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LINEAR TRANSFORMATIONS INTERTWINING WITH GROUP REPRESENTATIONS

POR

CARLOS EDUARDO FINOL

UNIVERSIDAD DE LOS ANDES

FACULTAD DE CIENCIAS

DEPARTAMENTO DE MATEMATICA

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BY

CARLOS EDUARDO FINOL

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CONTENTS

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ABSTRACT

As is apparent from the title, the aim of the present work is to investigate the form of linear operators intertwining with representation of groups, basically, the Euclidean and Poincaré groups. We thereafter undertake the examination of those linear operators which are invariant under the action of the group of homothetics as well.

The class of linear operators from one space of functions to another that obey the principle of relativity, which states that the laws and equations of Physics must maintain their forms under the action of the Poincare group, are a particular instance of intertwining operators.

Chapter I is of introductory nature. The main references are Boerner [2], Miller [16], Turnbull [20], Vilenkin [21] and Weyl [22].

In Chapter II we examine the form of linear transformations, on some linear spaces of functions, of a type that J. L. B. Cooper ([4],[5]) has called appropriate representations of groups. These are linear transformations of the form

$$W(\tilde{g})f(x) = Q(x,\tilde{g})\ f\ (V(\tilde{g})x),$$

where $\tilde{g}$ runs over some group $\tilde{G}$, $W(\tilde{g})$ is, for every $\tilde{g}$, defined on a linear space $A(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$, and $W(\tilde{g})f(x)$ is in some linear space of functions $B(\mathbb{R}^n)$. Also, $W(g)$ is a representation of $\tilde{G}$ on $A(\mathbb{R}^n)$ and $Q(x,\tilde{g})$ is a multiplier. Change of variable is an example of this.

We restrict ourselves to the case when the group $\tilde{G}$ is the semidirect product $G \times \tau(n)$ of some group G of transformations acting on $\mathbb{R}^n$ and the

group $\tau(n)$ of translations of $\mathbb{R}^n$. The operator $V(\tilde{g}) = V(g,h), g \in G$, $h \in \tau(n)$, is assumed to be affine, that is

$$V(g,h)x = A(g,h)x + B(g,h).$$

Further restrictions reduce the possible forms of $A(g,h)$ and we are led to consider the cases when $A(g,0) = A(g)$ is either g^{-1} or the identity. In either case $B(g,0)$ turns out to be zero. The examination of the operator $V(g,h)$ leads to some system of functional equations which we solve for a wide class of groups that embrace the rotation groups and the Lorentz group. This is the outcome of theorems 1 and 2 of Section II.2. As a consequence of these theorems the appropriate representations $W(g,h)$ are reduced to two canonical forms, namely

$$(1) \quad W(g,h)\, f\,(x) = Q(g)\, f\,(g^{-1}x - g^{-1}h)$$

$$(2) \quad W(g,h)\, f\,(x) = Q(x,g,h)\, f\,(x),$$

where in case (1) $Q(g)$ is a fixed representation of the group G on the space of values of the functions in $A(\mathbb{R}^n)$. We must point out that although we do not pursue the examination of the transformation of type (2) above, we consider that theorem II.2.2 has an interest of its own.

In Section II.3 we begin the discussion of linear operators intertwining between appropriate representations of type (1) above, that is, linear operators that satisfy the condition

$$(3) \quad \{T[Q(g)\, f\,(g^{-1}x - g^{-1}h)]\}(u) =$$

$$Q^*(g)\{T[f(x)]\}(g^{-1}u - g^{-1}h).$$

Cooper ([4],[5]) has called these equations, appropriate functional equations. The problem of linear operators intertwining with representations of the group of translations is dealt with in [4] and for higher dimensions in [5].

The expression (3) above can be split into the pair of expressions

(4) $[Tf(x-h)](u) = [Tf(x)](u-h)$

(5) $[TQ(g)f(g^{-1}x)](u) = Q^*(g)[Tf(x)](g^{-1}u)$.

The main feature of this section is theorem II.3.1, stated and proved by J. L. B. Cooper, which determines the form of linear operators acting on an α-space with values in a β-space that obey the expression (4) above. These are space of functions (or equivalence classes of functions) where spaces of infinitely differentiable functions are continuously embedded.

In Chapter III we examine the form of the solutions of the expression (4) that satisfy (5). We study separately the case when $Q(g)$ and $Q^*(g)$ are single valued representations of the special orthogonal group and the proper Lorentz group. It is shown that in both cases we can assume $Q(g)$ and $Q^*(g)$ to be irreducible. The problem of linear operators intertwining with appropriate representations of the Euclidean group has been investigated by J. L. B. Cooper.

In Section III.4 we investigate the form of the solutions of (3) above that are homothetic invariant in the sense that

$$[Tf(\lambda x)](u) = \lambda^h [Tf(x)](\lambda u),\ \lambda > 0.$$

CHAPTER I

INTRODUCTION

The purpose of this chapter is to present a brief introduction of well-known results and definitions of group representation and vector invariants, which will be needed in the subsequent chapters.

1. GROUP REPRESENTATIONS

Let G be a topological group. We write the product of the elements g_1, g_2 of G as $g_1 \cdot g_2$. By GL(A) we denote the linear space of all non-singular, linear, continuous transformations of the real or complex linear space A into itself.

DEFINITION 1. A representation T of a group G on the space A is a homomorphism

$$T: \quad G \to GL(A),$$

such that, for every x in A, we have that

$$T(g_r)x \to T(g)x,$$

whenever $g_r \to g$ with r in some set of indices.

Thus, by definition, for any $g \in G$, $T(g)$ belongs to GL(A), and $T(g_1g_2) = T(g_1)T(g_2)$ whenever $g_1, g_2 \in G$. Also, $T(g^{-1}) = [T(g)]^{-1}$ and $T(e) = I$, where e and I are the unit elements of G and GL(A) respectively.

The linear space A is called the space of the representation $T(g)$. If the space A is finite dimensional, then the representation

$T(g)$ is said to be finite dimensional. Otherwise the representation $T(g)$ will be called infinite dimensional. Unless we state the contrary, by a representation of a group we shall mean a finite representation.

