NOTAS DE MATEMATICAS

Nº 109

QUASI-SEMIGROUPS, EVOLUTION EQUATION AND CONTROLLABILITY

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MERIDA - VENEZUELA
1991

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Abstract. In this paper we generalize the notion of semigroup T_t $(t \geq 0)$ and infinitesimal generador A on a Banach space X to the notions of quasi-semigroup $K(t,s)$, $(t,s \geq 0)$ and generator $A(t)$ $(t \geq 0)$; respectively. In addition to prove several properties of $K(t,s)$ which are analogous to the ones of semigroups, we show that under certain conditions, the equation $\dot{x}(t) = A(t)\, x(t) + f(t)$ has a unique solution. We also consider the dual quasi-semigroup $K^*(t,s)$ and the non-autonomous control system $\dot{x}(t) = A(t)\, x(t) + Bu(t)$, where the controls belong to the space $L_p(0,T;U)$ with U a reflexive Banach space and $1 < p < \infty$. Finally, we give necesary and sufficient conditions for exact and approximate controllability.

Key Words. Quasi-semigroup, generator, evolution equation, exact and approximate controllability.

1. Introduction. The Sobolevski-Tanabe [7] and Kato [5] theory consider the non-autonomous evolution equation:

$$\dot{x}(t) = A(t)\, x(t),\ x(0) = x_o,\quad t > 0 \tag{1.1}$$

and to garantize the existence and uniqueness of its solution it is assumed the following hypotheses:

Hypothesis 1. For all $t \geq 0$, $A(t)$ is a closed operator on a Banach space X, with domain $D[A(t)] = D$ independent of t and dense in X.

Hypothesis 2. For each $t \geq 0$ $A(t)$ generates a strongly continuous semigroup.

Hypothesis 3. $A(t)$ is strongly continuous.

Hypothesis 4. For each fixed s, the operator $A(t)\, A^{-1}(s)$ is bounded and Hölder continuous in t, in the uniform topology of operators, i.e.

$$\| [A(t) - A(\tau)]\, A^{-1}(s) \| \leq C\, |t-\tau|^{\alpha},\ 0 < \alpha \leq 1,\ C > 0.$$

In this work we shall suppose that $A(t)$ is the generator of a strongly continuous quasi-semigroup $K(t,s)$ $(t,s \geq 0)$ and, as we shall see in section 2, $A(t)$ satisfies the Hypothesis 1 of Sobolevski-Tanabe and Kato theory but no necessarily verifies the hypotheses 2,3 and 4. Further, in example 2.3 we shall see that $A(t)$ does not generate a strongly continuous semigroup. However, we shall show that eq (1.1) has as a unique solution the function $x(t) = K(0,t)\,x_o$.

In this paper we investigate also the problem of the controllability of the non-autonomous and unbounded control system

$$(1.2)\qquad \begin{cases} \dot{x}(t) = A(t)\,x(t) + Bu(t), & 0 < t \leq T \\ x(0) = x_o \end{cases}$$

where $u(.) \in L_p(0,T;U)$ $(1 < p < \infty)$, $A(t)$ is the generator of a strongly continuous quasi-semigroup on X and U is a reflexive Banach Space.

There exists much literature on the controllability of autonomous and unbounded systems ([1], [2], [3] and [6]) but there is a little work, as far as we know, on non-autonomous and unbounded systems. In the case in which $A(t)$ is bounded and analytical, Korobov an Rabakh [6] give necessary and sufficient conditions for the exact controllability.

In this work, the notion of strongly continuous quasi-semigroup will allow us to give necessary and sufficient conditions for the exact and approximate controllability of the non-autonomous and unbounded system (1.2).

2. Quasi-semigroups. We now introduce what we consider is the most important definition of this paper.

DEFINITION 2.1. Let X be a Banach space and $L(X)$ a space of linear and continuous operator from X to X. A family of operators $K(t,s) \in L(X)$ $(t,s \geq 0)$ is a quasi-semigroup on X if it commutative and verifies:

a) $K(t,0) = I$, $(t \geq 0)$, I-identity on $L(X)$,

b) $K(r,t+s) = K(r+t,s)\, K(r,t)$, $(t,r,s \geq 0)$.

If, in addition, we have

c) $\lim_{(t,s)\to(t_o,s_o)} \|K(t,s)x_o - K(t_o,s_o)x_o\| = 0$, $(x_o \in X)$.

d) $\|K(t,s)\| \leq M(t+s)$, $(t,s \geq 0)$ where $M(.)$ is a continuous and non decreasing function from $[0,\infty)$ to $[1,\infty)$.

e) The subspace D formed by elements of X such that there exist the limits:

$$\lim_{s\to 0^+} \frac{K(t,s)x-x}{s} = \lim_{s\to 0^+} \frac{K(t-s,s)x-x}{s}, \quad t > 0,$$

$$\lim_{s\to 0} \frac{K(0,s)x - x}{s},$$

is dense on X, we shall say that the quasi-semigroup $K(t,s)$ is strongly continuous.

