Classical Limits Via Brownian Motion

H. Gzyl* and J.R. León**.

Escuela de Física y Matématicas.

Facultad de Ciencias U.C.V.

Caracas. Venezuela.

ABSTRACT

We present a simplified approach to the problem of obtaining classical mechanics from difussion equation. We consider the case of particles in static electromagnetic fields.

*Postal Address: A.P. 52120, Caracas 1050-A. Venezuela.

**Postal Address: A.P.47197. Caracas 1041-A. Venezuela.

I-Introduction.

In this note we present two simple approaches to obtainig the $h \dashrightarrow 0$ asymptotic behaviour of the solution to

(I.1-a) $$h\, \partial_t \psi_h = 1/2(h\nabla + A)^2\psi_h + V\, \psi_h \qquad x \in \mathbb{R}^n,\ t > 0$$

under the initial condition

(I.1-b) $$\psi_h(x,0) = f(x)\ \exp-(h^{-1} S_0)$$

We shall assume that $f(x)$ is bounded and continous, that $S_0(x)$, $V(x)$, $A_k(x)$ $k=1,2,\ldots,n$ are all bounded, continous and have two bounded continous derivatives in all variables. We shall moreover make the simplifying assumption (usually called the transversality condition or gauge in the physical literature) that $\nabla.A(x)=0$. We obtain the existence of the limits

(I-2) $$\lim_{h \dashrightarrow 0} -h\log \psi_h(x,t) = S(x,t)$$

(I-3) $$\lim_{h \dashrightarrow 0} \exp(S(x,t)/h)\ \psi_h(x,t) = f(x)\ \exp(1/2)\int_0^t \Delta S(x(s),t-s)ds$$

where $S(x,t)$ solves the Hamilton-Jacobi equation

(I-4) $$\partial_t S(x,t) + 1/2(\nabla S(x,t) - A(x))^2 + V(x) = 0 \qquad ,S(x,0)=S_0(x)$$

The first approach is somewhat similar to that of Elworthy and Truman in [1]. A similar problem, but in the quantun case was recently discussed in [2]. The second approach is a variation on the technique used by Shilder [5], adapted to the particular situation we deal with. Both approaches have (II-1) as starting point, so some explanation of our notation is in order. By $\{B_h(t);\ t \geq 0\}$ we denote the standard brownian motion on $\mathbb{R}^n$ of variance h. $E_h^x(.)$ denotes the path integral on $C([0,\infty),\mathbb{R}^n)$, the class of all continuos maps from $[0,\infty)$ to $\mathbb{R}^n$, constructed from the transition semigroup with density

$$p(x-y,t) = (\exp-(x-y)^2/2th)/(2\pi ht)^{n/2},$$

which is the fundamental solution to the heath equation $\partial_t u = h/2\ \Delta u$. All the probabilistic

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constructions and results that we shall need can be found in [3].

An important property of the process $\{B_h(t)\}$ that will be used below is that of the equidistribution of $\{B_h(t)\}$ and $h^{-1/2}B(t)$ where by $\{B(t):t\geq 0\}$ we denote the standard brownian motion (i.e. the h=1 case).

We shall be using the repeated indices sumation convention, and we shall not distinguish between lower and upper indices, which will be written according to typographical convenience. We will write ∂_i, ∂_{ij} for the derivatives with respect x_i; $x_i x_j$, etc...

II-First approach.

Under our assumptions on V(x) and A(x), see [3], the solution to (I.1) can be written as

(II.1) $$\psi_h = E_h^x\left[f(B_h(t)) \exp h^{-1}\{\textstyle\int_0^t A(B_h(s)).dB_h(s)+\int_0^t V(B_h(s))ds - S_0(B_h(t))\}\right]$$

For each fixed t in [0,T] apply Ito's formula to $S(B_h(s),t-s)$ to obtain

$$dS(B_h(s),t-s) = \nabla S(B_h(s),t-s).dB_h(s)-\partial_t S(B_h(s),t-s)+h/2\ \Delta S(B_h(s),t-s)ds$$

and now integrate both sides from 0 to t and make use of (I.4) to obtain for (II.1)

(II-2) $$\psi_h(x,t) = \exp-S(x,t)\ E_h^x\left[f(B_h(t))Z_h(t)\ \exp-(1/2)\textstyle\int_0^t \Delta S(B_h(s),t-s)ds\right]$$

where $Z_h(t)$ is the exponential martingale

$$Z_h(t) = \exp-h^{-1}\{\textstyle\int_0^t(\nabla S-A)(B_h(s),t-s).dB_h(s) +1/2\int_0^t(\nabla S-A)^2(B_h(s),t-s)ds\}.$$