It is natural and sometimes convenient to realize representations of groups as groups of non-singular matrices with complex or real coefficients, as follows. Let A be a linear space of dimension n. Let $T(g)$ be a representation of the group G on A. Choose a basis $\{v_1, \ldots v_n\}$ in A, then for every $g \in G$,

$$T(g)v_j = \sum_{i=1}^{n} t_{ij}(g)v_i.$$

Thus, with each operator $T(g)$ of the representation we associate the matrix

$$(T(g))_{n\times n} = (t_{ij}(g))_{n\times n}.$$

where the functions $t_{ij}(g)$ are defined on the group G.

The homomorphism property becomes thus,

$$t_{ij}(g_1g_2) = \sum_{k=1}^{n} t_{ik}(g_1)t_{kj}(g_2) \quad 1 \leq i,j \leq n$$

Obviously, the functions $t_{ij}(g)$ depend on the chosen basis. Under a change of basis with matrix $B_{n\times n}$, $(T(g))_{n\times n}$ becomes $B^{-1}(T(g))B$.

DEFINITION 2. Let $T(g)$ be a representation of the group G on the space A_1. Let B be a linear bijection from the space A_1 onto the space A_2. Define $T'(g)$ by

$$T'(g) = BT(g)B^{-1}$$

defines a representation of the group G on the space A_2. The representation $T'(g)$ is said to be equivalent to $T(g)$.

The equivalence of representations is reflexive, symmetric and transitive. Therefore the set of all representations of a given group is partitioned in equivalence classes. Thus, in order to determine all possible representations of a group G it is enough to find one representation in each class. The remain-ining representations $T'(g)$ in each class are obtained by the formula

$$T'(g) = BT(g)B^{-1},$$

where B runs over $GL(A,A_1)$, the set of all linear biyections from A onto A_1 and $T(g)$ is a representation of G on A.

DEFINITION 3. Let $T(g)$ be a representation of the group G on the linear space A and let A_1 be a proper subspace of A. If for every a in A and every g in G we have that $T(g)a$ is in A_1, then the subspace A_1 is said to be invariant under the representation $T(g)$.

DEFINITION 4. A representation $T(g)$ on space A is called irreducible if the only invariant subspaces of A under $T(g)$ are A itself and the null space $\{0\}$. Otherwise the representation $T(g)$ is reducible.

Let $(T(g))_{nxn}$ be a matrix realisation of some reducible representation $T(g)$ of a group G on an n-dimensional linear space A.

Let A_1 be a k-dimensional invariant subspace of A. Choose a basis $\{v_1, \ldots, v_{n_k}\}$ in A_1 and extend it to a basis $\{v_1, \ldots, v_k, v_{k+1}, \ldots v_n\}$ of the whole space A.

Then

$$\underset{n\times n}{(T(g))}\, v_i = \sum_{j=1}^{n} t_{ij}(g) v_j \quad . \qquad 1 \leqslant i \leqslant k$$

$$= \sum_{j=1}^{k} t_{ij}(g) v_j,$$

so that $t_{ij}(\lambda) = 0$ for $1 \leqslant i \leqslant k$ and $k + 1 \leqslant j \leqslant n$. Hence, the matrix $\underset{n\times n}{(T(g))}$ has, for this particular basis the form

$$\underset{n\times n}{(T(g))} = \begin{pmatrix} \underset{n\times k}{(T_1(g))} & \underset{k\times(n-k)}{(Q(g))} \\ 0 & \underset{(n-k)\times(n-k)}{(T_2(g))} \end{pmatrix} \qquad (I.1.1)$$

It is not always possible to find a basis in which the matrix $\underset{k\times(n-k)}{(Q(g))}$ is zero. We discuss below when this is possible.

DEFINITION 5. A linear space is said to be the direct sum of its subspaces A_1 and A_2 if every element a in A can be uniquely written as $a = a_1 + a_2$, where $a_1 \in A_1$, and $a_2 \in A_2$.

In particular the intersection of the subspaces A_1 and A_2 is the null space $\{0\}$.

If A is the direct sum of subspaces A_1 and A_2, then we write $A = A_1 \oplus A_2$.

Suppose that $T(g)$ is a representation of the group G on the n-dimensional space A and that A splits into the direct sum $A = A_1 \oplus A_2$ of two subspaces invariant under $T(g)$, that is

$$T(g)\ (A_1) \subset A_1,$$

$$T(g)\ (A_2) \subset A_2,$$

for all $g \in G$.

Thus, calling $T_1(g)$ and $T_2(g)$ the restrictions of $T(g)$ to the subspaces A_1 and A_2 respectively, we see that, if $a \in A$, $a = a_1 + a_2$ with $a_i \in A_i$, $i = 1, 2$, then

$$\begin{aligned} T(g)a &= T(g)\ (a_1 + a_2) \\ &= T(g)a_1 + T(g)a_2 \\ &= T_1(g)a_1 + T_2(g)a_2 \\ &= b_1 + b_2, \end{aligned}$$

with $b_i \in A_i$, $i = 1, 2$.

We see that by passing on to a matrix realization $(T(g))_{nxn}$ of $T(g)$ and then choosing a convenient basis, the matrix (I.1.1) becomes

$$(T(g))_{nxn} = \begin{pmatrix} (T_1(g))_{kxk} & 0 \\ 0 & (T_2(g))_{(n-k)x(n-k)} \end{pmatrix}, \qquad \text{(I.1.2)}$$

where k and n-k are the dimensions of A_1 and A_2 respectively.

In this case, we write

$$T(g) = T_1(g) \oplus T_2(g)$$

and we say that $T(g)$ is the direct sum of its restrictions $T_1(g)$, $T_2(g)$ to A_1 and A_2 respectively.

Similar expressions are obtained for the decomposition of A into the direct sum of s invariant subspaces.

In particular we have the following.

DEFINITION 6. A representation $T(g)$ of a group G on a space A is completely reducible if A splits into a direct sum of subspaces $A_1, A_2, \ldots, A_s$, each of which is invariant under $T(g)$ and such that the restriction, $T_i(g)$, of $T(g)$ to every A_i, $i = 1, 2, \ldots, s$ is itself irreducible.

