A RELATIVE DEGREE APPROACH FOR THE CONTROL IN SLIDING MODE OF NONLINEAR SYSTEMS OF GENERAL TYPE

Hebertt Sira-Ramirez[1]
Departamento Sistemas de Control
Universidad de Los Andes, Mérida-VENEZUELA

Abstract

In this article we treat the problem of inducing local sliding regimes on nonlinear smooth surfaces defined in the state space of general nonlinear controlled systems. A suitable notion of relative degree is found to be of crucial importance in establishing the most salient properties of nonlinear systems undergoing sliding motions. The relevance of sliding modes in several control problems as complementary "outer loop" feedback is examined.

Keywords : Sliding Regimes, Discontinuous control, Relative degree of Nonlinear Systems.

Mailing (home) Address : Avenida Las Américas, Edificio Residencias "El Roble" 6º Piso, Apt. D-6, Mérida 5101. VENEZUELA.

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[1] This work was supported by the Consejo de Desarrollo Científico, Humanístico y Tecnológico of the Universidad de Los Andes (Mérida-VENEZUELA) under Research Contract No. I-280-87.

I. INTRODUCTION

The notion of structure at infinity of dynamical controlled systems plays a fundamental role in the understanding of nonlinear controlled dynamics. Thus far, this concept has allowed the evolution, into a nonlinear setting, of many basic, and long standing, automatic control problems. Among these problems we find: local stabilization, feedback linearization, disturbance decoupling, interaction decoupling, systems invertibility and nonlinear adaptive control (See Byrnes and Isidori 1984, Isidori 1985a and the excellent introductory material in Isidori 1987, which is closely followed in this work).

In this article, we examine the relevance of the notion of the structure at infinity (or relative degree) and of its associated state coordinates transformation into normal form coordinates, in general single-input single-output nonlinear variable structure controlled systems operating in sliding mode (Utkin, 1978).

It is found that a sliding regime locally exists on the zero level set of the output function, if and only if the nonlinear system has relative degree equal to 1 (i.e., it exhibits the simplest structure at infinity). The corresponding n-1 dimensional zero dynamics precisely portrays the qualitative features of the ideal sliding dynamics (Utkin,1978) in local surface coordinates. The problem of inducing sliding regimes on systems with relative degree higher than one is also examined. The implications of the relative degree concept in sliding mode disturbance decoupling, variable structure control of feedback linearizable systems and model matching problems, via sliding modes, are also analyzed.

In section II we present background material about a generalization of the relative degree concept, normal forms and zero dynamics. New results on sliding motions, for general nonlinear systems, are also presented in that section. Section III presents applications of sliding modes in control areas such as: Disturbance decoupling, Feedback linearization and Model Matching. Section IV contains the conclusions and suggestions for futher work in this area.

II. BACKGROUND AND MAIN RESULTS

2.1 Relative Degree, Normal Forms and Zero Dynamics

Consider the nonlinear smooth system of the form:

$$dx/dt = X(x,u)$$
$$y = h(x) \quad (2.1)$$

locally defined for all $x \in O$, an open set in R^n, $u : O \to R$, is a (possibly discontinuous) scalar feedback input function, while, for each fixed smooth feedback control $u(x)$, X represents a locally smooth controlled vector field defined on O. The output function $h: O \to R$ is a locally smooth scalar function of the state. We often refer to (2.1) as the pair (X,h).

The level set $h^{-1}(0) = \{ x \in O : h(x) = 0 \}$, locally defines a smooth n-1 dimensional locally regular manifold of constant rank (i.e., an integrable manifold. See Boothby, 1975), addressed as the sliding manifold. The gradient of h(x), denoted by dh, is locally assumed to be nonzero on $h^{-1}(0)$ except, possibly, on a set of meaure zero. $h^{-1}(0)$ is oriented in such a way that dh locally points from the region where $h(x) < 0$ towards that where $h(x) > 0$.

We shall refer to a property as local around x^o, whenever it is valid on an open vicinity N of a given point $x^o \in O$, with $O \supset N$. If the point is located on $h^{-1}(0)$ we say the porperty is valid locally on $h^{-1}(0)$ if it is valid on an open set M of the submanifold $h^{-1}(0)$ (i.e., on an open subset of $h^{-1}(0) \cap N$).

The Lie derivative of a scalar function $\phi(x)$, with respect to a smooth vector field X, locally defined on O, is denoted by $L_X\phi$. One recursively defines, for any positive integer k, : $L^k_X\phi(x) = L_X[L^{k-1}_X\phi(x)]$.

<u>Definition 2.1</u> The pair (X,h) has, locally around x^o , a <u>zero at infinity of multiplicity r</u> if

$$L_{\partial X/\partial u}L^k{}_X h(x) = 0 \quad \text{for all x in N, and all } k < r-1$$
$$L_{\partial X/\partial u}L^{r-1}{}_X h(x^o) \neq 0 \qquad (2.2)$$

The integer r is also called the local <u>relative degree</u> of (X,h) at x^o. ■

<u>Example</u> Consider the nonlinear system (2.1) with $X(x,u) = f(x) + g(x)\,u$. Then $\partial X/\partial u = g(x)$. Suppose the system has local relative degree r at x^o. Using the definition 2.1 we compute, for any x in N:

$L_{\partial X/\partial u}h(x) = L_g h = 0$; $L_{\partial X/\partial u}L_X h(x) = L_g[L_{f+gu}h(x)] = L_gL_fh(x) + uL_g{}^2h(x) = L_gL_fh(x) = 0$; $L_{\partial X/\partial u}L^2{}_Xh(x) = L_gL_{f+gu}[L_fh(x)+uL_gh(x)] = L_g[L^2{}_fh(x) + uL_gL_fh(x) + uL_fL_gh(x) + u^2L^2{}_gh(x)\,] = L_gL^2{}_fh(x) = 0$ ... Generally, one obtains, $L_{\partial X/\partial u}L^k{}_Xh(x) = L_gL^k{}_fh(x) = 0$, for all $k < r-1$ and all x in N. Finally, $L_{\partial X/\partial u}L^{r-1}{}_Xh(x^o) = L_gL^{r-1}{}_fh(x^o) \neq 0$. Definition 2.1 thus generalizes the usual definition of local relative degree (See Byrnes and Isidori, 1984). ■

<u>Remark</u> The relative degree of (2.1) is interpreted as the minimum number of times one has to differentiate y, with respect to time, in order to have the derivative of y depend explicitly on u. Notice that if $y^{(k)}$ ($0 \le k < r$) is independent of u, then $\partial y^{(k)}/\partial u = 0$. Since $y^{(k)} = L^k{}_Xh(x) = L_X[L^{k-1}{}_Xh(x)] = \{\partial[L^{k-1}{}_Xh(x)]/\partial x\}X(x,u) = \{\partial y^{(k-1)}/\partial x\}X(x,u)$ and since $y^{(k-1)}$ is assumed to be independent of u, it then follows that, $\partial[L^k{}_Xh(x)]/\partial u = \partial(\{\partial[L^{k-1}{}_Xh(x)]/\partial x\}X(x,u))/\partial u = \{\partial[L^{k-1}{}_Xh(x)]/\partial x\}\,\partial X/\partial u = L_{\partial X/\partial u}[L^{k-1}{}_Xh(x)] = 0$. If $y^{(r)}$ is the first time derivative that explicitly depends on u then, in general, $\partial y^{(r)}/\partial u = L_{\partial X/\partial u}[L^{r-1}{}_Xh(x)] \neq 0$ (i.e., not identically zero). This completely justifies our definition of relative degree. ■

<u>Proposition 2.2</u> Let (2.1) have local relative degree r on x^o. Set $\phi_i(x) = L^{i-1}{}_X h(x)$ for i = 1,2,...,r, while the functions $\phi_{r+j}(x)$, j = 1,2,....,n-r, are chosen to be <u>functionally independent</u> of the first r functions, with the only additional requirement that, locally around x^o , $L_{\partial X/\partial u}\phi_{r+j} = 0$ for all j's. Define new z coordinates as $z = \Phi(x)$ with $\Phi(x) = col[\phi_1(x),...,\phi_n(x)]$ being a local diffeomorphism on N. The system (2.1) is locally expressed, around $z^o = \Phi(x^o)$ as:

$$
\begin{aligned}
&dz_i/dt = z_{i+1} \qquad i = 1,2,....,r-1 \\
&dz_r/dt = [L^r{}_X h](\Phi^{-1}(z), u) \\
&dz_{r+j}/dt = q_j(z) \qquad j = 1,2,..., n-r. \\
&y = z_1
\end{aligned}
\tag{2.3}
$$

<u>Proof</u> Obvious from the choice of coordinates (See also Isidori, 1987). ∎

System (2.3) is said to be in local <u>normal form</u> coordinates. A block diagram depicting the structure of the system (2.3) is shown in Figure 1.

