u>, <u>we assume that</u> $i : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \hookrightarrow N$ <u>is a monomorphic Noetherian envelope</u>, which we view as an inclusion. Recall that a ring A is called **indecomposable** when it cannot be properly decomposed as a product $A_1 \times A_2$ of two rings. That is equivalent to say that the only idempotent elements of A are 0 and 1.

Lemma 6.4 *There is a ring isomorphism*

$$\varphi : N \xrightarrow{\cong} \mathbf{Z}_{(p)} \times B_2 \times ... \times B_r$$

satisfying the following properties:

1. *φi is a matrix map* $\begin{pmatrix} \pi \\ \lambda_2 \\ . \\ . \\ \lambda_r \end{pmatrix} : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \longrightarrow \mathbf{Z}_{(p)} \times B_2 \times ... \times B_r$, *where $\pi : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \twoheadrightarrow \mathbf{Z}_{(p)}$*
   is the projection and each λ_i is an injective ring homomorphism into the indecomposable Noetherian ring B_i

2. *Each λ_i is a minimal morphism in $CRings$*

3. *If $\mu : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \rightarrowtail S$ is an injective ring homomorphism, with S an indecomposable Noetherian ring, then μ factors through some λ_i*

4. *There is no ring homomorphism $h : B_i \longrightarrow B_j$, with $i \neq j$, such that $h\lambda_i = \lambda_j$*

5. *The ring B_i does not contain a proper Noetherian subring containing $Im(\lambda_i)$.*

Proof. A simple observation will be frequently used, namely, that there cannot exist a proper Noetherian subring B of N containing R as a subring. Indeed, if such B exists and $u : R \hookrightarrow B$ is the inclusion, then we get a ring homomorphism $g : N \longrightarrow B$ such that $gi = u$. Then the composition $h : N \xrightarrow{g} B \hookrightarrow N$ is a non-bijective ring homomorphism such that $hi = i$, against the fact that i is an envelope.

The projection $\pi : R \twoheadrightarrow \mathbf{Z}_{(p)}$ is a retraction in the category *CRings* of commutative rings. Moreover, since i is a Noetherian envelope, we have a ring homomorphism $f : N \longrightarrow \mathbf{Z}_{(p)}$ such that $fi = \pi$. It follows that also f is a retraction in *CRings*, so that we have a $\mathbf{Z}_{(p)}$-module decomposition $N = \mathbf{Z}_{(p)} \oplus I$, where $I := Ker(f)$ is an ideal of N containing $i(0 \rtimes \mathbf{Q})$.

Also due to the fact that i is a Noetherian envelope, we have a factorization of the inclusion $j : R \xrightarrow{\ i\ } N \xrightarrow{\ \rho\ } \mathbf{Q} \rtimes \mathbf{Q}$ in *CRings*. Then $Im(\rho)$ is a subring of $\mathbf{Q} \rtimes \mathbf{Q}$ containing $R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q}$. One readily sees that $Im(\rho) = A \rtimes \mathbf{Q}$, where A is a subring of $\mathbf{Q}$ containing $\mathbf{Z}_{(p)}$ as a subring. But if $q \in \mathbf{Q} \setminus \mathbf{Z}_{(p)}$, then we can write $q = ap^{-t}$, for some invertible element $a \in \mathbf{Z}_{(p)}$) and some integer $t > 0$. Then we have $\mathbf{Z}_{(p)}[q] = \mathbf{Z}_{(p)}[p^{-t}]$. Given any integer $n > 0$, Euclidean division gives that $n = tm + r$, where $m > 0$ and $0 < r < t$. We then have $p^{-n} = p^{-r} \cdot (p^{-t})^m = p^{t-r}(p^{-t})^{m+1}$, which proves that $p^{-n} \in \mathbf{Z}_{(p)}[p^{-t}] = \mathbf{Z}_{(p)}[q]$, for every $n > 0$, and hence that $\mathbf{Z}_{(p)}[q] = \mathbf{Q}$. As a consequence, we get that either $A = \mathbf{Z}_{(p)}$ or $A = \mathbf{Q}$, and so that either $Im(\rho) = \mathbf{Z}_{(p)} \rtimes \mathbf{Q} = R$ or $Im(\rho) = \mathbf{Q} \rtimes \mathbf{Q}$. But the first possibility is discarded for, being a factor of Noetherian, the ring $Im(\rho)$ is Noetherian. Therefore any ring homomorphism $\rho : N \longrightarrow \mathbf{Q} \rtimes \mathbf{Q}$ such that $\rho i = j$ is necessarily surjective.

