SOME INEQUALITIES FOR THE FIRST EIGENVALUE OF THE LAPLACE-BELTRAMI OPERATOR

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1. Introduction.

In a recent paper on the biology of cell membranes [6] the following eigenvalue problem (for m=3) is studied

$$\frac{d^2F}{d\theta^2} + (m-2)\cot\theta \frac{dF}{d\theta} + \lambda F = 0, \quad 0<\theta<\theta_0<\pi$$
$$F(\theta_0)=0. \tag{1.1}$$

It is known ([1],[7]) that the solution of (1.1) gives the smallest value of J(f) where

$$J(f) = \frac{\int_0^{\theta_0} f'(\theta)^2 \operatorname{sen}^{m-2}\theta\, d\theta}{\int_0^{\theta_0} f(\theta)^2 \operatorname{sen}^{m-2}\theta\, d\theta}$$

and f belongs to the class ϑ_{θ_0} of functions defined in $[0,\pi]$, Lipschitzian, non-negative, non identically zero on $[0,\pi]$ and which vanish in $[\theta_0,\pi]$. The smallest value of J(f) in the class ϑ_{θ_0} is the smallest eigenvalue of the Laplace-Beltrami operator

$$\delta = r^2\Delta - r^2\frac{d^2}{dr^2} - r(m-1)\frac{d}{dr}$$

(where Δ is the Laplace operator in R^m) of a spherical cap. By a spherical cap we mean a set of the form $C_{\theta_0}=\{(x_1,x_2,\dots,x_m) \mid \cos\theta< x_1\leq 1\}\cap S_m$, where S_m is the unit sphere in R^m.

The following result is due to Pinsky [10].

Theorem 1. <u>If λ_1 is the smallest eigenvalue of</u> (1.1) <u>for</u> m=3, <u>then</u>

$$\frac{1}{2e\left|\log\cos\frac{\theta_0}{2}\right|} \leq \lambda_1 \leq \frac{j_0^2}{\theta_0^2},$$

where $j_0 \approx 2{,}4048\ldots$ is the first zero of the Bessel function J_0.

This theorem is a consequence of the following more general result (see [1]) which improves the lower bound and extends both the upper and lower bound.

2. Inequalities in R^m $(m \geq 3)$.

Theorem 2. If λ_1 is the smallest eigenvalue of (1.1) then

$$\frac{1}{\displaystyle\int_0^{\theta_0} \frac{dx}{\operatorname{sen}^{m-2} x} \int_0^x \operatorname{sen}^{m-2} t \, dt} \leq \lambda_1 \leq \frac{j^2_{(m-3)/2}}{j_0^2},$$

where $j_{(m-3)/2}$ is the first zero of the Bessel function $J_{(m-3)/2}$.

Remarks:

1. In the case m=3 the lower bound inTheorem 2 is $(2/|\log \cos\frac{\theta_0}{2}|)^{-1}$ which improves the one given in Theorem 1. We shall prove later on that this bound is sharp as θ approaches π both in the case m=3 and m>3 (see [1]).
2. The upper bound in Theorem 2 extends Pinsky's upper bound to the higher dimensional case. Later on we shall improve the upper bound.
3. We remark that in the case m=4 the smallest eigenvalue can be calculated explicitly giving

$$\lambda_1 = \frac{\pi^2}{\theta_0^2} - 1.$$

Since the proof given in [1] is short we shall include it here. More recently the author has given a new proof of the lower bound based on the elementary theory of compact operators in a Hilbert space [2a]

Proof of Theorem 2. Consider the eigenvalue problem

$$(\sin\theta)^{2-m} \frac{d}{d\theta}\left[(\sin\theta)^{m-2} \frac{dF}{d\theta}\right] + \lambda F = 0 \tag{2.1}$$

$$F(\theta_0)=0, \quad 0<\theta<\theta_0<\pi,$$

and let (X_t) be the diffusion on $(0,\pi)$ with generator

$$L = \frac{1}{2}[(\sin\,\theta)^{2-m}\frac{d}{d\theta}((\sin\,\theta)^{m-2}\frac{d}{d\theta})]$$

killed when it reaches θ_0. Define

$$T_{\theta_0}=\inf\{t>0\ :\ X_t=\theta_0\}$$

and let

$$Vf(\theta)=E^{\theta}\int_0^{T_{\theta_0}} f(X_t)dt$$

be the associated potential operator. We see that

$$||V|| = \sup_j\ E^{\theta}(T_\alpha) \le g(\theta_0),$$

where $g(\theta)=2\int_0^{\theta_0}\frac{dx}{\text{sen}^{m-2}x}\int_0^x \text{sen}^{m-2}t\ dt$.