<u>Remark</u> It is easy to see, from the definition of the normal coordinates, that if initial conditions of (2.1) are set on M (i.e., on $z_1 = 0$), then the components $z_1,...,z_r$ of the normal coordinate vector z are all zero. Hence, any point x^o on M is expressed, in normal coordinates, as : $(0,\eta)$ where $\eta = col(z_{r+1}, ..., z_n)$. If, furthermore, a feedback control $u = \alpha(z)$ is used such that $[L^r{}_X h](\Phi^{-1}(z),u(z)) = 0$ locally around z^o, the evolution of the controlled system locally remains on $h^{-1}(0)$. (Notice that from the definition of local relative degree and the implicit function theorem (Isidori, 1985a), such an $\alpha(z)$ locally exists around z^o and it is uniquely defined). Ú

Definition 2.3 The description, in normal coordinates, of (2.1) with inital conditions prescribed on an open set M of $h^{-1}(0)$ and feedback control input $\alpha(z)$ such that $dz_r/dt = [L^r{}_X h](\Phi^{-1}(z),\alpha(z)) = 0$, locally on M ,

$$d\eta/dt = q(0,\eta) = q_o(\eta) \tag{2.4}$$

is addressed as the zero dynamics. ■

The qualitative behavior of the zero dynamics is entirely governed by $q_o(\eta)$. The system is said to be minimum phase if the dimension of the stable manifold (See Guckenheimer and Holmes, 1983), around an equilibrium point in $h^{-1}(0)$, is n-r. The system is globally minimum phase if it is minimum phase and (2.4) is globally asymptotically stable (Byrnes and Isidori, 1984). From now on, we assume that $y = h(x)$ has been chosen to render the system globally minimum phase, i.e., the internal behavior of the system, while being forced to locally exhibit zero output value, is asymptotically stable to an equilibrium point located on the manifold $h^{-1}(0)$.

Lemma 2.4. The relative degree of a system is feedback invariant.

Proof For a system with feedback $u = u(x,v)$, the time derivatives of y are locally independent of u if and only if they are locally independent of v. ■

Remark Lemma 2.4 implies, in particular, that for a nonlinear system, with local relative degree r around x^o, given by : $dx/dt = X(x,u)$; $y = h(x)$, and feedback control law, $u = \alpha(x,v)$, which in closed loop form is written as: $dx/dt = X(x,\alpha(x,v)) = X^{\alpha}(x,v)$, the equality: $L^k{}_{X^\alpha} h = L^k{}_X h$, holds valid for $k = 0,1,2,\ldots,r-1$ and all x in a neighborhood of x^o.

2.2 Generalities about local sliding regimes

A local variable structure feedback control law for (X,h) is obtained by letting the control function u take one of two possible feedback function values in the set $U = \{ u^+(x) , u^-(x) \}$, with $u^+(x) > u^-(x)$ locally defined on a neighborhood N of x^o, according to the sign of the scalar output function h(x). i.e.,

$$u = \begin{cases} u^+(x) & \text{for } h(x) > 0 \\ u^-(x) & \text{for } h(x) < 0 \end{cases} \tag{2.5}$$

The feedback structures $u^+(x)$ and $u^-(x)$ are usually fixed beforehand, but they may also be part of the design problem.

Definition 2.5. (Utkin, 1978. Sira-Ramirez, 1988a) A sliding regime is said to locally exist on an open set M of $h^{-1}(0)$, if, as a result of the control policy (2.5), the state trajectories of (2.1) satisfy :

$$\lim_{h \to 0^+} L_{X(x,u^+(x))}h = \lim_{h \to 0^+} < dh, X(x,u^+(x)) > < 0 ;$$
$$\lim_{h \to 0^-} L_{X(x,u^-(x))}h = \lim_{h \to 0^-} < dh, X(x,u^-(x)) > > 0. \tag{2.6}$$ ■

Theorem 2.6 A sliding regime locally exists on an open set M of $h^{-1}(0)$, if and only if the system (X,h), has local relative degree equal to 1 (i.e., (X,h) has one zero at infinity on a point $x^o \in M$).

Proof If $L_{X(x,u)}h$ does not depend locally on u (i.e., $L_{\partial X/\partial u}h(x) = \partial\{L_X h\}/\partial u = 0$ for all x in N) then, changing the control u from $u^+(x)$ to $u^-(x)$, in the vicinity N of x^o, does not have any effect on the local sign of $L_{X(x,u)}h$. Therefore, a sliding regime can not locally exist on M .

To proof sufficiency, suppose $L_{\partial X/\partial u}h(x) = \partial\{L_X h\}/\partial u \neq 0$ locally around a neighborhood N of x^o. Let $\varepsilon^-(x)$ be a smooth, locally strictly positive function of x. Then, by virtue of the implicit function theorem, the equation $[L_X h](x,u) = \varepsilon^-(x)$ locally has a unique smooth solution $u = u^{-\varepsilon}(x)$ such that $L_{X(x,u^{-\varepsilon}(x))}h(x) = \varepsilon^-(x) > 0$. Similarly, by the same arguments, given a smooth locally strictly negative function $\varepsilon^{+}(x)$, a smooth control law $u = u_o{}^{+\varepsilon}(x)$ locally exists around x^o such that $L_{X(x,u^{+\varepsilon}(x))}h(x) = \varepsilon^{+}(x) < 0$. Hence, a sliding regime locally exists on an open set M of $h^{-1}(0)$ for the found variable structure feedback control law :

$$u = \begin{cases} u^{+}(x) = u^{-\varepsilon}(x) & \text{for} \quad h(x) < 0 \\ u^{-}(x) = u^{+\varepsilon}(x) & \text{for} \quad h(x) > 0 \end{cases}$$

■

Condition $L_{\partial X/\partial u}h \neq 0$, is a generalized local transversality condition (Sira-Ramírez 1988a).

Example: For systems of the form $X(x,u) = f(x) + g(x)\,u$; $y = h(x)$, the local transversality condition on an open set M in $h^{-1}(0)$ takes the form $L_g h < 0$. To see this, simply subtract the sliding regime conditions (2.6) on any point x of M : $L_f h + u^{+}L_g h < 0$ and $L_f h + u^{-} L_g h > 0$, to obtain: $[u^{+}-u^{-}]L_g h < 0$. ■

For all initial states located on a vicinity M of x^o in $h^{-1}(0)$, the unique control function, $u^{EQ}(x)$, locally constraining the system trajectories to the zero level set of $h(x)$, in the region of existence of a sliding regime, is known as the equivalent control. (i.e., the equivalent control locally turns the open set M in $h^{-1}(0)$, into an integral manifold for the controlled system trajectories starting on M). The resulting dynamics, ideally constrained to M, is the ideal sliding dynamics (Utkin, 1978). A coordinate free description of such dynamics is :

$$dx/dt = X(x,u^{EQ}(x)) \qquad (2.7)$$

A necessary and sufficient condition for an open set M of $h^{-1}(0)$ to be a local integral manifold of the controlled trajectories is that the gradient of h be locally orthogonal to the controlled vector field $X(x,u^{EQ}(x))$, i.e.,

$$L_{X(x,u^{EQ}(x))}h(x) = \langle dh, X(x,u^{EQ}(x)) \rangle = 0 \tag{2.8}$$

Theorem 2.7 A necessary condition for the local existence of a sliding regime of (X,h) on an open set M of $h^{-1}(0)$ is that the equivalent control locally exists and is uniquely defined on M.

Proof: If a sliding motion locally exists on M, then (X,h) has local relative degree 1, i.e., $L_{\partial X/\partial u}h(x) \neq 0$ for all x in a vicinity N of x^o. By the implicit function theorem the equation, $[L_X h](x,u) = 0$, has, locally, a unique solution $u^{EQ}(x)$ in M for which (2.8) holds valid. In other words, if a sliding regime exists, the equivalent control locally exists and is uniquely defined. ■

Example Let $X(x,u) = f(x) + ug(x)$ such that locally, on an open set M of $h^{-1}(0)$, $L_g h < 0$. $L_{X(x,u^{EQ})}h = L_f h + u^{EQ} L_g h = 0$ implies $u_{EQ} = - L_f h/L_g h$ i.e., the equivalent control locally exists and is uniquely defined. ■

Theorem 2.8 A necessary condition for the local existence of a sliding regime on an open set M of $h^{-1}(0)$ is that there locally exists a smooth equivalent control, satisfying:

$$u^-(x) < u^{EQ}(x) < u^+(x) \tag{2.9}$$

for the given smooth feedback functions $u^-(x)$ and $u^+(x)$.