We fix such a ρ from now on and also fix the decomposition $N = \mathbf{Z}_{(p)} \oplus I$ considered above. We claim that the restriction of ρ

$$\rho_{|I} : I \longrightarrow \mathbf{Q} \rtimes \mathbf{Q}$$

is a surjective map. Indeed $\rho(I)$ is a nonzero ideal of $\mathbf{Q} \rtimes \mathbf{Q}$ since ρ is a surjective ring homomorphism. Then we get that either $\rho(I) = 0 \rtimes \mathbf{Q}$ or $\rho(I) = \mathbf{Q} \rtimes \mathbf{Q}$. But the first possibility is discarded for it would produce a surjective ring homomorphism

$$\bar{\rho} : \mathbf{Z}_{(p)} \cong N/I \twoheadrightarrow \tfrac{\mathbf{Q} \rtimes \mathbf{Q}}{0 \rtimes \mathbf{Q}} \cong \mathbf{Q}.$$

Next we claim that $0 \rtimes \mathbf{Q} \subset (0 \rtimes \mathbf{Q})I$, which will imply that $0 \rtimes \mathbf{Q} \subset I^2$ and, hence, that $\mathbf{Z}_{(p)} + I^2$ is a subring of N containing R as a subring. Indeed we have $(0 \rtimes \mathbf{Q})N = (0 \rtimes \mathbf{Q})(\mathbf{Z}_{(p)} \oplus I) = (0 \rtimes \mathbf{Q}) + (0 \rtimes \mathbf{Q})I$ and, if our claim were not true, we would get:

$$0 \neq \tfrac{(0 \rtimes \mathbf{Q})N}{(0 \rtimes \mathbf{Q})I} \cong \tfrac{(0 \rtimes \mathbf{Q}) + (0 \rtimes \mathbf{Q})I}{(0 \rtimes \mathbf{Q})I} \cong \tfrac{\mathbf{Q}}{X},$$

where X is the $\mathbf{Z}_{(p)}$-submodule of $\mathbf{Q}$ consisting of those of those $q \in \mathbf{Q}$ such that $(0, q) \in (0 \rtimes \mathbf{Q})I$. It is routinary to see that the isomorphism of $\mathbf{Z}_{(p)}$-modules $\tfrac{(0 \rtimes \mathbf{Q})N}{(0 \rtimes \mathbf{Q})I} \cong \mathbf{Q}/X$ gives a bijection between the N-submodules of $\tfrac{(0 \rtimes \mathbf{Q})N}{(0 \rtimes \mathbf{Q})I}$ and the $\mathbf{Z}_{(p)}$-submodules of $\mathbf{Q}/X$. But this is impossible

for, due to the Noetherian condition of N, the N-module $\frac{(0 \rtimes \mathbf{Q})N}{(0 \rtimes \mathbf{Q})I}$ is Noetherian while $\mathbf{Q}/X$ does not satisfies ACC on $\mathbf{Z}_{(p)}$-submodules.