DEFINITION 2.2. Let $K(t,s)$ be a strongly continuous quasi-semigroup. The family of operators $A(t)$, $(t \geq 0)$ with common domain D, defined by:

$$A(t)x = \lim_{s\to 0^+} \frac{K(t,s)x - x}{s}, \quad (x \in D)$$

is called the generator of the quasi-semigroup K(t,s).

The following examples show that the class of strongly continuous quasi-semigroups is very broad.

EXAMPLE 2.1. Let T_t, $(t \geq 0)$ be a strongly continuous semigroup on X. If $K(t,s) = T_s$, $(t,s \geq 0)$ then K(t,s) is a strongly continuous quasi-semigroup.

EXAMPLE 2.2. Lets denote by X the Banach space of the uniformly continuous and bounded real functions defined on $[0,\infty)$ with the norm of supremum.

The family of operators $K(t,s) \in L(X)$ defined by:

$$(K(t,s)x)\ (\xi) = x(s^2 + 2st + \xi), \quad (t,s \geq 0)$$

is a strongly continuous quasi-semigroup of contractions.

In fact, the conditions a), b), c) and d) of the definition 2.1 can be verified easily. In order to verify e) we put

$$D = \{x \in X : \dot{x} \in X\}.$$

D is a dense subspace de X ([7] p. 24). If $x \in D$, then

$$\left(\frac{K(t,s)\ x-x}{s}\right)(\xi) = \frac{x(s^2+2st+\xi)-x(\xi)}{s}, \qquad (s,t \geq 0);$$

$$\left(\frac{K(t-s,s)x-x}{s}\right)(\xi) = \frac{x(s^2+2s(t-s)+\xi)-x(\xi)}{s}, \quad (0<s<t).$$

Let us put

$$F(s) = x(s^2+2st+\xi), \; G(s) = x(s^2+2s(t-s)+\xi).$$

Then, $\dot{F}(0) = 2t\,\dot{x}(\xi)$ and $\dot{G}(0) = 2t\,\dot{x}(\xi)$, thus

$$\lim_{s\to 0^+}\left(\frac{K(t,s)x-x}{s}\right)(\xi) = \lim_{s\to 0^+}\left(\frac{K(t-s,s)x-x}{s}\right)(\xi) = 2t\,\dot{x}(\xi)$$

$$\lim_{s\to 0^+}\left(\frac{K(0,s)x-x}{s}\right)(\xi) = 0.$$

Let us see that

$$\lim_{s\to 0^+}\left(\frac{K(t,s)x-x}{s}\right) = 2t\dot{x} \quad \text{uniformly:}$$

$$\left\|\frac{K(t,s)x-x}{s}\right\| = \sup_{\xi\geq 0}\left|\frac{x(s^2+2st+\xi)-x(\xi)}{s} - 2t\dot{x}(\xi)\right|$$

$$= \sup_{\xi\geq 0}\left|\frac{(s^2+2st)\dot{x}(\xi) + \theta(s^2+2st)}{s} - 2t\dot{x}(\xi)\right|$$

$$= \sup_{\xi\geq 0}\left|s\,\dot{x}(\xi) + \frac{\theta(s^2+2st)}{s}\right|$$

$$\leq s\|\dot{x}\| + \frac{\theta(s^2+2st)}{s} \to 0, \text{ when } s\to 0^+.$$

By the same procedure we can prove the existence of the rest of the limits. As consequence, the generator $A(t)$, $(t \geq 0)$ of $K(t,s)$ is given by

$$A(t) : D \to X, \; A(t)x = 2t\dot{x} .$$

EXAMPLE 2.3. Let T_t a strongly continuous semigroup on a Banach space X, and A its infinitesimal generator. The family of linear and continuous operators

$$K(t,s) = \exp(T_{t+s} - T_t) \quad (t,s \geq 0)$$

is a strongly continuous quasi-semigroup. In fact, the properties a), b) and c) of the definition of quasi-semigroup can be verified easily.