Proof: Suppose a sliding regime locally exists M for the switching feedback control law (2.5). Then, locally on M, the following three relations hold valid:

$$L_{X(x,u^+(x))}h = \langle dh, X(x,u^+(x)) \rangle < 0 \tag{2.10}$$

$$L_{X(x,u^{EQ}(x))}h = \langle dh, X(x,u^{EQ}(x)) \rangle = 0 \tag{2.11}$$

$$L_{X(x,u^-(x))}h = \langle dh, X(x,u^-(x)) \rangle > 0 \, . \tag{2.12}$$

Subtracting (2.11) from (2.10) and (2.12) from (2.11) one obtains :

$$< dh\ ,\ X(x,u^{+}(x)) - X(x,u^{EQ}(x)) > \ < 0$$
$$< dh\ ,\ X(x,u^{EQ}(x)) - X(x,u^{-}(x)) > \ < 0 \qquad (2.13)$$

From the mean value theorem (Boothby, 1975), there exists smooth functions $u^{+}_{o}(x)$ and $u^{-}_{o}(x)$, such that locally on M :

$$<dh, X(x,u^{+}(x)) - X(x,u^{EQ}(x))> = [u^{+}(x) - u^{EQ}(x)] < dh,\ \partial X(x,u^{+}_{o}(x))/\partial u > \ < 0$$
$$<dh, X(x,u^{EQ}(x)) - X(x,u^{-}(x))> = [u^{EQ}(x) - u^{-}(x)] < dh,\ \partial X(x,u^{-}_{o}(x))/\partial u > \ < 0 \qquad (2.14)$$

where $u^{+}_{o}(x)$ and $u^{-}_{o}(x)$, respectively, satisfy : $u^{EQ}(x) < u^{+}_{o}(x) < u^{+}(x)$ and $u^{-}(x) < u^{-}_{o}(x) < u^{EQ}(x)$, i.e., locally on M, $u^{-}(x) < u^{EQ}(x) < u^{+}(x)$. ■

<u>Example</u> For the linear in the control case, if a local sliding motion exists on an open set M of $h^{-1}(0)$, we locally have on M: $L_f h + u^{+} L_g h < 0$ and $L_f h + u^{-} L_g h > 0$. This implies that there exists smooth functions $a(x) > 0$ and $b(x) > 0$, such that: $a(x)[L_f h + u^{+} L_g h] + b(x)[L_f h + u^{-} L_g h] = [a(x)+b(x)]\ L_f h + [a(x)u^{+}(x)+b(x)u^{-}(x)]g(x) = 0$.i.e., $L_f h + L_g h\ [a(x)u^{+}(x)+b(x)u^{-}(x)]/[a(x)+b(x)] = 0$. i.e., $u^{-}(x) < [a(x)u^{+}(x)+ b(x)u^{-}(x)]/[a(x)+b(x)] = u^{EQ}(x) < u^{+}(x)$ locally on M. Ú

<u>Definition 2.9</u> Let (X,h) have local relative degree 1 on $x^{o} \in h^{-1}(0)$. The system (X,h) is said to exhibit a local <u>control foliation property</u> about the manifold $h^{-1}(0)$, if and only if given any smooth feedback funtions $u_1(x) < u_2(x) < u_3(x)$, defined on M, $[L_X h]\ (x,u_1) > [L_X h]\ (x,u_2) > [L_X h]\ (x,u_3)$, locally on M. (See also Sira-Ramirez, 1989) ■

<u>Theorem 2.10</u> Control law (2.5) locally induces a sliding regime, for system (X,h), on an open set M of $h^{-1}(0)$, if and only if (X,h) exhibits a control foliation property about $h^{-1}(0)$ and there exists a feedback control, $u^{EQ}(x)$, locally satisfying (2.8) and (2.9) on M.

Proof Suppose that, thanks to a control action of the form (2.5), the system locally exhibits a sliding regime on an open set M of $h^{-1}(0)$. Then, according to theorems 2.7 and 2.8, there necessarily exists a unique smooth $u^{EQ}(x)$ satisfying $u^-(x) < u^{EQ}(x) < u^+(x)$ locally on M. Since a sliding motion exists on M, it follows that: $[L_X h](x,u^-(x)) > 0$, $[L_X h](x,u^{EQ}(x)) = 0$ and $[L_X h](x,u^+(x)) < 0$. locally on M. i.e., $[L_X h](x,u^-(x)) > [L_X h](x,u^{EQ}(x)) > [L_X h](x,u^+(x))$ on M. Hence, (X,h) exhibits a control foliation property.

If, on the other hand the system (X,h) exhibits a control foliation property and $u^{EQ}(x)$ exists such that locally on an open set M in $h^{-1}(0)$: $[L_X h](x,u^{EQ}(x)) = 0$ and $u^+(x) > u^{EQ}(x) > u^-(x)$. Then, it follows that $[L_X h](x,u^-(x)) > [L_X h](x,u^{EQ}(x)) = 0 > [L_X h](x,u^-(x))$. Hence, necessarily, $[L_X h](x,u^+(x)) < 0$ and $[L_X h](x,u^-(x)) > 0$ holds true on M. It follows that there exists an open neighborhood N of $x^o \in M$, with nonempty intersection with $h^{-1}(0)$, where conditions (2.6) are satisfied. Thus, a sliding regime locally exists on $h^{-1}(0)$. ■

Example Notice that for $X(x,u) = f(x) + ug(x)$, and if the transversality condition $L_g h < 0$ holds locally true on an open set M of $h^{-1}(0)$, then the control foliation property is automatically satisfied. Since $L_X h = L_f h + uL_g h$, then for $u^+ > u^{EQ} > u^-$, $L_f h + u^+ L_g h < L_f h + u^{EQ} L_g h = 0 < L_f h + u^- L_g h$. It follows that, for the class of affine systems, the conditon $u^- < u^{EQ} < u^+$ is both a necessary and sufficient condition for the existence of a sliding regime (See Sira-Ramirez, 1988b). ■

Example Consider the nonlinear system:

$$dx_1/dt = \cos(ux_2) - x_1^2 \quad =: \quad X_1(x,u)$$

$$dx_2/dt = \sin(ux_1) \quad =: \quad X_2(x,u)$$

$$y = x_2 \quad =: \quad h(x)$$

In this case, the vector field $\partial X/\partial u = -x_2\sin(ux_2)\partial/\partial x_1 + x_1\cos(ux_1)\partial/\partial x_2$. Since $L_{\partial X/\partial u}\, h = x_1\cos(ux_1)$, the local relative degree of the system, with respect to the output $y = x_2$, is equal to 1 everywhere except on the line $x_1 = 0$. Thus, a local sliding motion exists on the manifold $x_2 = 0$ by use of an appropriate variable structure control law. Indeed, from $L_X\, h = \sin(ux_1)$, it is seen that the feedback control law: $u = -x_1 \text{sign}\, x_2$, locally creates a sliding regime on $x_2 = 0$, $0 < x_1 < \sqrt{\pi}$. The ideal sliding dynamics is obtained for the control law satisfying $L_X h = \sin(u^{EQ}(x)\ x_1) = 0$ on $x_2 = 0$. i.e., $u^{EQ}(x) = 0$. It follows that: $dx_1/dt = 1-x_1^2$, locally describes the ideal sliding motion. Figure 2 depicts the controlled phase trajectories. ■

2.3 Sliding Regimes in Variable Structure Systems with relative degree higher than one.

If, for the proposed output function $y = h(x)$, the system locally exhibits relative degree r, higher than 1, on x^o, then, an alternative to create a local sliding motion, which eventually reaches $h^{-1}(0)$, is to use the auxiliary output function (see Isidori (1987), for related ideas in local feedback stabilization):

$$w = k(x) = L^{r-1}{}_X h(x) + c_{r-2}L^{r-2}{}_X h(x) + \ldots + c_1 L_X h(x) + c_o h(x) \qquad (2.15)$$

or, in normal form coordinates:

$$w = z_r + c_{r-2}z_{r-1} + \ldots + c_1 z_2 + c_o z_1 \qquad (2.16)$$

Evidently, $L_{\partial X/\partial u}k(x^o) = L_{\partial X/\partial u}L^{r-1}{}_X h(x^o) \neq 0$. i.e., the system (X,k) has local relative degree 1, and a sliding motion can now be locally created on an open set of $k^{-1}(0)$. Then, ideally, $w = 0$ and $z_r = -c_{r-2}z_{r-1} - \ldots - c_1 z_2 - c_o z_1$. The ideal sliding system, associated with the new sliding surface $k^{-1}(0)$, is expressed as:

$$dz_i/dt = z_{i+1} \quad ; \quad i = 1,2,\dots,r-2$$

$$dz_{r-1}/dt = z_r = -c_{r-2}z_{r-1} - \dots - c_1z_2 - c_oz_1$$

$$dz_{r+j}/dt = q(z_1,z_2,\dots, z_{r-1},-(c_{r-2}z_{r-1} + \dots + c_1z_2 + c_oz_1),z_{r+1},\dots,z_n)$$

$$j = 1,\dots,n-r$$

$$y = z_1$$

$$w = 0 \qquad (2.17)$$

It is easy to see that by suitable choice of the parameters $c_o,\dots,c_{r-2}$, an asymptotically stable motion can be obtained for the first r-1 coordinates, z_1 through z_{r-1} (and hence, for z_r too). Thus, while a sliding motion locally takes place on $k^{-1}(0)$, the original output y and its first r-1 derivatives asymptotically approach zero (i.e., the state vector of the original system approaches the manifold $h^{-1}(0)$, as originally desired).