We next consider the subring $N' = \mathbf{Z}_{(p)} + I^2$ of N. If $\{y_1, ..., y_r\}$ is a finite set of generators of I as an ideal, then $\{1, y_1, ..., y_r\}$ generates $N = \mathbf{Z}_{(p)} + I$ as a N'-module. By Eakin's theorem (cf. [6], see also [8]), we know that N' is a Noetherian ring. By the first paragraph of this proof, we conclude that $N' = N$, from which it easily follows that $I^2 = I$. But then there is an idempotent element $e = e^2 \in N$ such that $I = Ne$ (cf. [1][Exercise 7.12]). Since $y(1 - e) = 0$ for all $y \in I$, we get that $(a + y)(1 - e) = a(1 - e)$, for all $a \in \mathbf{Z}_{(p)}$ and $y \in I$. Therefore $A := N(1 - e) = \mathbf{Z}_{(p)}(1 - e)$ is a ring (with unit $1 - e$) isomorphic to $\mathbf{Z}_{(p)}$ via the assignment $a \rightsquigarrow a(1-e)$. We put $B := I = Ne$, which is a ring (with unit e), and we have a ring isomorphism $\varphi : N \xrightarrow{\cong} \mathbf{Z}_{(p)} \times B$. Bearing in mind that $N = \mathbf{Z}_{(p)} \oplus I$ as $\mathbf{Z}_{(p)}$-modules, it is easy to see that $\varphi(a + b) = (a, ae + b)$, for all $a \in \mathbf{Z}_{(p)}$ and $b \in B = I$.

Then the composition $i' : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \xhookrightarrow{i} N \xrightarrow{\varphi} \mathbf{Z}_{(p)} \rtimes B$ is also a Noetherian envelope. Its two component maps are:

$$\pi : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \twoheadrightarrow \mathbf{Z}_{(p)}, \qquad (a, q) \rightsquigarrow a$$
$$\lambda : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \longrightarrow B, \qquad (a, q) \rightsquigarrow ae + (0, q).$$

We first note that every non-injective ring homomorphism $g : R \longrightarrow S$, where S is Noetherian indecomposable, factors through π. Indeed if $Ker(g)$ contains $0 \rtimes \mathbf{Q}$ that is clear. In any other case, we have $Ker(g) = 0 \rtimes M$, for some nonzero $\mathbf{Z}_{(p)}$-submodule M of $\mathbf{Q}$. Then the induced monomorphism $\frac{\mathbf{Z}_{(p)} \rtimes \mathbf{Q}}{0 \rtimes M} \cong \mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/M) \longrightarrow S$ factors through the Noetherian envelope of $\mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/M)$ which, by Example 5.3, is the projection $\mathbf{Z}_{(p)} \rtimes (\mathbf{Q}/M) \twoheadrightarrow \mathbf{Z}_{(p)}$. It then follows that g factors through π as desired.

Decompose now B as a finite product of indecomposable (Noetherian) rings $B = B_2 \times ... \times B_r$. Then λ is identified with a matrix map

$$\begin{pmatrix} \lambda_2 \\ . \\ . \\ \lambda_r \end{pmatrix} : R \longrightarrow B_2 \times ... \times B_r,$$

where the λ_i are ring homomorphisms. We claim that these λ_i are necessarily injective, thus proving property 1) in the statement. Indeed if, say, λ_2 is not injective then, by the previous paragraph, we have $\lambda_2 = u\pi$ for some ring homomorphism $u : \mathbf{Z}_{(p)} \longrightarrow B_2$. Now from the ring

endomorphism $\xi = \begin{pmatrix} 1 & 0 \\ u & 0 \end{pmatrix} : \mathbf{Z}_{(p)} \times B_2 \longrightarrow \mathbf{Z}_{(p)} \times B_2$ we derive a ring endomorphism

$$\Phi = \begin{pmatrix} \xi & 0 \\ 0 & 1 \end{pmatrix} : (\mathbf{Z}_{(p)} \times B_2) \times (B_3 \times ... \times B_r) \longrightarrow (\mathbf{Z}_{(p)} \times B_2) \times (B_3 \times ... \times B_r)$$

which is not bijective and satisfies that $\Phi i' = i'$ (here $1 : \prod_{3 \leq i \leq r} B_i \longrightarrow \prod_{3 \leq i \leq r} B_i$ is the identity map). That would contradict the fact that i' is an envelope.

Properties 2) and 3) in the statement will follow easily once we check the following two properties for λ:

a) λ is minimal, i.e. if $g : B \longrightarrow B$ is a ring homomorphism such that $g\lambda = \lambda$, then λ is an isomorphism

b) If $\mu : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \longrightarrow S$ is any injective ring homomorphism, with S an indecomposable Noetherian ring, then μ factors through λ.