From (2.2) we obtain

$$\lambda \ge \frac{2}{g(\theta_0)}$$

This gives the lower bound.

In order to find an upper bound for λ we follow Pinsky [10] and compare the equation

$$\frac{d^2F}{d\theta^2} + (m-2)\cot\theta\,\frac{dF}{d\theta} + \lambda F = 0, \qquad 0<\theta<\theta_0<\pi$$

with the equation

$$\frac{d^2F}{d\theta^2} + \frac{(m-2)}{\theta}\frac{dF}{d\theta} + \tilde{\lambda}F = 0, \qquad 0<\theta<\theta_0<\pi, \qquad (2.3)$$

whose solution is $F(\theta)=\theta^{3-m/2}\,J_{(3-m)/2}\,\sqrt{\tilde{\lambda}\theta}$, where $J_{(3-m)/2}$ is the Bessel function of order $(3-m)/2$. Thus the smallest eigenvalue $\tilde{\lambda}$, of

(2.3) is $\tilde{\lambda}=\frac{j^2_{(m-3)/2}}{\theta_0^2}$, $j_{(m-3)/2}$ being the first zero of the Bessel function $J_{(3-m)/2}$. Using now the comparison theorem in [3] we get the desired upper bound.

3. The characteristic constant of a spherical cap and asymptotic results.

In this section we would like to obtain asymptotic expressions for the first eigenvalue of a cap of the (m-1)-dimensional unit sphere (m≥3). The result that we obtain turns out to be different in the cases m=3 and m>3.

Following Hobson [9] we consider the Legendre's associated function of the first kind $P_n^l(m)$ defined for unrestricted values of the degree n and the order l. This function is a particular integral of the ordinary linear differential equation of the second order

$$(1-\mu^2)\frac{d^2u}{d\mu^2} - 2\mu\frac{du}{d\mu} + \{n(n+1)- \frac{l^2}{1-\mu^2}\}u=0,$$

which is known as Legendre's associated equation of degree n and order l. Next, we quote a result concerning the zeros of $P_n^{-l}(\cos j)$ considered as a function of n and when λ is near π. The proof of the following lemma can be found in Hobson [9].

Lemma 1. The smallest value of n for which $P_n^{-l}(\cos\theta)$ vanishes satisfies the following asymptotic relations as θ tends to π: if l=0 then

$$n \sim \frac{1}{2\log\frac{2}{\pi-\theta}},$$

if l>0 then

$$n-l \sim \frac{\Gamma(2l+1)}{\Gamma(l+1)\Gamma(l)} \tan^{2l}\{\frac{\pi-\theta}{2}\}.$$

In what follows we define the characteristic constant of a spherical cap of the (m-1)-dimensional unit sphere and see its relationship with both the above lemma and the first eigenvalue.

Let C_{θ_0} be the cap of the (m-1)-dimensional unit sphere $S_m(0,1)$ defined by $\cos\theta_0 < x_1 \leq 1$. The characteristic constant $\alpha(\theta_0)$ of such a cap is given by the positive root of the equation

$$\alpha(\theta_0)\{\alpha(\theta_0)+(m-2)\}=\lambda(\theta_0),$$

where

$$\lambda(\theta_0) = \inf_{f\in \vartheta_{\theta_0}} J(f) = \inf \frac{\int_{C_{\theta_0}} |\operatorname{grad} f|^2 d\sigma_\xi}{\int_{C_{\theta_0}} |f|^2 d\sigma_\xi},$$

and ϑ_{θ_0}={f, functions depending only on x_1, non-negative, Lipschitzian, non-identically zero on $S_m(0,1)$ and vanishing outside the cap C_{θ_0}}.This infimum is attained at the solution of the Laplace-Beltrami equation

$$\delta f + \lambda f = 0$$

on C_{θ_0}, where $\lambda=\lambda(\theta_0)$ is the lowest eigenvalue of this equation.