d) It is well-know that there exist constants $M, W > 0$ such that

$$\| T_t \| \leq M \exp(Wt), \quad (t \geq 0). \quad ([7]);$$

consequently,

$$\| K(t,s) \| \leq \exp(2Me^{W(t+s)}) = M(t+s).$$

e) We know that $D = D(A)$ is a dense subspace of X, in addition for each $x \in D$ we have that

$$\lim_{s\to 0^+} \frac{K(t,s)x-x}{s} = \lim_{s\to 0^+} \frac{K(t-s,s)x-x}{s} = AT_t\, x \quad (t > 0)$$

and

$$\lim_{s \to 0^+} \frac{K(0,s)x-x}{s} = Ax \qquad ([8]).$$

Consequently, the generator $A(t)$ $(t \geq 0)$ of $K(t,s)$ is given by:

$$A(t) : D \to X \ , \ A(t)x = AT_t x.$$

OBSERVATION. The evolution operator associated to (1.1) is defined ([2], [5], [7]) as the family $U(t,s) \in L(X)$, $(0 \leq s \leq t < \infty)$, wich satisfies the following four properties

i) $U(r,r) = I$ - identity on $L(X)$.

ii) $U(t,r)\ U(r,s) = U(t,s) \quad (0 \leq s \leq r \leq t < \infty)$

iii) $U(.,.)$ is strongly continuous

iv) The operator $\frac{\partial U(t,r)x}{\partial t}$ there exists and is continuous.

It can be shown, in addition, that if $A(t)$ satisfies the hypothesis 1,2,3 and 4 of the Sobolevski-Tanabe and Kato theory there exists tha evolution operator $U(t,s)$, hich verifies

$$\frac{\partial U(t,r)x}{\partial t} = A(t)\ U(t,r)x \ ; \ x \in D .$$

In this case, the only solution of (1.1) is

$$x(t) = U(t,0)x_o, \quad x_o \in D.$$

A strongly continuous quasi-semigroup $K(t,s)$, induces an evolution operator $U(t,s)$. In fact, if we write

$$U(t,s) = K(t,t-s) \qquad (0 \leq s \leq t < \infty)$$

we see that $U(t,s)$ is an evolution operator.

Conversely, given an evolution operator $U(t,s)$, it induces the quasi-semigroup defined by

$$K(t,s) = U(t+s,t), \qquad (0 \leq t \leq t+s < \infty).$$

THEOREM 2.1. Let $K(t,s)$ be a strongly continuous quasi-semigroup on the Banach space X. Then

a) If $x_o \in D$, $K(r,t)x_o \in D$ $(t,\ r \geq 0)$

b) For each $x_o \in D$ and $r \geq 0$,

$$\frac{\partial K(r,t)x_o}{\partial t} = A(r+t)\ K(r,t)x_o = K(r,t)\ A(r+t)x_o.$$

c) If $A(.)$ is locally strongly integrable, then for each $x_o \in D$ and $r \geq 0$, we have

$$K(r,t)x_o = x_o + \int_0^t A(r+s)K(r,s)x_o\, ds \qquad (t \geq 0).$$

PROOF. It is analogous to theorem 2.9 of [2]; by using the identities

$$K(r,t+s) = K(r+t,s)\ K(r,t)$$

$$K(r,t) = K(r+t-s,s)\ K(r,t-s),\ t > s.$$

THEOREM 2.2. Let $A(t)$ be the generator of a strongly continuous quasi-semigroup $K(t,s)$ on a Banach space X. Then for each $x_o \in D$ and $r \geq 0$, the problem

$$(2.1) \qquad \dot{x}(t) = A(r+t)\, x(t), \quad x(0) = x_o$$

admits a unique solution.

PROOF. By theorem 2.1, the function $x(t) = K(r,t)x_o$ is solution of (2.1). If $y(t)$ is another solution, we consider the function

$$F(s) = K(r+t,\, t-s)\, y(s), \quad s \in [0,t].$$

A routine calculation shows that $\dot{F}(s) = 0$, for each $s \in (0,t)$; therefore, F is constant and so

$$F(t) = F(0) \iff y(t) = K(r,t)x_o.$$

PROPOSITION 2.1. Let $A(t)$ be the generator of a strongly continuous quasi-semigroup and $f: [0,T] \to X$ a continuous function. If $\{x_n\} \subset D$ converges to x and $A(t)x_n$ converges uniformly to $f(t)$ on $[0,T]$, then for each $0 \leq r < T$, $A(r)x = f(r)$.

PROOF. It is consequence of uniform convergence and c) of the theorem 2.1.