The corresponding equivalent control is now locally given, in original coordinates, as the unique solution of :

$$[L_Xk](x,u^{EQ}(x)) = [L^r{}_Xh + c_{r-2}L^{r-1}{}_Xh + \dots + c_1L^2{}_Xh+ c_oL_Xh](x,u^{EQ}(x)) = 0 \qquad (2.18)$$

Notice that when $h^{-1}(0)$ is reached by the sliding controlled trajectory, the equivalent control locally becomes the unique solution of :

$$[L_Xk](x,u^{EQ}(x)) = [L^r{}_Xh](x,u^{EQ}(x)) = 0 \qquad (2.19)$$

<u>Example</u> : For systems with affine vector fields: $X = f + u\, g$ and output $y = h(x)$ which exhibit relative degree r on x^o, the function $w = k(x)$ is given by $k(x)= L^{r-1}{}_fh(x) + c_{r-2}L^{r-2}{}_fh(x) + \dots + c_1L_fh(x)+ c_oh(x)$ and the equivalent control (2.18) is computed as :

$$u^{EQ}(x) = - L_fk/L_gk = [L^r{}_fh + c_{r-2}L^{r-1}{}_fh + \dots + c_1L^2{}_fh + c_oL_fh]/L_gL^{r-1}{}_fh$$

Locally, on an open set M of $h^{-1}(0)$, this expression takes the form: $u^{EQ}(x) = L^r{}_fh /L_gL^{r-1}{}_fh$. ■

The use of the auxiliary output $w = k(x)$ implies the possibility of either being able to completely measuring the original state variables, and proceed to use (2.15), or else being able to generating r-1 derivatives of the original output function y. The last possibility is usually accomplished by means of a high gain, phase lead "post-processor" (Isidori,1987) fed by the output signal y(t). The transfer function of such a post-processor is given by :

$$\frac{-K\, n(s)}{(1+ Ts)^{r-1}} \tag{2.20}$$

with T being a sufficiently small positive constant, K a sufficiently large gain with, locally, the same sign as $L_{\partial X/\partial u} L^{r-1}{}_X h(x)$. n(s) is a stable polynomial built as: $n(s) = s^{r-1} + c_{r-2}s^{r-2} + \ldots + c_1 s + c_o$. (See Figure 3).

Example A simplified model of a spacecraft atempting a soft lunar landing is given by (Cantoni and Finzi, 1980)

$$dx_1/dt = x_2 \;;\quad dx_2/dt = g - (\sigma/x_3)\; u \quad;\quad dx_3/dt = -u$$

where x_1 is the position coordinate, oriented downwards with origin on the ground, x_2 is the downward velocity and x_3 representes the combined mass of the spacecraft and the residual fuel. σ is a constant of relative ejection velocity. The control parameter u , represents the rate of ejection per unit time and it is assumed to take values on the discrete set $\{0, \alpha\}$ with α a given constant such that $\sigma\alpha$ is the maximum thrust of the braking engine. Consider the output to be $y = h(x) = x_1$.

In this example, $X = x_2\partial/\partial x_1 + [g - (\sigma/x_3)\; u]\partial/\partial x_2 - u\; \partial/\partial x_3$ and $L_X h = x_2$ while $\partial X/\partial u = -(\sigma/x_3)\partial/\partial x_2 - \partial/\partial x_3$. Hence, $L_{\partial X/\partial u}h = 0$. $L^2{}_X h = g - (\sigma/x_3)\; u$ and $L_{\partial X/\partial u}L_X h = -(\sigma/x_3) \neq 0$ i.e., the system has relative degree 2. Consider then the auxiliary output $w = k(x) = x_2 + cx_1$, with $c > 0$. Then $L_X k = cx_2 + g - (\sigma/x_3)\, u$, and $L_{\partial X/\partial u} k = -(\sigma/x_3)$. A local sliding regime can now be created on $k^{-1}(0) := \{ x \in R^3 : x_2 = -cx_1 \}$. The equivalent control is found to be $u^{EQ} = (x_3/\sigma)[\, cx_2 + g]$. A local sliding motion exists on $k^{-1}(0)$ provided $0 < u^{EQ} = (x_3/\sigma)[cx_2 + g] = (x_3/\sigma)[-c^2 x_1 + g] < \alpha$. The first inequality is obviously satisfied since $x_1 < 0$ before landing, and the second inequality states that the net average descending force $x_3[g - c^2 x_1]$, along the sliding line, is to be bounded by the maximum thrust $\sigma\alpha$. Notice that since $x_2 = 0$ when $x_1 = 0$ on the sliding surface, the position coordinate of the ideal sliding dynamics, regulated by the asymptotically stable dynamics : $dx_1/dt = - c\, x_1$, guarantees a soft lunar landing. On the sliding line the mass evolution is governed by : $dx_3/dt = -(x_3/\sigma)[g - c^2 x_1]$. ■

III. AN "OUTER LOOP" SLIDING MODE CONTROL APPROACH TO SOME NONLINEAR CONTROL PROBLEMS

3.1 Robust Stabilization in Feedback Linearizable Systems

Suppose the local relative degree of the system (X,h) is n at x^o, with n being the dimension of the system state. It follows that, in normal form coordinates, the system may be locally expressed around z^o as:

$$dz_i/dt = z_{i+1} \; ; \; i = 1, 2, \ldots, n-1$$

$$dz_n/dt = [L^n{}_X h](\Phi^{-1}(z), u) \tag{3.1}$$

$$y = z_1$$

where, by definition of relative degee, $\partial\{[L^n{}_X h](\Phi^{-1}(z^o), u)\}/\partial u = L_{\partial X/\partial u} L^{n-1}{}_X h \neq 0$. It follows, from the implicit funciton theorem, that given any external independent scalar input function v, the equation : $[L^n{}_X h](\Phi^{-1}(z), u) = v$

locally has a unique smooth solution $u = \alpha(z,v)$ around z^o. Hence, using such a control law on (3.1), one obtains the linear controllable system:

$$dz_i = z_{i+1} \quad ; \; i = 1,2,\ldots,n-1$$
$$dz_n/dt = v \tag{3.2}$$
$$y = z_1$$

If one defines a new auxiliary linear output in new coordinates as :

$$w = k(z) = z_n + c_{n-2}z_{n-1} + \ldots + c_1z_2 + c_oz_1 \tag{3.3}$$

or in original coordinates as :

$$w = k(x) = L^{n-1}{}_Xh(x) + c_{n-2}L^{n-2}{}_Xh(x) + \ldots + c_1L_Xh(x) + c_oh(x). \tag{3.4}$$

then, the relative degree of (3.1) with respect to w is 1, as it is easily checked. Hence, a sliding motion can be created on an open set of $k^{-1}(0)$ by a suitable choice of smooth (possibly linear) "outer loop " feedback structures $v^+(z)$ and $v^-(z)$ such that, on an open set of $k^{-1}(0)$, $v^-(z) < v^{EQ}(z) < v^+(z)$, with :

$$v^{EQ}(z) = (c_{n-2}{}^2 - c_{n-3})\; z_{n-1} + (c_{n-2}c_{n-3} - c_{n-4})z_{n-2} + \ldots + (c_{n-2}c_1 - c_o)z_2 + c_{n-1}c_o \tag{3.5}$$

If such a sliding regime is created on $k^{-1}(0)$, w is ideally set to zero and hence, $z_n = -c_{n-2}z_{n-1} - \ldots - c_1z_2 - c_oz_1$. The ideal sliding dynamics is governed by :

$$dz_i/dt = z_{i+1} \quad ; \quad i = 1,2,\ldots, n-2$$
$$dz_{n-1}/dt = -c_{n-2}z_{n-1} - \ldots - c_1z_2 - c_oz_1 \tag{3.6}$$
$$y = z_1$$
$$w = 0$$

Evidently, system (3.6) can be made locally asymptotically stable by suitable choice of the design parameters c_i's. The result is the possibility of locally reaching the original surface $h^{-1}(0)$ while a sliding motion locally takes place on an open set of $k^{-1}(0)$. The variable structure control policy provides some degree of robustness to the exactly feedback linearized dynamics. (See Spong and Sira-Ramirez, 1986).