The fact that we may take axi-symmetric functions is due to Sperner [11]. If we write $x_1=\cos\theta$, then all functions f on the cap C_{θ_0} can be considered as functions of the variable $\theta(\theta\in[0,\pi])$. Therefore the elements of the class ϑ_{θ_0} are functions $f(\theta)$ defined in $[0,\pi]$, Lipschitzian, non-negative, non-identically zero on $[0,\pi]$ and which vanish in $[\theta_0,\pi]$. Furthermore

$$\lambda(\theta_0) = \inf_{f\in \vartheta_{\theta_0}} \frac{\int_0^{\theta_0} f'(\theta)^2 \operatorname{sen}^{m-2}\theta \, d\theta}{\int_0^{\theta_0} f(\theta)^2 \operatorname{sen}^{m-2}\theta \, d\theta}.$$

Regarding the minimum value of J(f) we have the following lemma due to Friedland and Hayman [7].

Lemma 2. Let $f(\theta)$ be a Lipschitzian function in $[0,\pi]$, not identically zero and such that $f(\theta)=0$, $\theta_0\le\theta\le\pi$. Then $J(f)\ge J(F)=\lambda(\theta_0)$, where $u=(\sin\theta)^{(m-2)/2}F(\theta)$ is a solution of the differential equation

$$\frac{d^2u}{d\theta^2}+\{\lambda+\frac{(m-2)^2}{4}+\frac{(m-2)(4-m)}{4\sin^2\theta}\}u=0 \qquad (3.1)$$

and the positive number λ is so chosen that F remains analytic at $\theta=0$

$F'(0)=0$, $F(\theta_0)=0$ and $F(\theta)>0$ for $0<\theta<\theta_0$.

The smallest zero of the function $F(\theta)$ is θ_0. The differential equation (3.1) can also be written in the following way

$$\frac{d}{d\theta}\{(\sin\theta)^{m-2}\frac{dF}{d\theta}\}=-\sin^{m-2}\theta\, F(\theta)$$

or

$$\frac{d^2F}{d\theta^2}+(m-2)\cot\theta\frac{dF}{d\theta}+\lambda F=0. \qquad (3.2)$$

Theorem 3. The following asymptotic relations hold as $\theta\to\pi$

$$\alpha(\theta)\sim\frac{1}{2\log\frac{2}{\pi-\theta}}, \quad \text{if } m=3$$

$$\alpha(\theta)\sim\frac{\Gamma(m-2)}{\Gamma(\frac{m-1}{2})\Gamma(\frac{m-3}{2})}\{\frac{\pi-\theta}{2}\}^{m-3} \quad \text{if } m>3.$$

Proof. The differential equation (3.2) is equivalent to

$$(1-z^2)w''-(2\nu+1)zw'+\alpha(\alpha+2\nu)w=0$$

for $z=\cos\theta$, $\nu=\frac{m-2}{2}$ and $\alpha(\alpha+m-2)=\lambda$.

This means that the function F in (3.2) is a Gegenbauer function of degree α and order ν. We also observe that α is the characteristic constant of C_{θ_0}. In the standard nomenclature this Gegenbauer function is denoted by $C_\alpha^\nu(z)$. These functions can be represented in terms of the Legendre's associated functions of the first kind in the following way (see[4])

$$C_{\alpha}^{\nu}(z) = \frac{2^{\nu-(1/2)}\Gamma(\alpha+2\nu)\Gamma(\nu+(1/2))(z^2-1)^{(1/4)-(1/2)\nu}}{\Gamma(2\nu)\Gamma(\alpha+1)} P_{\alpha+\nu-(1/2)}^{(1/2)-\nu}(z) \tag{3.3}$$