The martingale property of $Z_h(t)$ follows from the fact that our assumptions imply that $E_h^x[\exp(1/2)\int_0^t(\nabla S-A)^2(B_h(s),t-s)ds] < \infty$. Now rescale $B_h(t)$ as to have a fixed probability law on $C([0,\infty),\mathbb{R}^n)$ and consider for each t in [0,T] the equation

(II.3) $$dX_h(s) = -[\nabla S(X_h(s),t-s)- A(X_h(s))]ds+ h^{1/2}dB(s) \qquad , X_h(0)= x.$$

From our boundedness assumptions on A and S we obtain the uniqueness and existence of $X_h(s)$. Again by the Cameron-Martin-Girsanov translation formula rewrite (II.2)as

(II.4) $$\psi_h(x,t) = \exp -S(x,t)/h \; E\left[f(X_h(t))\exp -1/2\int_0^t \Delta S(X_h(s),t-s)ds\right].$$

Fron the uniform convergence in probability of $X_h(s)$, $t\geq s\geq 0$, as $h \to 0$ to the solution of (see [4])

(II.5) $$\dot{x}(s) = -(\nabla S(x(s),t-s) - A(x(s))) \qquad x(0)=x$$

we obtain (I.2) and (I.3) namely

$$\lim_{h\to 0} -h\log \psi_h(x,t) = S(x,t)$$

$$\lim_{h\to 0} \exp S(x,t)/h \; \psi_h(x,t) = f(x(t))\exp -1/2\int_0^t \Delta S(x(s),t-s).$$

The existence, uniqueness, and regularity of solutions to (II.5) follows easily from the boudedness and regularity of A and ∇S.

III.Second Approach.

In this section we isolate the limiting value by a different procedure. We translate the $B_h(s)$ by the curve $x(s)$ obtained by solving (II.5) with the extra end condition

$$\dot{x}(t) = -\nabla S_0(x(t)) + A(x(t))$$

in terms of which S(x,t) can be written (as we shall see below) as

(III.1) $$S(x,t) = S_0(x(t)) + \int_0^t 1/2\{\dot{x}^2(s) - A(x(s)) - V(x(s))\}ds$$

over the class of curves $y: [0,t] \to R^n$ such that $y(0)=x$.

Also, $x(s)$ satisfies the Lagrange (Newton) equation on $[0,t]$

(III.2) $$\ddot{x}_i = \dot{x}^j\partial_j A_i - \dot{x}^j\partial_i A_j - \partial_i V \qquad , i= 1,2\ldots,n.$$

with $$x(0) = x,\; \dot{x}(t) = -\nabla S_0(x(t)) + A(x(t)).$$

Now, by applying the Cameron-Martin translation formula, we can rewrite (II.1) as

(III.3) $$\psi_h(x,t) = E_h^x\left[f(x(t)+B_h(t))M_h(t)\ \exp -1/h\{\int_0^t \dot{x}(s).dB_h(s)+1/2\int_0^t \dot{x}^2(s)ds\}\right]$$

where $M_h(t)$ is given by

$$M_h(t) = \exp 1/h\left[\int_0^t A(x(s)+B_h(s)).(dx(s)+dB_h(s))+\int_0^t V(x(s)+B_h(s))ds - S_0(x(t)+B_h(t))\right].$$

Note that in (III.1) the integration is over brownian paths starting from the origin since the initial condition is included in $x(s)$. We shall now make use of the Taylor expansion formula

$$f(x+b) = f(x) + b^j\partial_j f(x) + b^{ij}\int_0^1(1-u)\partial_{ij}f(ub+x)u$$

valid for f in $C^2(\mathbb{R}^n)$ and we rewrite the terms in the exponent of $M_h(t)$ as

(III.4-a) $$S_0(x(t)+B_h(t)) = S_0(x(t))+B_h{}^i(t)\partial_i S_0(x(t)) + B_h{}^i(t)B_h{}^j(t)\tilde{S}_{ij}(t)$$

(III.4-b) $$\int_0^t V(x(s)+B_h(s))ds =$$

$$\int_0^t V(x(s))ds + \int_0^t B^i(s)\partial_i V(x(s)) + \int_0^t B_h{}^i(s)B_h{}^j(s)\tilde{V}_{ij}(s)ds$$

(III.4-c) $$\int_0^t A(x(s)+B_h(s)).(dx(s)+dB_h(s)) =$$

$$\int_0^t A(x(s)).(dx(s)+dB_h(s)) + \int_0^t B_h{}^i(s)B_h{}^j(s)\tilde{A}_k(s)(dx^k(s)+dB_h{}^k(s)).$$