We see that in this case we have further that

$$T(g) = T_1(g) \oplus \ldots \oplus T_s(g).$$

On the other hand, the matrix realization $(T(g))_{n\times n}$ of $T(g)$ in some convenient basis becomes

$$(T(g))_{n\times n} = \begin{pmatrix} (T_1(g)) & & \\ & \ddots & \\ & & (T_s(g)) \end{pmatrix}$$

We end this section with a very important result due to I. Schur.

THEOREM 1.1. (Schur's Lemma)

Let $T(g)$ and $Q(g)$ be finite dimensional irreducible representations of a group G on the complex linear spaces V_1 and V_2 respectively. Let $A : V_2 \to V_1$ be a linear mapping such that

$$T(g)Av = AQ(g)v,$$

for any $v \in V_2$ and any g in G.

Then either A is the zero operator or A is invertible.

2. TENSORS AND TENSOR REPRESENTATIONS

Let V be an n-dimensional linear space on a field K and let V* be its dual, that is, V* is the linear space of all linear forms v* on V, $v^* : V \to K$. The value of v* at $x \in V$ will be written as $[v^*,x]$.

Let $\{e_i\}_{i=1}^n$ be a basis for V, then $[v^*,x] = \sum_{i=1}^n x_i[v^*,e_1]$, for every v* in V* and x in V. Thus, v* is uniquely determined, with respect to the basis $\{e_i\}_{i=1}^n$ by the n scalars $[v^*,e_i]$ $i = 1,2,\ldots,n$.

Let e_i^* be the element in V* defined by $[e^*_i, e_j] = \delta_{ij}$, where δ_{ij} is the usual Kronecker delta. The elements $\{e_i^*\}_{i=1}^n$ are a basis for V*, this is the so-called dual basis to $\{e_j\}_{j=1}^n$.

We see that, if $v^* = \sum_i v^i e^*_i$ and $x = \sum_{i=1}^n x_i e_i$, then $[v^*,x] = \sum_{i=1}^n v^i x_i$.

The space $(V^*)^* = V^{**}$ is canonically isomorphic to V, it can thus be identified with V.

DEFINITION: 1 Let V^{*m} be the m-fold cartesian product $V^* \overset{m}{\times \ldots \times} V^*$ of of the dual V* to a linear space V. A covariant tensor of rank m is defined to be a multilinear form a,

$$a : V^{*m} \to K .$$

Let $a(v_1^*, \ldots, v_m^*)$ be a covariant tensor of rank m. The map $v_1^* \to a(v_1^*, \ldots, v_m^*)$ is linear, so that there is some x_1 in V such that

$$\begin{aligned} a(v_1^*, \ldots, v_m^*) &= [v_1^*,x_1] \\ &= \sum_{r_1} a_{r_1}(v_2^*,\ldots,v_m^*)[v_1^*, e_{r_1}] . \end{aligned}$$

By repeating the argument we see that $a(v_1^*, \ldots, v_m^*)$ can be written as

$$a(v_1^*, \ldots, v_m^*) = \sum_{r_1 \ldots r_m} a_{r_1 \ldots r_m} [v_1^*, e_{r_1}] \ldots [v_m^*, e_{r_m}] .$$

The linear form, $(v_1^*, \ldots, v_m^*) \to [v_1^*, e_{r_1}] \ldots [v_m^*, e_{r_m}]$, is a covariant tensor of rank m, and will be denoted by $e_{r_1} \otimes \ldots \otimes e_{r_m}$. Thus,

$$e_{r_1} \otimes \ldots \otimes e_{r_m} (v_1^*, \ldots, v_m^*) = [v_1^*, e_{r_1}] \ldots [v_m^*, e_{r_m}] .$$

The value of the tensor a at $(v_1^*, \ldots, v_m^*)$ becomes thus,

$$a(v_1^*, \ldots, v_m^*) = \sum_{r_1 \ldots r_m} a_{r_1 \ldots r_m} e_{r_1} \otimes \ldots \otimes e_{r_m} (v_1^*, \ldots, v_m^*).$$

We see that,

$$a = \sum_{r_1 \ldots r_m} a_{r_1 \ldots r_m} e_{r_1} \otimes \ldots \otimes e_{r_m} .$$

The set of all covariant tensors of rank m on V^{*m} is called the m-fold tensor product $V^{\otimes m}$ of V.

DEFINITION 2. Let V^m be the m-fold cartesian product $V_x \ldots m_x V$ of a linear space V. A contravariant tensor of rank m is defined to be a multilinear form b,

$$b : V^m \to K .$$

By arguments similar to those in the case of covariant tensors of rank m, we find that the value of a contravariant tensor of rank m at $(x_1, \ldots, x_m) \in V^m$ is given by

$$b(x_1, \ldots, x_m) = \sum_{i_1 \ldots i_n} b^{i_1 \ldots i_m} e_{i_1}^* \otimes \ldots \otimes e_{i_m}^* (x_1, \ldots, x_m)$$

where $b^{i_1..i_m} \in K$ and

$$e_{i_1}{}^* \otimes \ldots \otimes e_{i_m}{}^* (x_1,\ldots,x_m) = [e_{i_1}{}^*, x_1] \ldots [e_{i_m}{}^*, x_m].$$

Thus, b can be written as

$$b = \sum_{i_1\ldots i_m} b^{i_1..i_m} e_{i_1}{}^* \otimes\ldots\otimes e_{i_m}{}^*$$

The linear space of all contravariant tensors of rank m on V^m is called the m-fold tensor product $V^{*\otimes m}$ of V*.