We are interested in the zeros of $C_{\alpha}^{\nu}(\cos\theta)$ as a function of α. More precisely, we are interested in an asymptotic expression for α(as a zero of $C_{\alpha}^{\nu}(\cos\theta)$, in terms of θ, as θ approaches the value π. This asymptotic expression will lead us to the corresponding asymptotic expression for the characteristic constant of a spherical cap C_θ in terms of θ(as $\theta\to\pi$). Lemma 1 states these relations for the functions $P_n^{-l}(\cos\theta)$. Therefore by means of (3.3) we obtain the corresponding relations for the functions $C_{\alpha}^{\nu}(z)$. All we have to do is to write in Lemma 1 the values

$$-l = \frac{1}{2} - \nu = \frac{3-m}{2},$$

$$n = \alpha + \nu - \frac{1}{2} = \alpha + \frac{m-2}{2}.$$

Finally

$$\alpha(\theta) \sim \frac{1}{2\log\frac{2}{\pi-\theta}}\ , \quad \text{if } m=3$$

$$\alpha(\theta) \sim \frac{\Gamma(m-2)}{\Gamma(\frac{m-1}{2})\Gamma(\frac{m-3}{2})}\left(\frac{\pi-\theta}{2}\right)^{m-3} \quad \text{if } m>3.$$

<u>Remark</u>. In the case m=4 equation (3.1) is

$$\frac{d^2u}{d\theta^2} + (\lambda+1)u = 0$$

which is a Sturm-Liouville equation. The eigenfunctions of this equation are

$$u(\theta)=\sin\sqrt{\lambda+1}\,\theta.$$

The condition $u(\theta_0)=0$ implies that

$$\lambda = \frac{n^2\pi^2}{\theta_0^2} - 1, \ n=1,2,\ldots$$

so the least eigenvalue is

$$\lambda(\theta_0) = \frac{\pi^2}{\theta_0^2} - 1,$$

as was indicated in Section 1.

4. Sharpness of results.

The lower bound near $\theta=\pi$ for $m=3$:

According to Theorem 2 we have

$$\lambda(\theta) \geq \frac{1}{2\,|\log \cos\frac{\theta}{2}\,|} \tag{4.1}$$

Since $\lambda(\theta) \sim \alpha(\theta)$ for $m=3$ then the asymptotic behavior obtained in Section 3 for $\alpha(\theta)$ is the same for $\lambda(\theta)$. If we compare (4.1) with this asymptotic expression we see that this lower bound for $\lambda(\theta)$ is sharp. All one has to do is to prove that $[\log 2(\pi-\theta)^{-1}]/\,|\log \cos \frac{\theta}{2}\,| \to 1$ as θ tends to π. This is easy to do.

The lower bound near $\theta=\pi$ for $m\geq 4$.

In Theorem 2 we have the global lower bound

$$\lambda(\theta) \geq \frac{1}{\displaystyle\int_0^{\theta} \frac{dx}{\operatorname{sen}^{m-2}x} \int_0^{x} \operatorname{sen}^{m-2}t\, dt}.$$

We shall prove now that this lower bound is also sharp for $m\geq 4$. What we shall do is to compare this lower bound with the asymptotic expression that we obtained for a(j) in the case $m\geq 4$.

Since $\frac{\lambda}{\alpha} \sim m\text{-}2$ as $\theta \to \pi$, the sharpness of our lower bound comes as a consequence of the following

Lemma 3.

$$\lim_{\theta\to\pi} (\pi-\theta)^{m-3} \int_0^\theta \frac{dx}{\operatorname{sen}^{m-2}x} \int_0^x \operatorname{sen}^{m-2}t\, dt = \frac{2^{m-3}\Gamma(\frac{m-1}{2})\Gamma(\frac{m-3}{2})}{\Gamma(m-1)}.$$

Proof. We have as θ tends to π

$$(\pi-\theta)^{m-3} \int_0^\theta \frac{dx}{\operatorname{sen}^{m-2}x} \int_0^x \operatorname{sen}^{m-2}t\, dt \sim \frac{(-1)^m}{(m-3)\cos^{m-2}\theta} \int_0^\theta \operatorname{sen}^{m-2}t dt.$$