Indeed, let $g : B \longrightarrow B$ be a ring endomorphism such that $g\lambda = \lambda$, then the 'diagonal' map $\psi := \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} : \mathbf{Z}_{(p)} \times B \longrightarrow \mathbf{Z}_{(p)} \times B$ is a ring endomorphism such that $\psi i' = i'$. It follows that ψ is an isomorphism and, hence, that g is an isomorphism. Finally, if $\mu : \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \rightarrowtail S$ is an injective ring homomorphism, with S an indecomposable Noetherian ring, then the fact that i' is a Noetherian envelope gives a ring homomorphism $\upsilon = \begin{pmatrix} \upsilon_1 & \upsilon_2 \end{pmatrix} : \mathbf{Z}_{(p)} \times B \longrightarrow S$ such that $\upsilon i' = \mu$. The fact that S is indecomposable implies that either $\upsilon_1 = 0$ or $\upsilon_2 = 0$. But the second possibility is discarded for it would imply that $\mu = \upsilon_1 \pi$, and so that $0 \rtimes \mathbf{Q} = Ker(\pi) \subseteq Ker(\mu) = 0$.

It only remains to prove properties 4) and 5) in the statement. To prove 4), take any ring homomorphism $h : B_i \longrightarrow B_j$, with $i \neq j$. Without loss of generality, put $i = 2$ and $j = 3$. If $h\lambda_2 = \lambda_3$, then we consider the ring homomorphism given matricially in the form

$$\Psi = \begin{pmatrix} \psi & 0 \\ 0 & 1 \end{pmatrix},$$

where $\psi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h & 0 \end{pmatrix} : \mathbf{Z}_{(p)} \times B_2 \times B_3 \longrightarrow \mathbf{Z}_{(p)} \times B_2 \times B_3$ and $1 : \oplus_{4 \leq i \leq r} B_i \longrightarrow \oplus_{4 \leq i \leq r} B_i$ is the identity map. We have an equality $\Psi i' = i'$, but Ψ is not an isomorphism, which contradicts the fact that i' is an envelope.

Finally, suppose that $N' \subsetneq B_i$ is a proper Noetherian subring containing $Im(\lambda_i)$. Then, putting $i = 2$ for simplicity, we get that $\mathbf{Z}_{(p)} \times N' \times B_3 \times ... \times B_r$ is a proper Noetherian subring of $\mathbf{Z}_{(p)} \times B_2 \times B_3 \times ... \times B_r \cong N$ containing R as a subring. That is a contradiction. ■

Lemma 6.5 *Let* $\lambda : R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \hookrightarrow B$ *be an inclusion of rings, with* B *indecomposable Noetherian, and suppose that we have a decomposition* $B = \mathbf{Z}_{(p)} \oplus I$, *where* I *is an ideal of* B *containing* $0 \rtimes \mathbf{Q}$. *Then* B *admits a proper Noetherian subring* B' *containing* R.

Proof. We claim that $(0 \rtimes \mathbf{Q})B + I^2$ is an ideal of B propertly contained in I. Indeed the equality $I = (0 \rtimes \mathbf{Q})B + I^2$ would give an epimorphism of B-modules

$$(0 \rtimes \mathbf{Q})B \twoheadrightarrow I/I^2,$$

thus showing that I/I^2 is divisible as a $\mathbf{Z}_{(p)}$-module. But, on the other and, I/I^2 is finitely generated as a module over the ring $B/I \cong \mathbf{Z}_{(p)}$. Therefore we would get that $I/I^2 = 0$ and hence would find an idempotent $e = e^2 \in I$ such that $I = Be$. That would contradict the fact that B is indecomposable.