Let a group of transformations G, represented by matrices $g = \underset{n\times n}{(g_{ij})}$ act on V,

$$g : e_r \rightarrow \sum_{s=1}^{n} g_{sr}\, e_s.$$

The induced action of G on V* is given by

$$g : e_i{}^* \rightarrow \sum_r q_{ir}\, e_r{}^*,$$

where $g^{-1} = q = \underset{n\times n}{(q_{ir})}$, and $\{e_i{}^*\}_{i=1}^n$ is, as above, the dual basis to $\{e_j\}_{j=1}^n$. Then G can be extended to a group of transformations on $V^{\otimes m}$:

$$T^{\otimes m}(g)a = a' = \Sigma\, a_{r_1\ldots r_m}\, g\, e_{r_1} \otimes\ldots\otimes g e_{r_m}.$$

It can easily be verified that $T^{\otimes m}(g)$ is a representation of G on $V^{\otimes m}$. The representation $T^{\otimes m}(g)$ is called the m-fold tensor product of g.

From the definition of $T^{\otimes m}(g)$ we deduce that the quantities $a_{r_1\ldots r_m}$, $1 \leq r_1,\ldots r_m \leq n$, that determine the tensor a, transform according to the law

$$a'_{j_1..j_m} = \sum_{r_1..r_m} g_{j_1 r_1}\ldots g_{j_m r_m}\, a_{r_1..r_m},$$

when a is transformed into $T^{\otimes m}(g)a$.

Similarly, G can be extended to a group of transformations on $V^{*\otimes m}$:

$$T'^{\otimes m}(g)b = b' = \Sigma \; b^{i_1..i_m} \, ge^*_{i_1} \otimes \ldots \otimes ge^*_{i_m}$$

$$= \sum_{s_1..s_m} \Big(\sum_{i_1...i_m} b^{i_1..i_m} q_{i_1 s_1} \ldots q_{i_m s_m} \Big) e^*_{s_1} \otimes \ldots \otimes e^*_{s_m}$$

so that

$$b'^{s_1..s_m} = \sum_{i_1..i_m} b^{i_1..i_m} q_{i_1 s_1} \ldots q_{i_m s_m}$$

Mixed tensors of covariant rank p and contravariant rank m can be formed by considering the tensor product of a covariant tensor p and a contravariant tensor of rank m.

Under the induced action of the group G above, the components $b^{i_1..i_m}_{j_1..j_p}$ of a mixed tensor transform according to the law:

$$b'^{i_1...i_m}_{j_1..j_p} = \sum_{\substack{r_1..r_m \\ s_1..s_p}} b^{r_1...r_m}_{s_1..s_p} \, q_{r_1 i_1} \ldots q_{r_m i_m} \, g_{j_1 s_1} \ldots g_{j_p s_p}$$

3. REPRESENTATIONS OF THE FULL LINEAR GROUP GL(n,ℂ)

The full linear group GL(n,ℂ) consists of all invertible square matrices of rank n, whose entries are complex numbers. The full linear group GL(n,ℝ) of real matrices is defined similarly.

The tensor representations of GL(n,ℂ) can be decomposed in a manner closely related to that of the group S_n of all permutations of the set {1,...,n}. The representations of S_n are completely reducible. Indeed, the same is true for any finite group.

The irreducible non-equivalent representations of S_n are in a one to one correspondence with the conjugacy classes in S_n. Given an element t in S_n, its conjugacy class T is the set

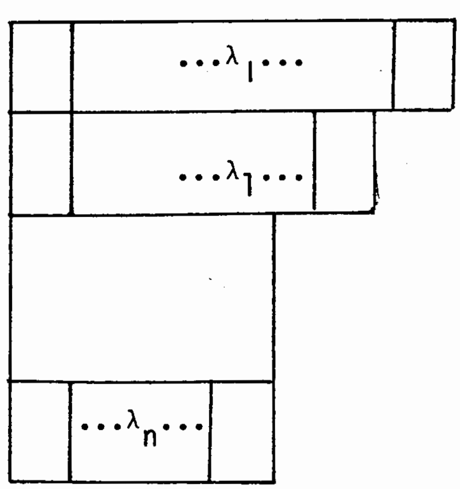
$$T = \{s \in S_n; \quad s\, t\, s^{-1} = t\}.$$

Two conjugacy classes T_1 and T_2 are either identical or their intersection is empty.

There is a one to one correspondence between sets of non-negative integers, $\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$ and $\lambda_1 + \lambda_2 + \ldots + \lambda_n = n$, and conjugacy classes in S_n. Every set $\{\lambda_1, \ldots, \lambda_n\}$ as defined above is called a partition of n.

The irreducible representations of S_n can be obtained by means of a graphical method due originally to A. Young and later modified by Von Neumann. The theory is rather complicated but the main results are easy to state, as we now pass on to describe.

Let $\{\lambda_1, \ldots \lambda_n\}$ be a partition of n. A frame is an arrangement of n squares in rows, so that there are λ_n squares in the top row, λ_2 in the row below and so on until the bottom row that contains λ_n squares



DEFINITION 1. YOUNG TABLEAUX

A Young tableaux is obtained by filling in the squares of the frame corresponding to the partition $\{\lambda_1,\ldots,\lambda_n\}$ with the numbers $1,\ldots,n$.

DEFINITION 2. STANDARD TABLEAUX

A tableaux is called a standard tableaux if the digits in each row increase from left to right and the digits in each column increase from top to bottom.

THEOREM 3.1

(a) There is a one to one correspondence between frames and irreducible representations of S_n. Different frames determine different irreducible representations, and representations corresponding to the same frame are equivalent.

(b) The dimension f of an irreducible representation corresponding to the frame $\{\lambda_j\}$ is equal to the number of standard tableaux $T_1,\ldots,T_f$ of this frame.

A maximal set of non-equivalent irreducible representations of S_n is called a fundamental set of irreducible representations.

Given a reducible representation of S_n, then it reduces completely into irreducible components. The number of times an irreducible representation appears in a reducible representation is called its multiplicity.

For the full linear group $GL(n,\mathbb{C})$ it is not true that every representation decomposes. However, it can be proved that the representation $T^{\otimes m}(g)$ on $V^{\otimes m}$, as defined in Section 2, decomposes. We now pass on to describe how this occurs.