Therefore,

$$\lim_{\theta\to\pi} (\pi-\theta)^{m-3} \int_0^\theta \frac{dx}{\operatorname{sen}^{m-2}x} \int_0^x \operatorname{sen}^{m-2}t\, dt = \frac{1}{m-3} \int_0^\pi \sin^{m-2}t dt$$

$$= \frac{\sigma_m}{(m-3)\sigma_{m-1}} = \frac{\sqrt{\pi}\,\Gamma(\frac{m-1}{2})}{(m-3)\Gamma(\frac{m}{2})}.$$

On the other hand, from the duplication formula for the Gamma function we have that

$$\Gamma(\tfrac{m-1}{2})\Gamma(\tfrac{m}{2}) = \frac{\sqrt{\pi}\,\Gamma(m-1)}{2^{m-2}}.$$

Thus

$$\frac{\sqrt{\pi}}{\Gamma(\frac{m}{2})} = \frac{2^{m-2}\Gamma(\frac{m-1}{2})}{\Gamma(m-1)} = \frac{2^{m-3}(m-3)\Gamma(\frac{m-3}{2})}{\Gamma(m-1)}.$$

Then

$$\frac{\sqrt{\pi}\,\Gamma(\frac{m-1}{2})}{(m-3)\Gamma(\frac{m}{2})} = \frac{2^{m-3}\Gamma(\frac{m-1}{2})\Gamma(\frac{m-3}{2})}{\Gamma(m-1)}.$$

Thus the lemma is proved.

The upper bound near $\theta=0$ for $m=3$.

We shall show that for m=3 the upper bound obtained by Pinsky [10] may be combined with the lower bound obtained by Friedland and Hayman [7] to show that

$$\lim_{\theta\to 0} \theta^2\lambda(\theta) = j_0^2,$$

where $j_0 = 2.4...$ is the first zero of the Bessel function J_0. In fact, Hayman and Friedland have shown that

$$\alpha(\theta) \geq \frac{1}{2} j_0 \sqrt{\frac{1}{\sin^2 \theta/2} - \frac{1}{2}} - \frac{1}{2}$$

for $0<\theta<\frac{\pi}{2}$. Since $\lambda(\theta)=\alpha(\theta)(\alpha(\theta)+1)$

then

$$\lambda(\theta) \geq \left[\frac{1}{2} j_0 \sqrt{\frac{1}{\sin^2 \theta/2} - \frac{1}{2}} - \frac{1}{2}\right]\left[\frac{1}{2} j_0 \sqrt{\frac{1}{\sin^2 \theta/2} - \frac{1}{2}} + \frac{1}{2}\right].$$

Therefore,

$$\lambda(\theta) \geq \frac{1}{4} j_0^2 \left(\frac{1}{\sin^2 \frac{\theta}{2}} - \frac{1}{2}\right) - \frac{1}{4}, \qquad 0<\theta<\frac{\pi}{2}.$$

On the other hand, Pinsky obtains

$$\lambda(\theta) \leq \frac{j_0^2}{\theta^2}.$$

The last two inequalities yield

$$\lim_{\theta\to 0} \theta^2\lambda(\theta) = j_0^2.$$

The upper bound near $\theta = 0$ for $m \geq 4$

We have obtained the upper bound

$$\lambda(\theta) \le \frac{j^2_{(m-3)/2}}{\theta^2}$$

where $j_{(m-3)/2}$ is the first zero of the Bessel function $J_{(m-3)/2}$. Friedland and Hayman [7] have obtained the following lower bound for the characteristic constant $\alpha(\theta)$ of C_θ, $0<\theta<\frac{\pi}{2}$, and $m\ge 4$,

$$\alpha(\theta) \ge j_{(m-3)/2}\left[\frac{1}{(m-1)\int_0^\theta \sin^{m-2}t\,dt}\right]^{1/m-1} - \frac{2}{5}j_{(m-3)/2} - \frac{1}{2}(m-2).$$

Since $\lambda(\theta)=\alpha(\theta)(\alpha(\theta)+m-2)$ one obtains

$$\left[j^2_{(m-3)/2}\left(\frac{1}{(m-1)\int_0^\theta \sin^{m-2}t\,dt}\right)^{1/m-1} - \frac{2}{5}\right]^2 - \frac{(m-2)^2}{4} \le \lambda(\theta) \le \frac{j^2_{(m-3)/2}}{\theta^2}.$$

From these inequalities one can deduce that also for $m\ge 4$

$$\lim_{\theta\to 0} \theta^2\lambda(\theta) = j^2_{(m-3)/2}.$$

5. <u>Upper and lower bounds near $\theta=0$</u>.