THEOREM 2.3. Let $K(s,t)$ be strongly continuous quasi-semigroup on a Banach space X, with $A(t)$ strongly continuous. If $f: [0,T] \to D$ is a continuous function and

$$\int_0^t K(r+s,\ t-s)\ f(s)ds \in D \qquad (0 \leq t \leq T),$$

then the problem

$$(2.2) \qquad \begin{aligned} \dot{x}(t) &= A(r+t)\ x(t) + f(t), \quad 0 < t \leq T \\ x(0) &= x_o \in D \end{aligned}$$

admits as unique solution, the function

$$(2.3) \qquad x(t) = K(r,t)x_o + \int_0^t K(r+t,t-s)\ f(s)ds$$

PROOF. It is analogous to theorem 2.2.3 of [7] by using the identity

$$K(r+s,\ t+h-s)\ f(s) = K(r+t,h)K(r+t,\ t-s)\ f(s).$$

DEFINITION 2.3. Let $f \in L_p(0,T;X)$, $p \geq 1$. The function

$$(2.4) \qquad x_r(t) = K(r,t)x_o + \int_0^t K(r+s,\ t-s)\ f(s)ds$$

is defined as the mild solution of (2.2) on $[0,T]$.

PROPOSITION 2.2. The function $x_r(t)$ is well defined and is strongly continuous on $[0,T]$.

PROOF. It is a consequence of the following relations:

a) $\quad \left\| \int_0^t K(r+s,\ t-s)\ f(s)ds \right\| \leq t^{1/q}\ M(r+t)\ \| f \|_p, \quad \frac{1}{p} + \frac{1}{q} = 1;$

b) $$x_r(t+h) - x_r(t) = K(r,\ t+h)x_o - K(r,t)x_o$$

$$+ (K(r+t,h)-I)\int_0^t K(r+s,\ t-s)\ f(s)ds$$

$$+ \int_t^{t+h} K(r+s,\ t+h-s)\ f(s)ds, \qquad h \geq 0, \quad t \geq 0,$$

c) $$x_r(t-h) - x_r(t) = K(r,t-h)x_o - K(r,t)x_o$$

$$+ (I - K(r+t-h,h))\int_0^{t-h} K(r+s,\ t-h-s)\ f(s)ds$$

$$- \int_{t-h}^{t} K(r+s,\ t-s)\ f(s)ds\ , \quad 0 < h < t.$$

PROPOSITION 2.3. Let $A(t)$ be the generator of a strongly continuous quasi-semigroup $K(t,s)$ on a Banach space X and $B \in L(X)$ such that $B^2 = B$ with $BK(t,s) = K(t,s)B$. Then $A(t)B$ is the generator of the strongly continuous quasi-semigroup

$$R(t,s) = B[K(t,s) - I] + I.$$

PROOF. It is very easy.

PROPOSITION 2.4. Let $A(t)$ and $K(t,s)$ be as in the preceding proposition and $B \in L(X)$ such that $K(t,s)B = BK(t,s)$. Then $A(t) + B$ is the generator of the strongly continuous quasi-semigroup defined by

$$R(t,s) = e^{sB}\ K(t,s).$$

PROOF. It is very easy.

PROPOSITION 2.5. With the same hypothesis of the preceding proposition, the function $x_r(t) = R(r,t)x_o$ is solution of the integral equation

$$(2.5) \qquad z(t) = K(r,t)x_o + \int_0^t K(r+s,t-s)\ B\ z(s)ds$$

PROOF.

$$x_r(t) = R(r,t)x_o = K(r,t)e^{tB}\ x_o$$

$$= K(r,t)x_o + K(r,t)\int_0^t B\ e^{sB}\ ds$$

$$= K(r,t)x_o + \int_0^t K(r+s,\ t-s)\ K(r,s)\ e^{sB}\ B\ x_o\ ds$$

$$= K(r,t)x_o + \int_0^t K(r+s,\ t-s)\ x_r(s)ds.$$

EXAMPLE 2.4. Let $r \geq 0$ and $\mu \in \mathbb{R}$. Consider the problem

$$P_{\mu,r}) \qquad \begin{cases} \partial_t\ x(t,\xi) = 2(t+r)\partial_\xi\ x(t,\xi) + \mu\ x(t,\xi) \\ x(0,\xi) = x_o(\xi)\ ;\quad \xi,\ t \geq 0. \end{cases}$$

If X,D and K(t,s) are as in example 2.2, then

$$K(t,s)x(\xi) = x(s^2 + 2st + \xi).$$

Let us define $B: X \to X$ by means of $Bx = \mu x$. If is clear that $K(t,s)B = BK(t,s)$ and $A(t): D \to X$, the generator of $K(t,s)$ is defined by

$$A(t)\phi = 2t\,\dot{\phi} \quad , \quad (t \geq 0)$$

Hence, the problem $P_{\mu,r})$ can by written as

$$P_r) \qquad \begin{cases} \dot{x}(t) = (A(r+t) + B)\,x(t), & t > 0 \\ x(0) = x_o. \end{cases}$$

If $x_o \in D$, then applying theorem 2.2 and proposition 2.4, we obtain that

$$x(t) = R(r,t)x_o = e^{Bt}\,K(r,t)x_o$$

is the unique solution of the problem $P_r)$.