Let S_m be the group of permutations of the digits {1,2,...,m}. By T(s), $s \in S_m$, we mean the representation of S_m on $V^{\otimes m}$ defined by

$$T(s)w = w_{s^{-1}(1)} \otimes \ldots \otimes w_{s^{-1}(m)}.$$

Let $\{S^{(1)}, \ldots, S^{(P)}\}$ be a fundamental set of non-equivalent irreducible representations of S_m. Then T(s) decomposes, say,

$$T(s) = a_1 S^{(1)} \oplus \ldots \oplus a_p S^{(P)}, \qquad (I.3.1)$$

where a_i, i = 1,2,...,p is the multiplicity of $S^{(i)}$ in T(s).

THEOREM: 3.2

The tensor representation $T^{\otimes m}(g)$ of $GL(n,\mathbb{C})$ on $V^{\otimes m}$ can be decomposed into a direct sum of irreducible representations $T^{(i)}(g)$ i = 1,2,...,P. More precisely,

$$T^{\otimes m}(g) = b_1 T^{(1)}(g) \oplus \ldots \oplus b_p\, T^{(P)}_{(g)}. \qquad (I.3.2)$$

where b_i is the dimension of the irreducible component $S^{(i)}$ in (I.3.1) and the dimension of the irreducible component $T^{(i)}$ in (I.3.2) is the multiplicity a_i of the irreducible component $S^{(i)}$ in (I.3.1), for i = 1, ..., p.

The decomposition (I.3.2) is essentially unique, that is, the irreducible components $T^{(i)}$ occurring in (I.3.2) and their multiplicities are uniquely determined. Also, (I.3.2) carries over to any subgroup of GL. However, it is not true that the irreducible components $T^{(i)}$ in (I.3.2) are always irreducible when restricted to representations of subgroups of GL.

The subspaces of $V^{\otimes m}$ that transform according to some irreducible component of $T^{\otimes}(g)$ are called a symmetry classes of tensors.

4. LIE GROUPS AND THEIR LIE ALGEBRAS

Let K denote either the field $\mathbb{C}$ of the complex numbers or the field $\mathbb{R}$ of real numbers. By K^m we denote the m-fold cartesian product of K.

DEFINITION 1. Let N be an open connected neighbourhood of $0 = (0,\ldots,0) \in K^m$. Let G be a set of matrices.

$$A(t) = A(t_1,\ldots,t_m) = (a_{ij}(t))_{n \times n}, \quad t \in N,$$

such that the mapping $t \to A(t)$ of N into G is one to one. The group G is called an m-dimensional local linear Lie group if G satisfies the following properties.

(a) The entries $a_{ij}(t)$. $1 \leq i,j \leq n$, are analytic functions of the parameters $t_1,\ldots,t_m$.

(b) The matrices $\frac{\partial}{\partial t_i} A(t)$. $i = 1, 2, \ldots, m$ are linearly independent.

(c) There is some neighbourhood N′ of $(0,\ldots,0) \in K^m$ contained in N, such that given t, t′ in N′ there is an element t″ in N such that

$$A(t)A(t') = A(t'').$$

DEFINITION 2. An analytic manifold X of dimension n is a Hausdorff space, such that there is a family of open sets $\{U\alpha;\ \alpha \in A\}$ that covers X; and a family of mappings $\{f\alpha;\ \alpha \in A\}$, such that

$f\alpha: U\alpha \to \mathbb{R}^n$ is a homeomorphism onto $f\alpha(U\alpha)$ and, if $U\alpha \cap U\alpha_2 \neq \emptyset$, then $f\alpha_1 o f_2^{-1}$ is analytic.

The pair $(f_\alpha m_\alpha)$ is called a coordinate system; f_α is called the coordinate mapping and, if $p \in U_\alpha$, then $x = f_\alpha(p)$ is called the coordinate of p.

DEFINITION 3. A global Lie group G is an analytic manifold, endowed with a group structure such that mapping

$$(g_1, g_2) \to g_1 g_2^{-1} ,$$

from G x G onto G, is analytic.

A local linear Lie group that is also a global Lie group in the sense above is called a global linear Lie group. One can always construct a global linear Lie group $\hat{G}$ from a local linear Lie group G. The elements of $\hat{G}$ are all the products of finite sequences of elements in G.

Let G be a linear Lie group defined in $N \in K^m$, and let $\phi : t \to \phi(t) \in N$ be an analytic function defined in some neighbourhood of $0 \in K$, such that $\phi(0) = (0,\ldots,0)$. The mapping $t \to A(\phi(t))$ is called a curve at $A(0) = I_{\substack{m \\ nxn}}$.

DEFINITION 4. Let $A(\phi(t)) \in G$ be a curve at $I_{\substack{m \\ nxn}}$, the matrix a,

$$a = \frac{d}{dt} A(\phi(t)) \Big|_{t=0} = \sum_{j=1}^{m} \frac{\partial}{\partial \phi_j} A(\phi(t)) \Big|_{\phi(0)} \frac{d}{dt} \phi i(t) \Big|_{t=0}$$

is called a tangent vector at A(0).

The set $\{\frac{\partial}{\partial \phi_j} A(\phi(t)) \Big|_{\phi(0)} = A_j ; j = 1,\ldots,m\}$ is by definition linearly independent, so that the set of all matrices $\sum_{j=1}^{m} \lambda_j A_j$, $\lambda_j \in K$

is an m-dimensional linear space.

DEFINITION 5. The linear space L(G) of all matrices $a = \sum_{j=1}^{m} \lambda_j A_j$, $\lambda_j \in K$ is called the Lie algebra of G.

THEOREM 4.1 If a and B are in L(G), then $[a,B] = aB - Ba \in L(G)$.

The expression $[a,B]$ is called the Lie bracket or commutator.

DEFINITION 6. A representation T of a linear Lie group G on a space V is a representation of G in the sense of section 1 with the further requirement that given any matrix realization of T(A), A ∈ G, the entries $T_{ij}(A)$ be analytic.

DEFINITION 7. Let T be a representation of the linear Lie group G on V by matrices (T(A)). Given an element $a \in L(G)$, the infinitesimal operator $\hat{a}$ on V is defined by

$$\hat{a} = \frac{d}{dt} T(A(t)),$$

where A(t) is an analytic curve in G with tangent vector a at A(0). The operators $\hat{a}$ form a Lie algebra homeomorphic to L(G).