In this section we shall improve the upper bound given in Theorem 2. We shall also give upper and lower bounds near zero. Since λ is explicitly given for m=4 we shall consider the cases m=3 and m≥5. The technique to do this is due to Friedland and Hayman [7].

<u>Theorem 4</u>. <u>In the case</u> $m=3$ <u>and</u> $0<\theta_0\le\frac{\pi}{2}$ <u>we have</u>

$$\frac{j_0^2}{\theta_0^2} - \left(\frac{1}{2}-\frac{1}{\pi^2}\right) \le \lambda_1 \le \frac{j_0^2}{\theta_0^2} - \frac{1}{3}.$$

<u>The upper bound holds</u> for $0<\theta_0<\pi$.

Proof. First of all we note that $(\sin\theta)^{-2} - \theta^{-2}$ increases from $\frac{1}{3}$ to $1 - \frac{4}{\pi^2}$ as θ increases from 0 to $\frac{\pi}{2}$. This can be seen from the inequality

$$\frac{d}{d\theta}\left(\frac{1}{\sin^2\theta} - \frac{1}{\theta^2}\right) > 0$$

taking into account that

$$\frac{1}{\sin^2\theta} = \sum_{n=-\infty}^{+\infty}\left(\frac{1}{\theta+n\pi}\right)^2$$

The differential equation (1.1) can also be written in the following way

$$\frac{d}{d\theta}\left\{\sin^{m-2}\theta\,\frac{dF}{d\theta}\right\} = -\lambda\,\sin^{m-2}\theta\,F(\theta)$$

or

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{(m-2)^2}{4} + \frac{(m-2)(4-m)}{4\sin^2\theta}\right\}u = 0, \qquad (5.1)$$

where $u=(\sin\theta)^{(m-2)/2}\,F(\theta)$.

The equation (5.1) for m=3 is

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{1}{4} + \frac{1}{4\sin^2\theta}\right\}u = 0. \qquad (5.2)$$

Since $(\sin\theta)^{-2} > \theta^{-2} + \frac{1}{3}$, $0<\theta<\pi$, then we can compare (5.2) with the differential equation

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{1}{3} + \frac{1}{4\theta^2}\right\}u = 0 \qquad (5.3)$$

using the comparison theorem in the Sturm-Liouville theory (see [2]). A solution of (5.3) is $v(\theta)= \theta^{1/2}J_0\left(\sqrt{\lambda_1+\frac{1}{3}}\,\theta\right)$, where J_0 is the Bessel's function of order zero. From the comparison theorem we deduce that the first zero of $J_0\left(\sqrt{\lambda_1+\frac{1}{3}}\,\theta\right)$, that is $\frac{j_0}{\sqrt{\lambda_1+\frac{1}{3}}}$ is greater or equal to θ_0.

Therefore

$$\theta_0 \leq \frac{j_0}{\sqrt{\lambda_1+\frac{1}{3}}}$$

This inequality gives the upper bound in Theorem 4. In order to obtain the lower bound we start with the inequality

$$\frac{1}{\sin^2\theta} \leq \frac{1}{\theta^2} + \left(1-\frac{4}{\pi^2}\right), \quad 0 < \theta \leq \frac{\pi}{2}\ . \tag{5.4}$$

This inequality allows us to compare (5.2) with

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{1}{4} + \frac{1}{4}\left(1 - \frac{4}{\pi^2}\right) + \frac{1}{4\theta^2}\right\}u = 0\ . \tag{5.5}$$

A solution of (5.5) is $v(\theta)= \theta^{1/2}\, J_0(C\theta)$, where $C = \left[\lambda_1+ \frac{1}{2} - \frac{1}{\pi^2}\right]^{1/2}$. From the comparison theorem we deduce that the first zero of $J_0(C\theta)$, namely j_0/C, is less than or equal to θ_0. Thus

$$\theta_0 \geq \frac{j_0}{\lambda_1 + \frac{1}{2} - \frac{1}{\pi^2}}\ .$$

This completes the proof of Theorem 4.