Consequently, the problem $P_{\mu,r})$ admits a unique solution. This solution is:

$$x(t,\xi) = e^{\mu t}\,x_o(t^2 + 2rt + \xi).$$

EXAMPLE 2.5. Consider the problem

$$P_\mu) \qquad \begin{cases} \partial_t x(t,\xi) = \partial_\xi\, x(t,t+\xi) + \mu x(t,\xi), \\ x(0,\xi) = x_o(\xi), \quad t,\xi \geq 0, \end{cases}$$

where $\mu \in \mathbb{R}$ is fixed.

Using the example 2.4 and the proposition 2.4, we obtain that the unique solution of $P_\mu)$ is

$$x(t,\xi) = e^{\mu t-1}.\sum_{n=0}^{\infty} \frac{x_o(nt+\xi)}{n!}, \quad x_o \varepsilon D.$$

The following Theorem is a generalization of the proposition 2.4.

THEOREM 2.4. Let A(t) be the generator of a strongly continuous quasi-semigroup K(t,s) on a Banach space X and B ε L(X). Then A(t) + B is the generator of a strongly continuous quasi-semigroup R(t,s) defined by

$$(2.6) \qquad R(r,t)x_o = K(r,t)x_o + \int_0^t K(r+s,t-s)\ BR(r,s)x_o ds,$$

in addition if

$$\| K(r,t) \| \leq M(r+t),$$

then

$$\| R(r,t) \| \leq M(r+t)\ \exp(\| B\|\ M(r+t)t).$$

PROOF. The solution of the integral equation (2.6) will be of the form

$$R(r,t) = \sum_{n=0}^{\infty} R_n(r,t);$$

where $R_o(r,t) = K(r,t)$, and

$$\text{(2.7)} \qquad R_n(r,t)x_o = \int_0^t K(r+s,\ t-s)BR_{n-1}(r,s)x_o\, ds, \quad n=1,2,3\ldots$$

First we have that

$$\| R_o(r,t) \| \leq M(r+t);$$

and proceding by induction we see that

$$\| R_n(r,t) \| \leq M(r+t). \frac{[\| B\| M(r+t)t]^n}{h!}, \quad n=0,1,2,3,\ldots.$$

consequently, the series

$$R(r,t) = \sum_{n=0}^{\infty} R_n(r,t)$$

is bounded by the convergent series

$$M(r+t) \sum_{n=0}^{\infty} \frac{[\| B\| M(r+t)]^n}{h!} = M(r+t). \exp(\| B\| M(r+t)t),$$

and hence it is convergent in the topology of the uniform convergence of $L(X)$, uniformly on compacts of $[0,\infty)$.

Now, we have that

$$R(r,t) = \sum_{n=0}^{\infty} R_n(r,t)x_o$$

$$= K(r,t)x_o + \sum_{n=1}^{\infty} R_n(r,t)x_o$$

$$= K(r,t)x_o + \sum_{n=1}^{\infty} \int_0^t K(r+s,t-s)BR_{n-1}(r,s)x_o \, ds$$

$$= K(r,t)x_o + \int_0^t K(r+s,t-s)B \sum_{n=0}^{\infty} R_n(r,s)x_o \, ds$$

$$= K(r,t)x_o + \int_0^t K(r+s,t-s)BR(r,s)x_o \, ds.$$

Thus, for each $r \geq 0$ fixed, $R(r,t)x_o$ is solution of (2.6).

If $x_r(t)$ is another solution of (2.6), then

$$\| R(r,t)x_o - x_r(t) \| \leq \int_0^t M(r+t) \| B \| \; \| R(r,s)x_o - x_r(s) \| \, ds,$$

and by Gronwall's Lemma we get that

$$R(r,t)x_o = x_r(t), \quad (t \geq 0).$$

Now let us see that $R(r,t)$ satisfies the conditions of the definition (2.1).

Clearly the condition a) is satisfied

c) From expression (2.7) it follows that $R(r,.)$ is strongly continuous. Now we will prove that $R(.,s)$, is strongly continuous:

$$\| R(r,s)x_o - R(r_o,s)x_o\| \leq \| K(r,s)x_o - K(r_o,s)x_o\|$$

$$+ \left\| \int_0^s K(r+\alpha,s-\alpha)BR(r,\alpha)x_o - K(r_o+\alpha,s-\alpha)\ BR(r_o,\alpha)x_o\ d_\alpha \right\|$$

$$\leq \| K(r,s)x_o - K(r_o,s)x_o\| + \int_0^s \| K(r+\alpha,s-\alpha)BR(r_o,\alpha)x_o -$$

$$- K(r_o+\alpha,s-\alpha)BR(r_o,\alpha)x_o\| \, d\alpha$$

$$+ \int_0^s \| K(r+\alpha,s-\alpha)B(R(r,\alpha)x_o - R(r_o,\alpha)x_o)\| \, d\alpha.$$