Since our claim is true we can take the proper subring $B' = \mathbf{Z}_{(p)} \oplus [(0 \rtimes \mathbf{Q})B + I^2]$ of B. An argument already used in the proof of Lemma 6.4 shows that B is finitely generated as B'-module, and hence that B' is Noetherian. ∎

Lemma 6.6 *Let* $\lambda : R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \hookrightarrow B$ *be an inclusion of rings, with* B *indecomposable Noetherian. Suppose that* $\mathbf{m}$ *is a maximal ideal of* B *and that* $g : A \longrightarrow B$ *is a homomorphism of Noetherian* $\mathbf{Z}_{(p)}$-*algebras such that the composition*

$$A \xrightarrow{g} B \xrightarrow{pr} B/\mathbf{m}$$

is surjective. Then $\tilde{B} = A \oplus \mathbf{m}$ *has a structure of Noetherian ring, with multiplication* $(a,m)(a',m') = (aa', g(a)m' + mg(a') + mm')$, *such that the map* $\psi : \tilde{B} \longrightarrow B$, $(a,m) \rightsquigarrow g(a) + m$, *is a (surjective) ring homomorphism.*

Proof. Clearly the multiplication given on $\tilde{B}$ makes it into a ring, and the canonical map $\psi : \tilde{B} \twoheadrightarrow B$ is a surjective ring homomorphism. Its kernel consists of those pair $(a,m) \in \tilde{B}$ such that $g(a) + m = 0$, which gives the equality

$$Ker(\psi) = \{(a, -g(a)) : a \in g^{-1}(\mathbf{m})\}.$$

Note that every $\tilde{B}$-submodule of $Ker(\psi)$ is canonically an A-submodule and that we have an isomorphism of A-modules $Ker(\psi) \cong g^{-1}(\mathbf{m})$. Since $g^{-1}(\mathbf{m})$ is an ideal of the Noetherian ring A, we conclude that $Ker(\psi)$ is a Noetherian $\tilde{B}$-module. This and the fact that the ring $\tilde{B}/Ker(\psi) \cong B$ is Noetherian imply that $\tilde{B}$ is a Noetherian ring. ∎

Lemma 6.7 *Suppose that in the situation of last lemma, we have $A = \mathbf{Z}_{(p)}[X]$ and $B/\mathbf{m} \cong \mathbf{Q}$. If the homomorphism $\psi : \tilde{B} \twoheadrightarrow B$, $(a, m) \rightsquigarrow g(a) + m$, is a retraction in $CRings$, then either B contains a proper Noetherian subring containing R or there is a maximal ideal $\mathbf{m}'$ of B such that $B = \mathbf{Z}_{(p)} + \mathbf{m}'$.*

Proof. We fix a section $\varphi : B \longrightarrow \tilde{B}$ for ψ in $CRings$. Then we put $\mathbf{q}' := \varphi^{-1}(0 \oplus \mathbf{m})$ and $A' := B/\mathbf{q}'$. We get a subring A' of $\mathbf{Z}_{(p)}[X]$ (whence A' is an integral domain) containing $\mathbf{Z}_{(p)}$. Moreover, since p is not invertible in $\mathbf{Z}_{(p)}[X]$ it cannot be invertible in A'. Therefore $pA' \neq A'$ and we have an induced ring homomorphism $\bar{\varphi} : A'/pA' \longrightarrow \mathbf{Z}_{(p)}[X]/p\mathbf{Z}_{(p)}[X] \cong \mathbf{Z}_p[X]$. We denote by C its image, which is then a subring of $\mathbf{Z}_p[X]$ isomorphic to $B/\mathbf{q}$, for some $\mathbf{q} \in Spec(B)$ such that $\mathbf{q}' + pB \subseteq \mathbf{q}$. Then the composition $\mathbf{Z}_{(p)} \hookrightarrow B \twoheadrightarrow B/\mathbf{q} = C$ has kernel $p\mathbf{Z}_{(p)}$.