3.2 Disturbance Rejection Properties of Systems undergoing Sliding Regimes.

Consider a smooth nonlinear perturbed system of the form:

$$dx/dt = X(x,u,w)$$
$$y = h(x) \qquad (3.7)$$

where w is a scalar perturbation signal affecting the system behavior. Let us assume that the system locally has relative degree 1 on x^o, and that $\partial(L_X h)/\partial w = L_{\partial X/\partial w}h(x) = 0$. (i.e., the intput w is assumed to have local relative degree higher than 1 with respect to the output function h). In normal form coordinates, the perturbed system is written as :

$$dz_1/dt = [L_X h]\ (\Phi^{-1}(z),u,w) = [L_X h]\ (\Phi^{-1}(z),u)$$
$$d\eta/dt = q(z_1,\eta,w) \qquad (3.8)$$
$$y = z_1$$

If a sliding motion can be locally created on $z_1 = 0$, the equivalent control $u^{EQ}(z)$, obtained by zeroing the first equation of (3.8), is clearly unaffected by the perturbation signal w. Only the ideal sliding dynamics is influenced by the perturbation input w. The following lemma follows immediately.

Lemma 3.1 : Let (X,h) have local relative degree 1. The existence of a local sliding motion on $h^{-1}(0)$ is independent of the perturbation signal w if and only if w has local relative degree, at least, equal to 2, with respect to the output function h(x). ∎

However, notice that w does, in general, affect the evolution of the ideal sliding dynamics (zero dynamics), unless the normal form coordinates $z_2 = \phi_2(x), \ldots, z_n = \phi_n(x)$, are chosen in such a way that $L_{\partial X/\partial w}\phi_i = 0$ for $i=2,\ldots,n$. But, due to the fact that the normal form description demanded that the ϕ_i's were chosen to also satisfy: $L_{\partial X/\partial u}\, \phi_i = 0$, for $i = 2,\ldots,n$, it follows that $\partial X/\partial w$ is, at least locally, exactly in the range of $\partial X/\partial u$. On the other hand, $\partial X/\partial w$ is locally in the range of $\partial X/\partial u$ if and only if there exists a smooth function $b(x)$ such that locally on M , $\partial X/\partial w = b(x)[\partial X/\partial u]$. Hence, for $i = 2,3,\ldots,n$ we have $L_{\partial X/\partial w}\phi_i = L_{[b(x)\partial X/\partial u]}\phi_i = b(x) \cdot L_{\partial X/\partial u}\, \phi_i = 0$. It follows that w does not affect the zero dynamics. We have thus proved the following general theorem.

Theorem 3.2 System (3.7) is totally unaffected by perturbation signals w, of any kind, if and only if the matching condition :

$$\partial X/\partial w \in \text{range}\ \{\ \partial X/\partial u\ \} \qquad (3.9)$$

is satisfied. ∎

Remark For the case of affine vector fields of the form $X(x,u,w) = f(x) + g(x)u + p(x)w$. Condition (3.9) is equivalent to : $p(x) \in \text{range}\ g(x)$ which is a well known "invariance condition" (Sira-Ramirez, 1988a,1988c, See also Drazenovic, 1969, for the linear time-invariant case). ∎

Notice however that even in the case of a perturbation signal with relative degree equal to 1, it is still possible to create a local sliding motion on the zero level set of the output function. For this, bounds are to be known for the perturbation signal. In genral, the extreme values of the variable structure control law u^+, u^- will depend on the bounds of the perturbation signal. Usually, however, the feedback functions $u^+(x)$, $u^-(x)$ are fixed at the outset. In this case the following theorem applies.

Theorem 3.3 Let the system $dx/dt = X(x,u,w)$ and $y = h(x)$ have local relative degree 1 both in u and w, and let the system exhibit a local control foliation property on $h^{-1}(0)$. Suppose the scalar perturbation w is known to be restricted to the bounded interval $W = [\, w_{min} , w_{max} \,]$ of the real line. A sliding regime locally exists on $h^{-1}(0)$ if and only if for all $w \in W$

$$u^{-}(x) < u^{EQ}(x,w) < u^{+}(x)$$

Example Consider the dynamic model of an ideal, separately excited, direct current motor (Rugh, 1981 pp. 98-99):

$$dx_1/dt = -(R_a/L_a)x_1 - (K/L_a)x_2 u + (V_a/L_a) = X_1(x,u)$$

$$dx_2/dt = -(B/J)x_2 + (K/J)x_1 u + (1/J)\, T^L = X_2(x,u)$$

$$y = x_2 = h(x)$$

with x_1 being the armature current, x_2 the angular velocity of the motor shaft moving against a viscous torque characterized by a damping coefficient B and J is the moment of inertia of the mechanical load. The control u is the controlled current in the field circuit. V_a is the constant armature voltage. R_a, L_a represent armature circuit resistence and inductance while K is the torque constant of the motor. T^L is a load perturbation torque.

Here, $\partial X/\partial u = -(K/L_a)x_2\partial/\partial x_1 + (K/J)x_1\partial/\partial x_2$ and $L_{\partial X/\partial u}h = (K/J)x_1$ and the system has local relative degree equal to 1 everywhere except on $x_1 = 0$. However, the perturbation torque, which also acts as an input, exhibits relative degree also equal to 1, since $\partial X/\partial T^L = (1/J)\partial/\partial x_2$ and $L_{\partial X/\partial T^L}\, h = 1/J \neq 0$. Moreover, since $(1/J)\partial/\partial x_2 \notin$ range $\partial X/\partial u$, the load perturbation torque, T^L, can not be decoupled from the angular velocity output. A sliding regime does exist on $x_2 = 0$ but its creation has to take into account the magnitude bounds of the perturbation torque. The control foliation property is trivially satisfied in this example and thus a sliding regime can be created whenever a variable structure feedback field current law with extreme values $u^{+}(x)$, $u^{-}(x)$ can be prescribed such that:

$$u^-(x) < \min_{w \in W} (Bx_2 - w)/Kx_1 < \max_{w \in W} (Bx_2 - w)/Kx_1 < u^+(x)$$

In spite of this possibility, the ideal sliding dynamics can not be made totally independent of the perturbation load torque. ■

3.3 Robust Disturbance Decoupling in the absence of the Matching Condition

Suppose that the input w, in (3.7), has local relative degree larger than r with respect to the output function h(x). Let the vector ξ denote the firt r normal coorinates $z_1, z_2, \ldots, z_r$. In such coordinates the system (3.7) is expressed, locally aournd z^o, as:

$$dz_i/dt = z_{i+1} \quad i = 1, 2, \ldots, r-1$$

$$dz_r/dt = [L^r{}_X h](\Phi^{-1}(z), u)$$

$$d\eta/dt = q(\xi, \eta, w) \quad ; \quad y = z_1 \tag{3.10}$$

Given a smooth feedback control law $u = \alpha(z,v)$ such that, locally on M, : $[L^r{}_X h](\Phi^{-1}(z), \alpha(z,v)) = v$, with v being an external independent sacalar input, then, the output y is totally decoupled from the perturbation input w (See figure 4). Notice, once more, that such a scalar control law exists by virtue of the implicit function theorem and the definition of relative degree. This proves the " if" part of the following theorem:

Theorem 3.4 . Let (X,h) have relative degree r on x^o. There exists a smooth feedback control law of the form $u = \alpha(x,v)$ which locally decouples the output $y = h(x)$ from the disturbance input w, if and only if the input w has relative degree strictly greater than r on x^o .i.e., for all x in a neighorhood N of x^o :

$$L_{\partial X/\partial w} L^{i-1}{}_X h(x) = 0 \quad ; \quad i = 1, 2, \ldots, r \tag{3.11}$$