The first two terms go to zero according to c) and d) of the definition 2.1 and by dominated convergence theorem. Hence

$$\| R(r,s)x_o - R(r_o,s)x_o\| \leq \varepsilon + \int_0^s M(r+s)\|B\| \, \|R(r,\alpha)x_o - R(r_o,\alpha)x_o\| \, d\alpha$$

therefore the strongly continuity of $R(r,.)$ and Gronwall's lemma it follows that

$$\| R(r,s)x_o - R(r_o,s)x_o\| \leq \varepsilon \exp(\|B\| \, M(r+s)s);$$

which implies that $R(.,s)$ is continuous.

Now

$$\| R(r,t)x_o - R(r_o,t_o)x_o\| \leq \|K(r,t)x_o - K(r_o,t_o)x_o\|$$

$$+ \left\| \int_0^{t_o} |K(r+s,t-s)BR(r,s)\,x_o - K(r_o + s,t_o-s)BR(r_o,s)x_o| \, ds \right\|$$

$$+ \left\| \int_0^{t_o} K(r+s,t-s) \left[R(r,s)x_o - R(r_o,s)x_o\right] \right\|$$

$$+ \left\| \int_{t_o}^{t} K(r+s,t-s)BR(r,s)x_o \, ds \right\| ,$$

which converges to zero if (r,t) converges to (r_o, t_o) which proves c).

Now we will prove the part d) of the definition 3.1:

$$R(r,t+s)x_o - R(r+t,s) \; R(r,t)x_o$$

$$= K(r,t+s)x_o + \int_0^{t+s} K(r+\alpha,t+s-\alpha)BR(r,\alpha)x_o \, d\alpha$$

$$-(K(r+t,s) + \int_0^{s} K(r+t+\alpha,s-\alpha)BR(r+t,\alpha)d\alpha) \; (K(r,t)x_o +$$

$$+ \int_0^{t} K(r+\alpha, t-\alpha)BR(r,\alpha)x_o \, d\alpha)$$

$$= \left[K(r,t+s)x_o - K(r+t,s) \; K(r,t)x_o\right]$$

$$+ \int_0^{t+s} K(r+\alpha, t+s-\alpha)BR(r,\alpha)x_o \, d\alpha$$

$$- \int_0^{s} K(r+t+\alpha, s-\alpha)BR(r,t)x_o \, d\alpha$$

$$- \int_0^{t} K(r+t,s) \; K(r-\alpha,t-\alpha)BR(r,\alpha)x_o \, d\alpha$$

$$= \int_t^{t+s} K(r+\alpha,\ t+s-\alpha) BR(r,\alpha) x_o \, d\alpha$$

$$- \int_0^s K(r+t+\alpha,\ s-\alpha) BR(r+t,\alpha) R(r,t) x_o \, d\alpha$$

Since

$$K(r+\alpha, t+s-\alpha) = K(r+t,s) K(r+\alpha,\ t-\alpha).$$

Changing variables in the first integral we obtain

$$R(r,t+s)x_o - R(r+t,s)R(r,t)x_o$$

$$= \int_0^t K(r+t+\alpha, s-\alpha) B\left[R(r,t+s)x_o - R(r+t,\alpha)\ R(r,t)x_o\right] d\alpha,$$

and by Gronwall's lemma it follows that

$$R(r,t+s) = R(r+t,s)R(r,t)$$

e) Let $t \geq 0$ and $x_o \in D$;

$$\lim_{s\to 0} \frac{R(t,s)x_o - x_o}{s} = \lim_{s\to 0} \left[\frac{K(t,s)x_o - x_o}{s} + \frac{1}{s}\int_0^s K(t+\alpha, s-\ ) BR(t,\alpha) x_o \, d\alpha\right]$$

$$= A(t)x_o + K(t,0)BR(t,0)x_o$$

$$= (A(t) + B)x_o.$$

Let us suppose that $t > 0$. Then

$$\lim_{s\to 0^+} \frac{R(t-s,s)x_o - x_o}{s} = \lim_{s\to 0^+} \left[\frac{K(t-s,s)x_o - x_o}{s} \right.$$

$$\left. + \frac{1}{s}\int_0^s K(t-s+\alpha,s-\alpha)BR(t-s,\alpha)x_o \, d\alpha \right]$$

$$= A(t)x_o + K(t,0)BR(t,0)x_o$$

$$= (A(t) + B)x_o,$$

Since D is dense on X, then $A(t) + B$ is the generator of $R(t,s)$.