We have introduced the notations $\tilde{S}_{ij}=\int_0^1(1-u)\partial_{ij}S_0(uB_h(s)+x(s))ds$ and $\tilde{V}_{ij}(s)$, $\tilde{A}_k(s)$ are similarly defined. With this (III.3) becomes

(III.5) $$\psi_h(x,t) = \exp -S(x,t)/h\ E_h\left[f(x(t)+B_h(t))N_h(t)\right]$$

where E_h is the path integral with respect to the brownian motion starting in 0 and

$$N_h(t)=\exp 1/h\{\int_0^t B_h{}^i(s)B_h{}^j\tilde{V}_{ij}(s)ds + \int_0^t B_h{}^i(s)B_h{}^j(s)\tilde{A}_{kij}(s)(dx^k(s)+dB^k(s))$$

$$+ \int_0^t B_h{}^i(s)\partial_i A_j(x(s))dB_h{}^j(s) - B_h{}^i(t)B_h{}^j\tilde{S}_{ij}(s)\}.$$

We must verify that the terms not appearing in the exponent of $N_h(t)$ nor in $S(x,t)$ drop away. Consider

$$\int_0^t B_h{}^i\partial_i V(x(s))ds + \int_0^t B_h{}^i\dot{x}^i\partial_i A_j ds$$ 87

which equals

$$-\int_0^t B_h{}^i(\ddot{x}_i-\dot{x}_j\partial_j A_i)ds$$

as follows from (III.1). This added to

$$-\int_0^t \dot{x}_i dB_h{}^i(s)+\int_0^t A_i(x(s))dB_h{}^i(s)$$

yields

$$-\int_0^t d\left[B_h{}^i(s)(\dot{x}_i(s)-A_i(x(s))\right]$$

which cancels

$$-B_h{}^j.\partial_j S_0(x(s))$$

since at t

$$\nabla S_0.(x(s)) + \dot{x}(t) - A(x(t)) = 0.$$

Again, rescale (III.5) to obtain

(III.6) $$\psi_h(x,t) = \exp-h^{-1}S(x,t)E^0\left[f(x(t)+h^{1/2}B(t))N_h''(t)\right]$$

where $N_h''(t)$ is given by

$$N_h''(t) = \exp(\int_0^t B^i(s)B^j(s)V''_{ij}(s)ds + \int_0^t B^i(s)B^j(s)A''_{kij}(dx^k+h^{1/2}dB^k(s))$$
$$+ \int_0^t B^i(\partial_i A_j(x(s)))dB^j(s) - B^i(t)B^j(t)S''_{ij}(t))$$

where, for example, $V''_{ij}(s) = \int_0^1(1-u)\partial_{ij}V(uh^{1/2}B(s)+x(s))du$

similar expressions hold for $S''_{ij}(t)$, $A''_{kij}(s)$. It is now an easy to show that

$$-\lim_{h\to 0} h\log\psi_h(x,t) = S(x,t)$$
$$\lim_{h\to 0}\exp h^{-1}S(x,t)\psi_h(x,y) = f(x(t))\,E^0\left[N''_0(t)\right].$$

This limiting procedure can be justified as follows. The family

$$f(x(t) + h^{1/2}B(t))\,N_h''(t)$$

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is uniformly integrable. To see this we must proof that

(III.8) $$E\,\{f(x(t)+h^{1/2}B(t))\,N_h''(t)\}^k$$

is bounded uniformly in $0<h<\epsilon$, for some $0<k$. But it is an easy consecuence of the Hölder inequality and of the expectation of the exponential martingale for the function

$$b(s,y)=\int_0^1 \partial_j A(x(s)+h^{1/2}u\,y)\,y_j\,du.$$

We still have to verify (III.1) . For that let $S(x,t)$ be the solution to (I.4), and differentiate $S(x(s),t-s)$ with respect to s, use (I.4) and (II.5) to obtain

$$\partial_s S(x(s),t-s)=-1/2\,\dot{x}^2(s)+V(x(s))+A(x(s)).\dot{x}(s)$$

now integrate in s from to t to obtain

$$S(x,t)=S_0(x(t))+\int_0^t\left[1/2\,\dot{x}^2(s)-A(x(s)).\dot{x}(s)-V(x(s))\right]ds.$$

Comparing the procedures in sections II and III we obtain the result

$$E^0\left[N_0(t)\right]=\exp-1/2\int_0^t \Delta S(x(s),t-s)ds.$$

When $A\equiv 0$, and $S_c(x)=x.\,P$ for some constant vector P, then $S''_{ij}\equiv 0$ and

$$E^0\left[\exp 1/2\int_0^t B^i(s)B^j(s)\,\partial_{ij}V(x(s))ds\right]=\exp-1/2\int_0^t\Delta S(x(s),t-s)ds.$$

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