<u>Proof</u>: to prove necessity, suppose $u = \alpha(x,v)$ is any feedback control law locally decoupling the output h from the perturbation input w. The closed loop system is expressed as:

$$dx/dt = X(x,\alpha(x,v),w) =: X^{\alpha}(x,v,w) \tag{3.12}$$
$$y = h(x)$$

If in the system (3.12) w is locally decoupled from the output, then, necessarily, the normal form coordinates, $z_i = L^{i-1}_{X^{\alpha}}h$; $i = 1,\ldots,r$ are all locally independent of w. From (3.12) it follows that the quantities $L^{i-1}_{X}h$; $i=1,\ldots,r$ are also independent of w. Hence, locally around x^o, $\partial(L^{i-1}_{X}h)/\partial w = L_{\partial X/\partial w}L^{i-2}h = 0$ for $i=1,2,\ldots,r$. Since $dz_r/dt = [L^{r}_{X^{\alpha}} h](x,v,w)$ must also be locally independent of w, then, one has, $\partial[L^{r}_{X^{\alpha}} h](x,v,w)/\partial w = 0$. Hence $\partial L^{r}_{X}h /\partial w = 0$ and therefore $L_{\partial X/\partial w}L^{r-1}_{X}h = 0$ locally around x^o. ■

Since $L_{\partial X/\partial w}L^{i-1}_{X}h = [\partial(L^{i-1}_{X}h)/\partial x]\partial X/\partial u = 0$; $i = 1,2,\ldots,r$. Condition (3.11) is equivalent to the condition of having $\partial X/\partial u$ locally contained in the null space of the matrix $\Omega(x)$ given by :

$$\Omega(x) = \begin{bmatrix} \partial h/\partial x \\ \partial(L_{X} h)/\partial x \\ \vdots \\ \partial(L^{r-1}_{X} h)/\partial x \end{bmatrix} \tag{3.13}$$

which constitutes a generalization of the condition found in Isidori (1987) for systems linear in the control.

The sliding mode disturbance decoupling problem can be formulated as follows:

Consider the perturbed system (3.7) with local relative degree r. It is desired to find a variable structure control law, inducing a local sliding regime on an open set of the zero level set $k^{-1}(0)$ of an auxiliary output function $\psi = k(x)$, such that the original output y is locally stabilized to zero while being unaffected by the perturbation signal w.

The variable structure control law will constitute an "outer loop" feedback control inducing locally desirable robustness properties into an "inner loop" feedback control law. Such control law, of the form $u = \alpha(x,v)$, is assumed to be devised for "exact" disturbance decoupling. The sliding mode approach is especially useful to obtain a robust design with respect to small modeling errors and other external perturbations signals. This discontinuous control scheme can be accomplished by proposing an auxiliary output function of the form:

$$\psi = k(x) = L^{r-1}{}_{X}{}^{\alpha} h(x) + c_{n-2}L^{r-2}{}_{X}{}^{\alpha} h(x) + \ldots + c_1 L_X{}^{\alpha} h(x) + c_o h(x) \qquad (3.14)$$

which in normal form coordinates is a linear output function given by :

$$\psi = k(z) = z_r + c_{r-2}z_{r-1} + \ldots + c_1 z_2 + c_o z_1 \qquad (3.15)$$

Devising a variable structure control law that locally creates a sliding regime on an open set of $k^{-1}(0)$, the "outer loop" closed loop system, in normal form coordinates, would be expressed, locally around z^o, as :

$$dz_i/dt = z_{i+1} \quad ; \quad i = 1,2,\ldots,r-2$$

$$dz_{r-1}/dt = z_r = w - c_{r-2}z_{r-1} - \ldots - c_1 z_2 - c_o z_1$$

$$dz_r/dt = v = 0.5(1+\text{sign } k(z))v^+(z) + 0.5(1-\text{sign } k(z))v^-(z)$$

$$d\eta/dt = q(\xi,\eta,w)$$

$$y = z_1 \quad ; \quad \psi = k(z). \qquad (3.16)$$

The corresponding ideal sliding dynamics is thus :

$$dz_i/dt = z_{i+1} \qquad i = 1,2,\ldots,r-2$$
$$dz_{r-1}/dt = -c_{r-2}z_{r-1} - \ldots - c_1 z_2 - c_o z_1$$
$$d\eta/dt = q(\xi,\eta,w)$$
$$y = z_1 \quad ; \quad \psi = 0 \qquad (3.17)$$

If the coefficients c_i, in (3.15), are appropriately chosen, an asymptotically stable motion is obtained, towards $z_1 = 0$, which is totally independent of the perturbation input w.

The following trivial lemma will be useful:

Lemma 3.5 Let $m = m(x,u,w)$ with $u = \alpha(x,v,w)$ such that locally $\partial\alpha/\partial w \neq 0$. If m is locally independent of w, then, m is also locally independent of u. ∎

If one is allowed to conduct measurements on the disturbance signal w (this will be the case in the model matching problem), one can relax somewhat the hypothesis impossed on the formulation of the disturbance decoupling problem.

Indeed, suppose that both the perturbation input w, and the control input u, have local relative degree r (notice that a smaller local relative degree of w renders the problem unsolvable) and consider the control law $u = \alpha(x,v,w)$. The closed loop system becomes:

$$dx/dt = X(x,u,w) = X(x,\alpha(x,v,w),w) = X^{\alpha}(x,v,w)$$
$$y = h(x) \qquad (3.18)$$

Let $\Omega^{\alpha}(x)$ denote the matrix in (3.13) with X substituted by X^{α}. Using the result of Theorem 3.4, a feedback control law exists that locally solves the disturbabnce decoupling problem, if and only if :

$$\partial x^{\alpha}/\partial w \in \text{ Null space of } \Omega^{\alpha}(x) \qquad (3.19)$$

It follows, from the definition of local relative degree, that the first r-1 entries of the vector : $\Omega^{\alpha}(x)\ \partial x^{\alpha}/\partial w$, are all identically zero. The last entry, which must also be zero, is given by:

$$[\partial(L^{r-1}{}_{X^{\alpha}}h)/\partial x]\ \partial x^{\alpha}/\partial w = L_{\partial X^{\alpha}/\partial w}L^{r-1}{}_{X^{\alpha}}h = 0 \qquad (3.20)$$

Notice that since the quantities $[\partial(L^{i-1}{}_{X^{\alpha}}h)/\partial x]\ \partial x^{\alpha}/\partial w = L_{\partial X^{\alpha}/\partial w}\ L^{i-1}{}_{X^{\alpha}}h$ $= \partial\{\ L^{i}{}_{X^{\alpha}}h\ \}/\partial w = 0$; $i=0,1,2,\ldots,r-1$, are independent of w, it follows, accroding to lemma 3.5, that they are also independent of α . Hence $L^{i}{}_{X^{\alpha}}h = L^{i}{}_{X}h$ for $i = 0,1,2,\ldots,r-1$. Condition (3.20) is then rewritten as :

$$L_{\partial X^{\alpha}/\partial w}L^{r-1}{}_{X}h = L_{[\partial X/\partial w + (\partial X/\partial u)\partial\alpha/\partial w]}\ L^{r-1}{}_{X}h$$

$$= L_{\partial X/\partial w}\ L^{r-1}{}_{X}h + (\partial\alpha/\partial w)\ L_{(\partial X/\partial u)}L^{r-1}{}_{X}h = 0 \qquad (3.21)$$

If a control law, $\alpha(x,v,w)$, exists satisfying (3.21), then, one may, essentially, eliminate all possible influence of w on the r-th diferential equation of the normal form model. The solution of (3.21), with respect to α, is explicitely found only in special cases, as the next example shows.

<u>Example</u> : If $X(x,u) = f(x) + g(x)u + p(x)w$, condition (3.21) translates into: $L_pL^{r-1}{}_fh + (\partial\alpha/\partial w)\ L_gL^{r-1}{}_fh = 0$. In this case $L_gL^{r-1}{}_fh$ and $L_pL^{r-1}{}_fh$ are independent of w, and, hence , $\partial\alpha/\partial w = -(L_pL^{r-1}{}_fh)/\ L_gL^{r-1}{}_fh$. Integrating with respect to w one finds that: $\alpha(x,v,w) = -(L_pL^{r-1}{}_fh\ /\ L_gL^{r-1}{}_fh)\ w + \gamma(x,v)$. Choosing $\gamma(x,v) = -(L^{r}{}_fh\ /L_gL^{r-1}{}_fh\ ) + (1/L_gL^{r-1}{}_fh\ )v$, the controlled system, expressed in normal form coordinates, is reduced to :

$$dz_i/dt = z_{i+1} \quad ; \; i = 1,2,\ldots,r-1$$

$$dz_r/dt = v$$

$$d\eta/dt = q(\xi,\eta,w) \quad ; \quad y = z_1 \qquad (3.22)$$

■

Once again, v can be designed as a variable structure feedback control law switching on the basis of the sign of an auxiliary output function of the form, $h(z) = z_r + c_{r-2} z^{r-1} + \ldots + c_o z_1$, with appropriately chosen coefficients.