Theorem 5. If $m \geq 5$ and $0<\theta_0\leq\frac{\pi}{2}$ we have

$$\frac{j_k^2}{\theta_0^2} - \frac{(m-2)(m-1)}{6} \leq \lambda_1 \leq \frac{j_k^2}{\theta_0^2} - \frac{(m-2)(4m-16+2\pi^2)}{4\pi^2}\ .$$

The lower bound holds for $0<\theta_0<\pi$.

Proof. The proof follows the same lines as that of Theorem 4. Again, since $(\sin\theta)^{-2}>\theta^{-2} + \frac{1}{3}$, $0<\theta<\pi$ we can compare the equations

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{(m-2)^2}{4} + \frac{(m-2)(4-m)}{4\sin^2\theta}\right\}u = 0 \qquad (5.6)$$

and

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{(m-2)^2}{4} + \frac{(m-2)(4-m)}{12} + \frac{(m-2)(4-m)}{4\theta^2}\right\}u = 0 \qquad (5.7)$$

taking into account that a solution of (5.7) is $v(\theta) = \theta^{1/2}J_k(C\theta)$, where J_k is the Bessel's function of order $k=\frac{(m-3)}{2}$ and $C^2 = \lambda + \frac{(m-2)^2}{4} + (m-2)(4-m),12)$. The first zero of $v(\theta)$ is j_k/C. Using the comparison theorem we get the lower bound. The upper bound follows by comparing (5.6) with

$$\frac{d^2u}{d\theta^2} + \left\{\lambda + \frac{(m-2)^2}{4} + \frac{(m-2)(4-m)}{4}\left(1-\frac{4}{\pi^2}\right) + \frac{(m-2)(4-m)}{4\theta^2}\right\}u = 0$$

taking into consideration that $\frac{1}{\sin^2\theta} - \frac{1}{\theta^2} \leq 1 - \frac{4}{\pi^2}$, $0<\theta<\frac{\pi}{2}$.

Remark. Theorem 5 also holds for m=4. In this case both the upper and lower bounds are equal to $\lambda_1 = \frac{\pi^2}{\theta_0^2} - 1$.

6. Some questions.

There are a series of questions that would be interesting to answer.

(a) (Convexity of the characteristic constant). The characteristic constant α of a spherical cap C_θ defined in Section 3 can be considered as a function of θ or as a function of $S = \frac{\sigma_{m-1}}{\sigma_m}\int_0^\theta \sin^{m-2}t\,dt$, where σ_k is the surface measure of the unit ball in R^k. From the point of view of the theory of growth of subharmonic functions (see [7]) it would be nice to prove that $\alpha(S)$ is a convex function of S. If this is not the case one can ask for the region in [0,1] where α is a convex function of S.

(b) (An inequality of Friedland-Hayman-Ortiz). Friedland, Hayman and Ortiz have proved ([7],[8]) that $\alpha(S)\geq 2(1-S)$, $\frac{1}{2} \leq S \leq 1$. There is no

analytic proof of this inequality. Would it be possible to prove it using the lower bound in Theorem 2, expressing it in terms of S? We remark that the proof of the inequality $\alpha(S)\geq 2(1-S)$ is highly technical and relies to some extent on the use of computer techniques.
(c) (Upper bound for λ_1 near π). The upper bound for λ_1 given in Theorem 2 is not good as θ approaches π so it would be interesting to give an upper bound for λ_1 near π with the same order of magnitude as the lower bound given there
(d) (Harmonic measure and the first eigenvalue). To extend Tsuji's estimate for the harmonic measure [12] to the higher dimensional case and use the lower bound in Theorem 2.
(e) (Riemannian Geometry). To extend Theorem 2 to Riemannian Manifolds. An answer to this question as regard asymptotic estimates has been given in [6a].
(f) (Asymptotic paths for subharmonic functions). It would be interesting to study the connection between Theorem 2 and the recent work by Erëmenko [5].
(g) Sperner has proved [11] that among all regions of a given measure (on a sphere) the spherical cap of that measure possesses the smallest λ_1. One may ask if this result can be extended to Riemannian manifolds and geodesic balls.

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