We distinguish two situations. In case the last composition is surjective, and hence $\mathbf{Z}_p \cong C$, we have that $\mathbf{q}$ is a maximal ideal of B such that $B = \mathbf{Z}_{(p)} + \mathbf{q}$ and the proof is finished. In case the mentioned composition is not surjective, there exists a nonconstant polynomial $f = f(X) \in \mathbf{Z}_p[X]$ such that $f \in C$. There is no loss of generality in taking f to be monic, so that X is integral over $\mathbf{Z}_p[f]$ and, hence, the inclusion $C \subseteq \mathbf{Z}_p[X]$ is an integral extension. In particular, we have $K - dim(C) = 1$ and the assignment $\mathbf{q} \rightsquigarrow C \cap \mathbf{q}$ gives a surjective map $Max(\mathbf{Z}_p[X]) \twoheadrightarrow Max(C)$ (cf. [10][Corollary II.2.13]).

If $\mathbf{n}$ is a maximal ideal of C and we put $\mathbf{n} = C \cap \hat{\mathbf{n}}$, with $\hat{\mathbf{n}} \in Max(\mathbf{Z}_p[X])$, then we get a field homomorphism $C/\mathbf{n} \longrightarrow \mathbf{Z}_p[X]/\hat{\mathbf{n}}$. In particular, $C/\mathbf{n}$ is a finite field extension of $\mathbf{Z}_p$. Take now $\mathbf{n}' \in Max(B)$ such that $\mathbf{n} = \mathbf{n}'/\mathbf{q}$. One easily sees that $B' = \mathbf{Z}_{(p)} + \mathbf{n}'$ is a subring of B such that B is finitely generated as B'-module, and then, by Eakin's theorem, we know that B' is Noetherian. But $0 \rtimes Q$ is contained in all maximal ideals of B since it consists of (2-)nilpotent elements. In particular, we get that B' contains $R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ and the proof is finished. $\blacksquare$

We are now ready to give the desired proof.

Proof of Theorem 6.3:

Put $R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q}$ as usual and suppose that it has a Noetherian envelope, represented by a matrix map as in Lemma 6.4. We first prove that at least one of the B_i of Lemma 6.4 has a maximal ideal $\mathbf{m}$ such that $B_i/\mathbf{m} \cong \mathbf{Q}$. Indeed, preserving the notation of the proof of Lemma 6.4, we see that the map $\rho : I = B \longrightarrow \mathbf{Q} \rtimes \mathbf{Q}$ is a surjective ring homomorphism. But, since $\mathbf{Q} \rtimes \mathbf{Q}$ is indecomposable, ρ necessarily vanishes on all but one of the B_i appearing in the decomposition $B = B_2 \times ... \times B_r$. Then we get a unique index i such that $\rho_{|B_i} : B_i \longrightarrow \mathbf{Q} \rtimes \mathbf{Q}$ is nonzero, and hence $\rho_{|B_i}$ is surjective. Now $\mathbf{m} = \rho_{|B_i}^{-1}(0 \rtimes \mathbf{Q})$ is a maximal ideal of B_i such that

$B_i/\mathbf{m} \cong \mathbf{Q}$.

Let fix now $i \in \{2, ..., r\}$ such that B_i admits a maximal ideal $\mathbf{m}$ with $B_i/\mathbf{m} \cong \mathbf{Q}$. For simplification, put $C = B_i$. We fix a surjective ring homomorphism $\Psi : C \longrightarrow \mathbf{Q}$ with kernel $\mathbf{m}$ and fix an element $x \in C$ such that $\Psi(x) = p^{-1}$. If X is now a variable over $\mathbf{Z}_{(p)}$, then the assignment $X \rightsquigarrow x$ induces a homomorphism of Noetherian $\mathbf{Z}_{(p)}$-algebras, $g : \mathbf{Z}_{(p)}[X] \longrightarrow C$ such that the composition