3.4 Sliding Mode stabilization in Nonlinear Model Matching Schemes.

In the model matching problem (Isidori,1985a, 1985b,1987) one wishes to obtain a feedback control law for the system : $dx/dt = X(x,u)$, $y = h(x)$ such the input-output behavior coincides with that of a linear system characterized by: $dz/dt = Az + bw$, $y = cz$. In the design of such a feedback control law, we are allowed to measure the input w of the reference linear system and its state vector z, i.e., $u = \alpha(x,z,w)$.

The problem can be solved by seeking the required feedack so that the output error signal, $e = h(x) - Cz$, is decoupled from the input w. One can also impose, after a local decoupling law has been found, that the nonlinear system output y, be robustly stabilized to zero from the input w.

Consider the "extended" system $dx^e/dt = X^e(x^e,u,w)$; $y = h^e(x^e)$, with $x^e = col[z,x]$ given by :

$$\frac{d}{dt}\begin{bmatrix} z \\ x \end{bmatrix} = \begin{bmatrix} Az \\ X(x,u) \end{bmatrix} + \begin{bmatrix} bw \\ 0 \end{bmatrix} \quad ; \quad e = h(x) - cz \qquad (3.23)$$

If we let $u = \alpha(x,z,w)$, and denote by $X^\alpha(x^e,w) = X(x,\alpha(x,z,w))$, it is easy to see that the matrix $\Omega(x)$ of (3.13), corresponding to the extended system (3.23), and denoted by $\Omega^e(x)$ is given by :

$$\Omega^{e}(x) = \begin{bmatrix} \partial h^{e}/\partial x^{e} \\ \partial(L_{X^e} h^{e})/\partial x^{e} \\ \vdots \\ \partial(L^{r-1}_{X^e} h^{e})/\partial x^{e} \end{bmatrix} = \begin{bmatrix} -c & \partial h/\partial x \\ -cA & \partial(L_{X^\alpha} h)/\partial x \\ & \\ -cA^{r-1} & \partial(L^{r-1}_{X^\alpha} h)/\partial x \end{bmatrix} \tag{3.24}$$

For the extended system $\partial X^{e}/\partial w = \text{col}\ [b, \partial X^{\alpha}/\partial w]$. Using the result of theorem 3.4, this column vector must belong to the null space of $\Omega^{e}(x)$. This implies that the following conditions must be satisfied:

$$-cb + L_{\partial X^{\alpha}/\partial w} h = 0;\ -cAb + L_{\partial X^{\alpha}/\partial w} L_{X^{\alpha}} h = 0;\ \ldots\ ; -cA^{r-1}b + L_{\partial X^{\alpha}/\partial w} L^{r-1}_{X^{\alpha}} h = 0 \tag{3.25}$$

Since the relative degree is invariant under feedback, the terms of the form: $L_{\partial X^{\alpha}/\partial w} L^{k}_{X^{\alpha}} h$, $(k = 1,\ldots, r-2)$ in (3.25) vanish. This means that, necessarily, $cb = cAb =, \ldots, cA^{r-2}b = 0$. The linear reference model must exhibit, at least, the same relative degree as the nonlinear system. Since, by Lemma 3.5, $L^{k}_{X^{\alpha}}h = L^{k}_{X}h$, for $k = 0,1,\ldots, r-1$. The last equality in (3.25) implies that the nonlinear feedback control law must satisfy :

$$-cA^{r-1}b + L_{\partial X^{\alpha}/\partial w} L^{r-1}_{X^{\alpha}} h = -cA^{r-1}b + L_{(\partial X/\partial u)(\partial \alpha/\partial w)} L^{r-1}_{X}h$$

$$= -cA^{r-1}b + (\partial\alpha/\partial w)\ L_{(\partial X/\partial u)} L^{r-1}_{X}h = 0$$

i.e., $(\partial\alpha/\partial w) = (cA^{r-1}b)/L_{\partial X/\partial u}L^{r-1}_{X}h$. Since in this case $\partial X/\partial u$ is independent of w (and so is $L^{r-1}_{X}h$), one can integrate with respect ot w, to obtain:

$$\alpha(x,z,w) = [cA^{r-1}b/\ L_{\partial X/\partial u}L^{r-1}_{X}h]\ w + \gamma(x,z) \tag{3.26}$$

To determine the unspecified part, $\gamma(x,z)$, of the feedback control law (3.26), one imposes the equality among the r-th derivatives of the output, y, of the nonlinear system, and the corresponding one of the reference model output (See Isidori, 1987). This is equivalent to setting to zero the r-th differential equation of the normal form model of the extended system. This procedure leads to :

$$cA^r z + cA^{r-1}bw = [L_X{}^r h](x, \alpha(x,z,w)) = [L_X{}^r h](x, [cA^{r-1}b / L_{\partial X/\partial u} L^{r-1}{}_X h]\ w + \gamma(x,z)) \tag{3.27}$$

The definition of relative degree and the implicit function theorem guarantee the local existence of a unique solution for $\gamma(x,z)$, i.e., for $\alpha(x,z,w)$.

<u>Example</u> for controlled vector fields X of the form: f+gu, equation (3.27) results in:

$$cA^r z + cA^{r-1}bw = L_f{}^r h + ([cA^{r-1}b/L_{\partial X/\partial u}L^{r-1}{}_X h]\ w + \gamma(x,z))\ L_g L^{r-1}{}_f h$$

i.e.,

$$\alpha(x,z,w) = [cA^{r-1}b/L_{\partial X/\partial u}L^{r-1}{}_X h]w + \gamma(x,z) = [cA^r z + cA^{r-1}bw - L^r{}_f h\ ]/\ L_g L^{r-1}{}_f h$$

■

The closed loop system makes the nonlinear model behave in the same manner as the linear system from the input-output viewpoint, except for a term depending on the initial condition that can also be appropriately set to zero. Indeed, the output of the nonlinear controlled system is of the form:

$$y(t) = h(x(t)) = e(t) + cz(t) = e(t) + \int_0^t ce^{A(t-\sigma)}bw(\sigma)\,d\sigma \tag{3.28}$$

Writing the normal form equations for the extended system, it is easy to see that if a control law obtained from (3.27) is used, the term e(t), above, only depends on the initial conditions of the extended sytstem. Its effect can therefore be cancelled.

The output y(t) generated by the closed loop extended system can now be stabilized by appropriate choice of the input w. Since the nonlinear system behaves in a linear fashion and responds according to (3.28), a variable structure control law (which properly takes into account the relative degree of the linear system) can now be devised for robust stabilization of the nonlinear system with arbitrarily prespecified eigenvalues. The details are left for the reader.

IV CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this article the relevance of the relative degree concept has been examined in the analysis and design issues related to the creation of sliding regimes for general nonlinear systems. The results indicate that the simplest possible structure at infinity must be exhibited by nonlinear systems undergoing sliding motions on the zero level set of the output feedback function. General necessary as well as necessary and sufficient conditions for the existence of sliding regimes have been presented. The disturbance rejection properties of sliding mode control were examined and a generalization of the matching condition was found. The implications of sliding mode control as an "outer loop" feedback strategy was also examined in a variety of control problems including; Local Stabilization of Feedback Linearizable Systems, Disturbance Decoupling problems -with and without measurement of the disturbance input- and Nonlinear Model Matching.

Several important research areas may be pursued in the future, within the context of this article. For instance, one may wish to extend the general results about sliding motions to the case of nonlinear multivariable systems.