DEFINITION 8. A representation of a Lie algebra G on a linear space V is a map ρ.

$$\rho : G \rightarrow GL(V)$$

such that

$$\rho(\lambda a + \mu B) = \lambda \rho(a) + \mu \rho(B),$$

$$\rho([a,B]) = [\rho(a), \rho(B)]$$

for any $\lambda, \mu \in K$ and $\alpha, \beta \in G$.

We now describe the connection between the representations of the Lie group G and the representations of its Lie algebra L(G).

Let G be a connected Lie group and let L(G) be its Lie algebra. Every element A in G can be written as

$$A = \exp a_1 \exp a_2 \ldots \exp a_r,$$

where $\exp a = \sum_{j=0}^{\infty} \frac{a^j}{j!}$ and the matrices a_i, i = 1,... r are in a suitable neighbourhood w of the zero matrix 0 in L(G).

If $\rho(a)$ is a representation of L(G) on a linear space V, then a representation T(A), A ∈ G of G is uniquely defined by

$$T(A) = \exp \rho(a_1) \ldots \exp \rho(a_r)$$

5. REPRESENTATIONS OF THE ORTHOGONAL GROUPS

The usual realization of the complex orthogonal group $O(n,\mathbb{C})$ is the group of complex matrices $g = (g_{ij})_{n \times n}$ such that

$$g^t g = gg^t = I_n,$$

where I_n is the identity matrix of rank n and g^t denotes the transpose of g.

The group $O(n,\mathbb{C})$ is also realized as the set of all linear operators g on a linear space V, where a nondegenerate bilinear form (- , -) is defined on V x V, such that $(gx,gy) = (x,y)$ for all x,y in V. There always exists a basis in V such that, with respect to this basis, the matrices $(g_{ij})_{n \times n}$ of orthogonal transformations g

satisfy

$$\underset{n\times n}{(g_{ij})^t} \underset{n\times n}{K} \underset{n\times n}{(g_{ij})} = \underset{n\times n}{K}$$

where $K = \begin{pmatrix} O_{\frac{n}{2}\times\frac{n}{2}} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & O_{\frac{n}{2}\times\frac{n}{2}} \end{pmatrix}$ if n is even,

and

$$K = \begin{pmatrix} I & O\ldots & O \\ O & O_{\frac{n-1}{2}\times\frac{n-1}{2}} & I_{\frac{n-1}{2}} \\ O & I_{\frac{n-1}{2}} & O_{\frac{n-1}{2}\times\frac{n-1}{2}} \end{pmatrix}$$

if n is odd.

In this realization the Lie algebra $o(n)$ of $O(n)$, which is the same as that of $SO(n)$, the set of orthogonal matrices with det $g = 1$, consists of all n x n complex matrices Q such that

$$Q^t K + KQ = \underset{n\times n}{0} .$$

The dimension of $o(n)$ with n even is $2(\frac{n}{2})^2 - \frac{n}{2}$ and, if n is odd $2(\frac{n-1}{2}) + (\frac{n-1}{2})$. A basis for $o(2m)$ is given by

$$e_{jk} - e_{k+m,\, j+m}\,, \quad j_1k = 1, \ldots, m$$

$$\left.\begin{array}{l} e_{j+m,k} - e_{k+m_1 j} \\ \\ e_{jk+m} - e_{k,j+m} \end{array}\right\} \quad s \neq k\,,$$

where e_{rs} is the 2m x 2m matrix whose entry in the column r and row s is one and all other entries are zero.

If we label the top row and column of every element in O(2m +1) as zero, then a basis for O(2m + 1) is given by

$$e_{jk} - e_{k+m,\, j+m} \qquad j,k = 1,2,\ldots,m$$

$$\left.\begin{array}{l} e_{ko} - e_{o,k+m} \\ \\ e_{ok} - e_{k+m,0} \end{array}\right\} \qquad k = 1, \ldots m$$

$$\left.\begin{array}{l} e_{j+m,k} - e_{k+m,j} \\ \\ e_{j,k+m} - e_{k,j+m} \end{array}\right\} \qquad j \neq k$$

Let $A_j = e_{j,j} - e_{j+m,j+m,j}$; we define H_m as the aggregate of all diagonal matrices $A(\lambda_1,\ldots,\lambda_m) = \sum_{j=1}^{m} \lambda_j A_j$

DEFINITION 1

Let $C_1,\ldots, C_m$ be complex numbers. The linear functional L defined on H_n by

$$L(A(\lambda_1,\ldots,\lambda_m)) = \sum_{j=1}^{m} \lambda_j C_j,$$

is called a weight.

Let T be a representation of $o(n)$ by operators T(Q), $Q \in o(n)$ on the complex vector space V.

DEFINITION 2. A non-zero vector x in V such that

$$T(A)\, x = L(A)x,$$

for all A in H_m is called a weight vector.

The weights of a given irreducible representation of $o(n)$ can be totally ordered in the following manner: a weight $L = \sum_{j=1}^{m} \lambda_j C_j$ is greater than a weight $L' = \sum_{j=1}^{m} \lambda_j C_j'$ if the first non-zero element in the set $\{C_1 - C_1', \ldots, C_m - C_m'\}$, reading from left to right is positive.

THEOREM 5.1

(a) Two irreducible representations of $o(n)$ with the same highest weight are equivalent.

(b) The highest weight of an irreducible representation of $o(n)$ is of the form

$$P_1\lambda_1 + \ldots + P_n\lambda_n = L(A(\lambda_1, \ldots, \lambda_m)),$$

where P_j is integer for $j = 1, 2, \ldots, m$, and such that

$$P_1 \geqslant P_2 \geqslant \ldots \geqslant P_{m-1} \geqslant | P_m | \qquad \text{if } n = 2m,$$

$$P_1 \geqslant P_2 \geqslant \ldots \geqslant P_m \qquad \text{if } n = 2m + 1$$

Also, every irreducible representation of $o(n)$ is determined by one such sequence.