$$\mathbf{Z}_{(p)}[X] \xrightarrow{g} C \xrightarrow{pr} C/\mathbf{m}$$

is surjective. According to Lemma 6.6, we know that $\tilde{C} = \mathbf{Z}_{(p)}[X] \oplus \mathbf{m}$ has a structure of Noetherian ring such that the canonical map $\psi : \tilde{C} \longrightarrow C$, $(a, m) \rightsquigarrow g(a) + m$, is a surjective ring homomorphism. Note that we have an obvious (injective) ring homomorphism $h : R = \mathbf{Z}_{(p)} \rtimes \mathbf{Q} \longrightarrow \tilde{C} = \mathbf{Z}_{(p)}[X] \oplus \mathbf{m}$ induced by the inclusions $\mathbf{Z}_{(p)} \hookrightarrow \mathbf{Z}_{(p)}[X]$ and $0 \rtimes \mathbf{Q} \overset{\lambda}{\hookrightarrow} \mathbf{m}$. Such a ring homomorphism has the property that $\psi h = \lambda_i : R \longrightarrow B_i = C$. It is not difficult to see that the only idempotent elements of $\tilde{C}$ are the trivial ones, so that $\tilde{C}$ is an indecomposable ring. By Lemma 6.4, the morphism h factors through some λ_j $(j = 2, ..., r)$. Fix such an index j and take then a ring homomorphism $h' : B_j \longrightarrow \tilde{C}$ such that $h'\lambda_j = h$. Then we have that $\psi h' \lambda_j = \psi h = \lambda_i$. Again by Lemma 6.4, we get that $i = j$ and that $\psi h'$ is an isomorphism. In particular, we get that ψ is a retraction in $CRings$.

Now from Lemmas 6.7 and 6.4, we conclude that $C = B_i$ has a maximal ideal $\mathbf{m}'$ such that $\mathbf{Z}_{(p)} + \mathbf{m}' = C$. Then, according to Lemma 6.6, $\tilde{C} = \mathbf{Z}_{(p)} \oplus \mathbf{m}'$ gets a structure of Noetherian (indecomposable) ring, with multiplication $(a, m)(a', m') = aa' + am' + ma' + mm'$, so that the canonical map $\psi : \tilde{C} \longrightarrow C$, $(a, m) \rightsquigarrow a + m$, is a surjective ring homomorphism. An argument similar to the one in the previous paragraph shows that ψ is a retraction in $CRings$. We again fix a section for it $\varphi' : C \longrightarrow \tilde{C}$. Notice that the composition

$$\varphi_1 : C \xrightarrow{\varphi'} \tilde{C} = \mathbf{Z}_{(p)} \oplus \mathbf{m}' \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} \mathbf{Z}_{(p)}$$

is a ring homomorphism such that $\varphi'(b) = (\varphi_1(b), b - \varphi_1(b))$, for all $b \in B$. The universal property of localization with respect to multiplicative subsets implies that the only ring endomorphism of $\mathbf{Z}_{(p)}$ is the identity map, so that $\varphi_{1|\mathbf{Z}_p} = 1_{\mathbf{Z}_{(p)}} : \mathbf{Z}_{(p)} \longrightarrow \mathbf{Z}_{(p)}$. Therefore we get that $\varphi'(a) = (a, 0)$, for all $a \in \mathbf{Z}_{(p)}$. That proves that $\mathbf{Z}_{(p)} \cap Ker(\varphi') = 0$. But since φ_1 is surjective we conclude that we have a $\mathbf{Z}_{(p)}$-module decomposition $B = \mathbf{Z}_{(p)} \oplus I$, where $I = Ker(\varphi_1)$. On the other hand, since $(0 \rtimes \mathbf{Q})^2 = 0$ and $\mathbf{Z}_{(p)}$ is an integral domain, we conclude that $\varphi_1(0 \rtimes \mathbf{Q}) = 0$, and so that

$0 \rtimes \mathbf{Q} \subseteq I$. By Lemma 6.5, we get that $C = B_i$ contains a proper Noetherian subring containing R. That contradicts Lemma 6.4 and ends the proof.

We end the paper by proposing:

Conjectures 6.8 *1. There does not exist any non-Noetherian commutative ring having a monomorphic Noetherian envelope*

    2. A commutative ring R has a Noetherian envelope if, and only if, it has a nil ideal I such that R/I is Noetherian and $\mathbf{p}I_{\mathbf{p}} = I_{\mathbf{p}}$, for all $\mathbf{p} \in Spec(R)$.

By Theorem 5.2 and our comments at the beginning of this section, the two conjectures above are equivalent.

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