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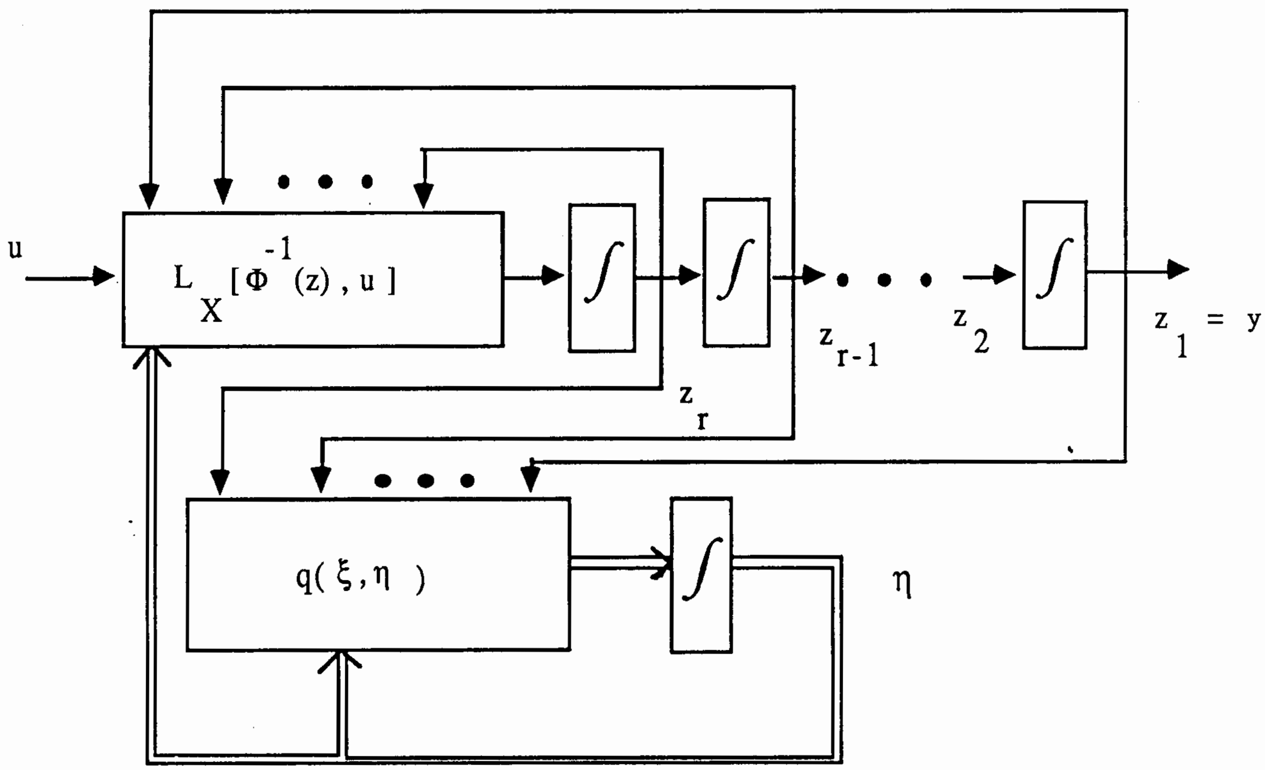
FIGURE CAPTIONS

Figure 1. Block diagram of a nonlinear system in normal form

Figure 2. Controlled trajectories with local sliding motion on $x_2 = 0$

Figure 3. Sliding Regime creation in systems of relative degree > 1

Figure 4. Confinement of perturbations to zero dynamics block



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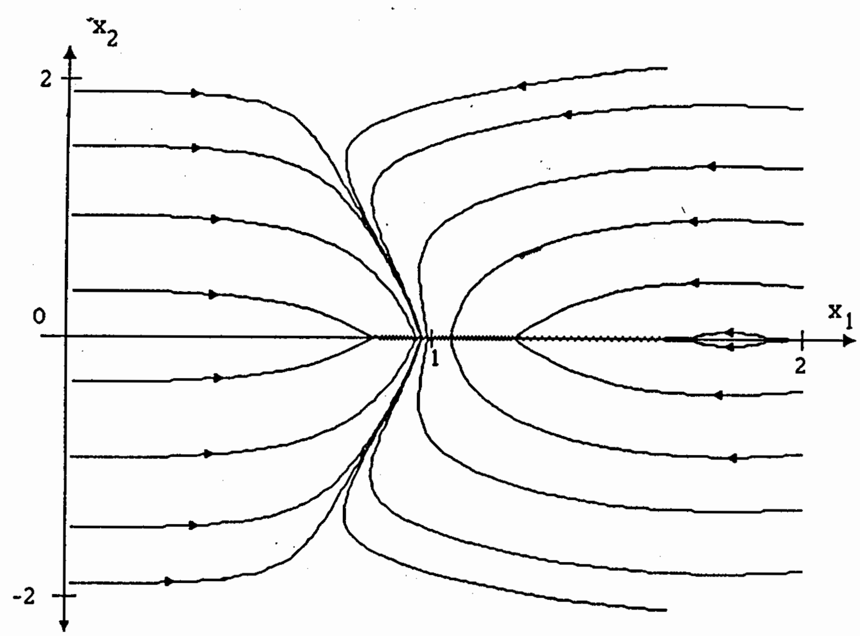

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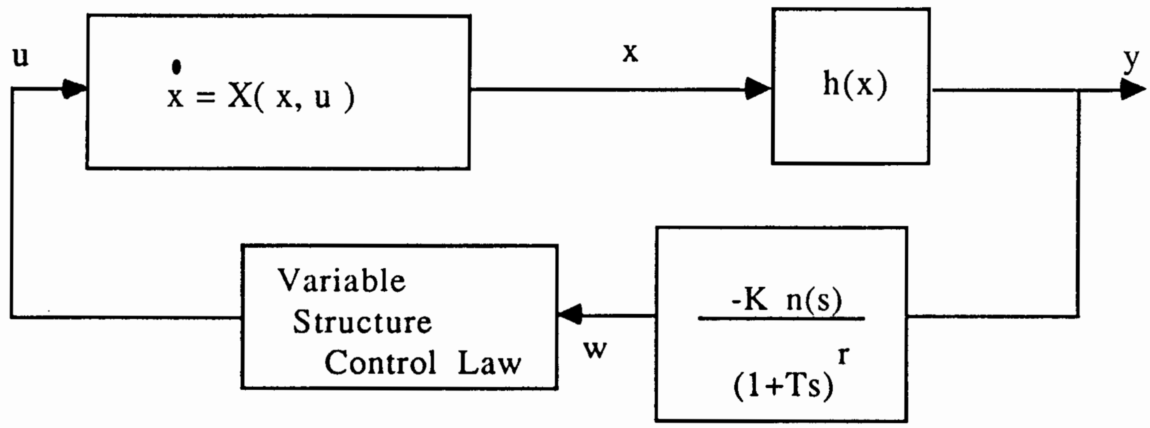

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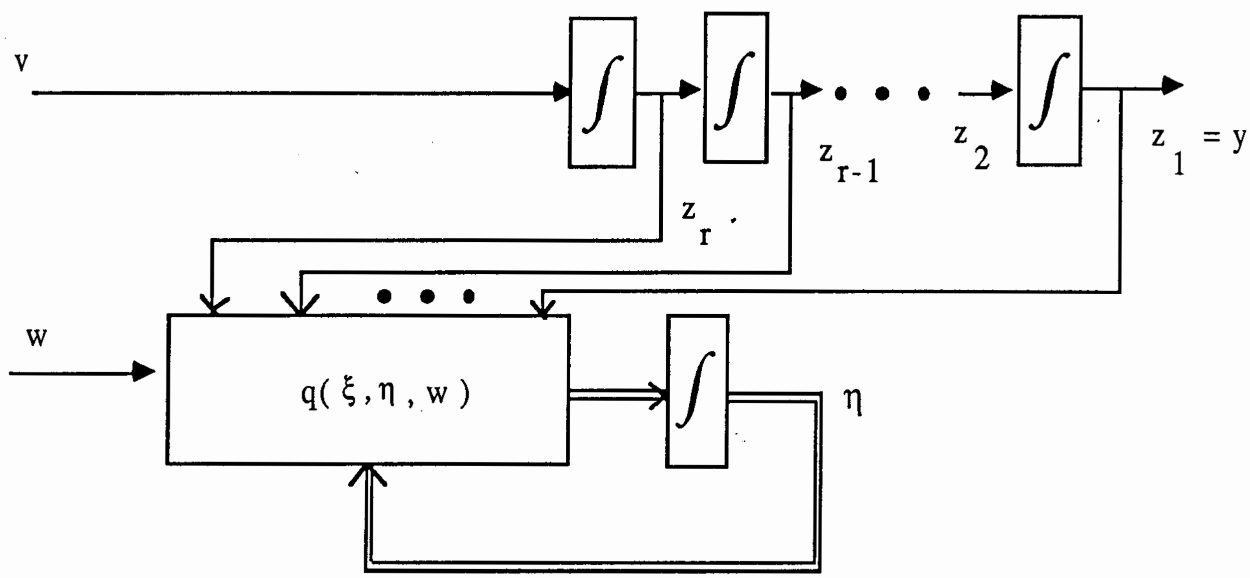

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