The sequences $(P_1, \ldots, P_m)$ appearing in the theorem above are called signatures. The signatures $(1, \overset{r}{\ldots}, 1, 0, \ldots, 0)$ are written as (1^r).

THEOREM 5.2

(a) The representations with signature (1^r) are irreducible. The highest weight is (1^m) where $m = \frac{n}{2}$ if n is even, and $m = \frac{n-1}{2}$ if n is odd.

(b) The representation of signature $(P_1, \ldots, P_m)$ is contained exactly once in the tensor representation

$$(1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \ldots \otimes (1^{m-1})^{\otimes k_{m-1}} \otimes (1^m)^{\otimes k_m},$$

where $k_m = P_m$, $k_{m-1} = P_{m-1} - P_m$, ..., $k_1 = P_1 - P_2$, whenever n is odd. Similarly, if n is even, then the representation of signature $(P_1, \ldots, P_m)$ is contained in the tensor representation

$$(1)^{\otimes k_1} \otimes (1^2)^{\otimes k_2} \otimes \ldots \otimes (1^{m-1})^{\otimes k_{m-1}} \otimes (1^{m-1}, -1)^{\otimes k_m} \otimes (1^m)^{\otimes k_{m+1}}$$

with multiplicity one. Here

$$k_j = p_j - p_{j+1}, \quad 1 \leqslant j \leqslant m-2, \text{ and}$$

$$k_{m-1} = P_{m-1} - P_m,\ k_m = 0,\ k_{m+1} = P_m \text{ if } P_m \geqslant 0,$$

$$k_{m-1} = P_{m-1} + P_m,\ k_m = -P_m,\ k_{m+1} = 0 \text{ if } P_m < 0$$

Since the Lie algebra of $SO(n,\mathbb{R})$ is a real form of the complex Lie algebra of $O(n,\mathbb{C})$, then there is a one to one correspondence between the representations of $SO(n,\mathbb{R})$ and $SO(n,\mathbb{C})$.

6. REPRESENTATIONS OF THE PROPER LORENTZ GROUP

We begin this section with some facts about the special linear group $SL(2,\mathbb{C})$ of all matrices g of the form

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where a, b, c, d are complex numbers and det $g = 1$, because this group is important for the representations of the Lorentz group.

The Lie algebra of SL(2,$\mathbb{C}$) is the 3 dimensional space of all 2 x 2 complex matrices α,

$$\alpha = \begin{pmatrix} a_1 & a_2 \\ a_3 & -a_1 \end{pmatrix}$$

Let 2n be a non-negative integer, by $P_{2n}(Z)$ we mean the 2n+1 dimensional linear space of all polynomials

$$f(Z) = \sum_{j=0}^{2n} c_j Z^j,$$

where c_j, j = 1, 2, ... 2n are complex numbers.

The set of all operators $T(g)$, $g \in SL(2,C)$ defined on $P_{2n}(Z)$ by

$$[T(g)f](Z) = (bZ + d)^{2n} f\left(\frac{aZ + c}{bZ + d}\right)$$

is an irreducible representation of SL(2,C) on P_{2n}.

This representation shall be denoted throughout this section by $D^{(u)}$.

The tensor product of two irreducible representations $D^{(u)}$, $D^{(v)}$; $D^{(v)} \otimes D^{(u)} = D^{(u,v)}$ can be completely reduced into a direct sum of irreducible representations,

$$D^{(u,v)} = D^{(u)} \otimes D^{(v)} = \sum_{w=|u-v|}^{u+v} \oplus D^{(w)}$$

This expression is the so-called Clebsch-Gordon series of $D^{(u)} \otimes D^{(v)}$.

DEFINITION 1. The full or homogeneous Lorentz group L(4) is the group of all linear mappings of the real linear space $\mathbb{R}^4$ that leaves the form

$$\langle x,x \rangle = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

invariant.

The usual matrix realization of L(4) is the group of matrices $g = (g_{ij})_{4\times4}$ such that

$$g^t \; I_+ \; g = I_+$$

with $I_+ = (S^{ij})_{4\times4}$, where S^{ij} is defined by

$$S^{ij} = \begin{cases} 1 & \text{if} \quad i = j = 3 \\ -1 & \text{if} \quad i = j = 4 \\ 0 & \text{otherwise} , \end{cases}$$

and g^t is the transpose of g.

The inverse g^{-1} of an element $g \in L(4)$ is given by

$$g^{-1} = q = (q_{ij})_{4\times4} = I_+ \, g^t \, I_+$$

This relation can also be written as

$$q_{ij} = g_{ji} \; S^{jj} \, S^{ii}$$

DEFINITION 2. The proper Lorentz group $L^1(4)$ is the subgroup of L(4) that consists of those elements $g \in L(4)$ such that det $g = 1$ and $g_{44} \geq 1$.

The set consisting of all matrices of the form

$$\begin{pmatrix} & & & 0 \\ & \Theta & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\theta \in SO(3)$, is a subgroup of $L^1(4)$. We shall identify this subgroup with SO(3).

The full Lorentz group L(4) is a linear Lie group, its Lie algebra is six dimensional. The Lie algebra of L(4) is isomorphic to the Lie algebra of $SL(2,\mathbb{C})$ considered as a six dimensional real Lie algebra. Therefore the Lie groups $SL(2,\mathbb{C})$ and $L^1(4)$ are locally isomorphic. There is a one to one relationship between single valued representations of $L^1(4)$ and representations T of $SL(2,\mathbb{C})$ such that $T(-I_2) = T(I_2)$ is the identity operator. Here I_2 is the identity element in $SL(2,\mathbb{C})$.

Therefore, to find the analytic irreducible representations of $L^1(4)$ we compute the analytic irreducible representations T of $SL(2,\mathbb{C})$ considered as a real Lie group and then we single out those that satisfy $T(-I_2) = T(I_2)$.

In particular, the representations $D^{(u,v)}$ determine a single valued representations of $L^1(4)$ if and only if $u + v$ is an integer.