3. Dual quasi-semigroup.

In this section we shall define the dual quasi-semigroup and prove some properties analogous the ones of dual semigroups.

PROPOSITION 3.1. Let $K(t,s)$ be a strongly continuous quasi-semigroup on a Banach space X. Then

a) $K^*(t,0) = I^*$, I^* the identity operator on X^*.

b) $K^*(r,t+s) = K^*(r+t,s)K^*(r,t)$.

c) $\lim_{(t,s)\to(t_o,s_o)} K^*(t,s)x^* = K^*(t_o,s_o)x^*$ $\quad (x^* \in X^*)$

in the weak* topology of X^*.

d) $\| K^*(t,s) \| \le M(t+s)$.

PROOF. The proof is analogous to the case of semigroups [2].

DEFINITION 3.1. The function $K^*(t,s)$ of the preceding proposition is called the dual quasi-semigroup of $K(t,s)$. In general, $K^*(t,s)$ is not strongly-continuous however, what follows is true.

PROPOSITION 3.2. If $K(r,t)$ is weakly continuous, then for each $r > 0$ fixed, the application $K(r,.)$ is strongly continuous in $(0, +\infty)$.

PROOF. If $x(t) = K(r,t)t_o$, then is weakly continuous and therefore Bochner integrable on compact intervals in $(0, +\infty)$.

Let $\xi > 0$ be and consider

$$0 \leq \alpha < \eta < \beta < \xi - \varepsilon < \xi, \quad \varepsilon > 0,$$

$$x(\xi) = K(r,\xi)x_o = K(r+\eta, \xi-\eta)K(r,\eta)x_o,$$

$$= K(r+\eta, \xi-\eta)x(\eta);$$

therefore

$$(\beta-\alpha)\ x(\xi) = \int_\alpha^\beta K(r+\eta, \xi-\eta)x(\eta)d\eta \ ;$$

$$(\beta-\alpha)\ x(\xi\pm\varepsilon) = \int_\alpha^\beta K(r+\eta, \xi \pm \varepsilon - \eta)x(\eta)d\eta$$

consequently

$$(\beta-\alpha)\left[x(\xi \pm \varepsilon) - x(\xi)\right] =$$

$$= \int_\alpha^\beta K(r,\eta) \left[K(r+\eta,\ \xi \pm \varepsilon-\eta)x_o - K(r+\eta,\ \xi-\eta)x_o\right] d\eta\ ,$$

therefore,

$$(\beta-\alpha)\|x(\xi\pm\varepsilon)-x(\xi)\| \le M(r+\beta) \int_\alpha^\beta \|K(r+\eta,\xi \pm \varepsilon-\eta)x_o - K(r+\eta,\ \xi-\eta)x_o\|\, d\eta\ .$$

If K is continuous in the second variable, then

$$\lim_{\varepsilon\to 0} \int_\alpha^\beta \|K(r+\eta,\xi \pm \varepsilon-\eta)x_o - K(r+\eta,\varepsilon-\eta)x_o\|\, d\eta = 0.$$

The general case is consequence of the density of the continuous function in $L_1(0,T;X)$.

THEOREM 3.1. Let $K(r,t)$ be a strongly continuous quasi-semigroup on a Banach space X. Then

a) If $x^* \in D(A^*(r+t))$, then $K^*(r,t)x^* \in D(A^*(r+t))$

and

$$A^*(r+t)K^*(r,t)x^* = K^*(r,t)A^*(r+t)x^*$$

b) $x^* \in D(A^*(t)) \iff w^* - \lim_{s\to 0^+} \dfrac{K^*(t,s)x^* - x^*}{s}$

$$= w^* - \lim_{s\to 0^+} \frac{K^*(t-s,s)x^* - x^*}{s}$$

$$= A^*(t)x^* \qquad (t > 0)$$

c) If $A(t)$ is strongly integrable,

$$K^*(r,t)x^* - x^* = \int_0^t A^*(r+s)\ K^*(r,s)x^* ds$$

$$x^* \varepsilon \bigcap_{t \geq 0} D(A^*(t)) = D^*$$

PROOF. It is analogous to theorem 2.2 of [2].

PROPOSITION 3.3. Let $K(t,s)$ be strongly continuous quasi-semigroup of isometries on a Hilbert space H. Then its generator $A(t)$ is skew-symmetric.

4. Controllability.

Consider the non-autonomous and unbounded system.

$$\text{(4.1)} \qquad \dot{x}(t) = A(t)x(t) + Bu(t),\ x(0) = x_o, \quad 0 \leq t \leq T$$

where $A(t)$ is the generator of a strongly continuous quasi-semigroup $K(t,s)$ on a Banach space X, $B \varepsilon L(U,X)$ where U is a Banach space and control function $u(.) \varepsilon L_p(0,T;U)$, $(p > 1)$. According to definition 2.3, the mild solution of (4.1) is given by

$$x(t) = K(0,t)x_o + \int_0^t K(s,t-s)Bu(s)ds, \quad 0 \leq t \leq T.$$

DEFINITION 4.1. We shall say that the system (4.1) is exactly controllable in time $T > 0$, if for each x_o, $x_1 \varepsilon X$ there exists a control $u \varepsilon L_p(0,T;U)$ such that the mild solution of (4.1) $x(t)$ corresponding to u, verifies: $x(T) = x_1$.

Consider the operator

$$G_T\colon L_p(0,T;U) \to X,$$

defined by

$$G_T u = \int_0^T K(s,T-s)Bu(s)ds.$$

It is easy to see that G_T is linear and continuous and that (4.1) is exactly controllable if and only if G_T is onto, that is

$$G_T\, L_p(0,T;U) = \text{Range } G_T = X.$$

DEFINITION 4.2. We shall say that the system (4.1) is approximately controllable in time $T > 0$ (approximality controllable in free time) if

$$\overline{\text{Range } G_T} = X \;, \left(\overline{G}_\infty = \overline{\bigcup_{T>0} G_T\, L_p} = X. \right)$$

In what follows we shall suppose that X and U are reflexive Banach spaces.

THEOREM 4.1. If $u \in L_p(0,T;U)$, $1 < p < \infty$, then (4.1) is exactly controllable in time $T > 0$, if and only if there exists $r > 0$ such that

$$r\| B^* K^*(.,T-.)x^*\|_{L_q} \geq \| x^*\|, \quad \frac{1}{p} + \frac{1}{q} = 1, \; x^* \in X^*.$$

PROOF. If we put $W = L_p(0,T;U)$, then $G_T \in L(W,X)$ and by Theorem 3.3 of reference [2], we have that

$$\text{Range } G_T = X \iff \exists \; r > 0: r\| G_T^* x^*\| \geq \| x^*\|, \; (x^* \in X^*).$$

Let us calculate G_T^*

$$< x^*, G_T u >_{X^*,X} = <x^*, \int_0^T K(s,T-s)B\ u(s)ds>_{X^*,X}$$

$$= \int_0^T <B^*\ K^*(s,T-s)x^*,\ u(s)>\ ds$$

$$= < B^*\ K^*(0,T-.)x^*, u(.)>_{W^*,W} ;$$

Therefore ,

$$G_T^*\ x^* = B^*\ K^*(.,T-.)x^* \ \varepsilon\ L_q(0,T;U^*),$$

This ends the proof.

The following theorems are immediate consequence of theorem 3.6 of [2].

THEOREM 4.2. The system (4.1) is approximately controllable in time $T > 0$ if an only if

$$B^*\ K(t,T-t)\ x^* = 0, \quad 0 \le t \le T, \text{ implies } \quad x^* = 0.$$

THEOREM 4.3. The system (4.1) is approximately controllable in free time if and only if

$$B^*\ K^*(t,T-t), \ \forall T > 0, \quad 0 \le t \le T, \text{ implies } \quad x^* = 0.$$

Supported by C.D.C.H.T proyect C-391-89 of Universidad de los Andes.

R E F E R E N C E S

[1] A.V. BALAKRISHNAN. Applied Functional Analysis, Application of Mathematics, Vol. 3 Springer Verlag, Berlin (1976).

[2] R.F. CURTAIN AND A.J. PRITCHARD. Infinite Dimensional Linear Systems, Lecture Notes in control and information Sciences, Vol. 8. Springer Verlag, Berlin (1978).

[3] H.O. FATTORINI. Some Remarks on complete controllability, Siam J. Control 4 (1966) pp. 686-694.

[4] E. HILLE, AND R.S. PHILLIPS. Functional Analysis and Semigroups, Colloq. Amer. Math. Soc. 31 (1957).

[5] T. KATO. Integration of equation of evolution in Banach Space, J. Math. Soc. Japan, 5,208-234 (1953).

[6] V.I. KOROBOV AND R. RABAKH. Exact controllability in Banach Space, A.M. Gorki Kharkov State university. Translated from Differentsial'nye Uravheniya, Vol. 15 Nº 12, pp. 2142-2150, December (1979).

[7] C.E. LADAS AND LASHMIKANTHAM "Differential Equations in Abstract Spaces", Vol. 85, Academic Press, New York. (1972).

[8] H. LEIVA Y D.BARCENAS. Sobre la existencia y unicidad de las soluciones de una cierta clase de ecuaciones diferenciales en espacios de Banach, Notas de Matemáticas Nº 87, Universidad de los Andes, Mérida,Venezuela (1988).