THEOREM 6.1

Let V be a four dimensional real space. Let $V^{\otimes n}$ be the n-fold tensor product of V. The representation

$$(ga)_{i_1 \ldots i_n} = \sum_{j_- - j_{n=1}}^{4} g_{i_1 \ldots i_n,\, j_i \ldots j_n} \; a_{j_i \ldots j_n}$$

on $V^{\otimes n}$ is equivalent to $(D^{(\frac{1}{2},\frac{1}{2})})^{\otimes n}$. Every irreducible component of $(D^{(\frac{1}{2},\frac{1}{2})})^{\otimes n}$ is equivalent to an irreducible single valued representation of $L^1(4)$, and every single valued irreducible representation of $L^1(4)$ can be so obtained.

7. VECTOR INVARIANTS

In this section we state some well-known definitions and facts about vector invariants of the rotation and Lorentz group.

DEFINITION 1 Let G be a group of transformations on a set X. A relative invariant of G of weight χ is a mapping $f: X \to \mathbb{R}$ such that

$$f(gx) = \chi(g)\ f(x),$$

for all $g \in G$ and $X \in X$.

A relative invariant whose weight X is constant on G is called an absolute invariant.

A situation that holds in most important cases is that every invariant is expressible in terms of a finite number of them. That is, there is a finite, complete table, of basic invariants of G $\{\psi_1, \ldots \psi_n\}$ such that every invariant $f(x)$ can be expressed as $f(x) = F(\psi_1(x), \ldots, \psi_n(x))$.

In the case with $X = \mathbb{R}^n$ and $G = SO(n)$ a complete table of basic invariants consists of the inner product $(x,y) = x_1 y_1 + \ldots + x_n y_n$ and the determinant of n vectors

$$[x_1 \ldots x_n] = \begin{vmatrix} x_1^1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & & \vdots \\ x_n^1 & x_n^2 & \cdots & x_n^n \end{vmatrix}$$

DEFINITION: 2 An homogeneous form of degree r of n vectors $x_1 \ldots x_m$ is an expression of type

$$f(x_1 \ldots x_m) = \Sigma\, a_{r_1 \ldots r_n}\, x_1^{r_1} \ldots x_m^{r_m}$$

where $r_1 + \ldots + r_m = r$.

THEOREM: 7.1 Every invariant form for $SO(n)$ of a number of vectors $x_1 \ldots x_m$ in $\mathbb{R}^n$ can be expressed as a sum of a function $F((x_r, x_s))$ of the inner products and terms of the form

$$[y_1, \ldots, y_n] \times F^*(x_1, \ldots, x_m)$$

where the vectors $y_1, \ldots y_n$ are selected from $\{x_1, \ldots, x_m\}$ and $F^*((x_r, x_s))$ depends only on the inner products $(x_r, x_s))$.

Terms of the form $[y_1, \ldots, y_n] \times F^*((x_r, x_s))$ appear if and only if $m \geqslant n$.

In the case of the Lorentz group a complete table of basic invariants consists of mixed products $[x, u] = \sum_i x_i u^i$ of a covariant vector $x = (x_1, \ldots, x_4)$ and a contravariant $u = (u^1, \ldots, u^4)$. The form $\langle x, y \rangle = \sum_{i=1}^{4} x_i y_i\, S^{ii}$ and determinants $[u_1 \ldots u_4]$ of contravariant vectors and determinants $[x^1 \ldots x^4]$ of covariant vectors $x^1, \ldots, x^4$.

The product of a determinant with covariant vectors and a determinant with contravariant vectors is equal to a determinant whose entries are mixed products of covariant and contravariant vectors, that is

$$[U_1 \ldots U_4][x^1 \ldots x^4] = \begin{vmatrix} [U_1,x^1] & \ldots & [U_1,x^4] \\ \vdots & & \vdots \\ [U_4,x^1] & \ldots & [U_4,x^4] \end{vmatrix}$$

THEOREM 7.2 Every invariant form for the proper Lorentz group $L^1(4)$ of a number of covariant vectors $x_1,\ldots,x_m$ and a number of contravariant vectors $y^1, \ldots, y^p$ can be expressed as a linear combination of products of the èlements in its complete table of basic invariants.

CHAPTER II

APPROPRIATE FUNCTIONAL EQUATIONS AND THEIR FORMS

INTRODUCTION

In this chapter we study linear transformations, on linear spaces $A(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$, of a type that J.L.B. Cooper [4], [5] has termed appropriate representations of groups.

We restrict ourselves to transformations in which the operator acting on $\mathbb{R}^n$ is affine in a sense which we define below. Since our ultimate aim is to study appropriate representations of specific groups, further restrictions appear naturally.

After reducing these transformations into two canonical forms, we undertake the study of functional equations for linear operators T on spaces of functions that intertwine between appropriate representations of groups. These equations are split into a system of functional equations. One of these equations asserts that the linear operators T are translation invariant.

In Section 3 we include results about translation invariant operators due to J.L.B. Cooper, yet unpublished.

Section 1: APPROPRIATE REPRESENTATIONS

DEFINITION 1.1.

Let $A(\mathbb{R}^n)$ be a linear space of functions on $\mathbb{R}^n$ with values on a linear space E, such that $A(\mathbb{R}^n)$ separates the points of $\mathbb{R}^n$. A linear transformation W on $A(\mathbb{R}^n)$ to itself is called an appropriate transformation if for each x in $\mathbb{R}^n$ and f in $A(\mathbb{R}^n)$ the value of $Wf(x)$

depends exclusively on the value of f at some point in $\mathbb{R}^n$, say Vx, or, (as in the case of space of functions defined only up to sets of measure zero) if a similar statement is true in the limit for functionals on $A(\mathbb{R}^n)$ whose support tend to x.

Equivalently, W is an appropriate transformation, if there is for each $x \in \mathbb{R}^n$ a linear map

$$Q(x) : A(\mathbb{R}^n) \to A(\mathbb{R}^n)$$

and a map $V: \mathbb{R}^n \to \mathbb{R}^n$ such that

$$Wf(x) = Q(x)\, f(Vx)$$

Change of variable is an example of this.

DEFINITION 1.2. Let $A(\mathbb{R}^n)$ be a linear space of functions defined on $\mathbb{R}^n$ with values in a linear space E such that $A(\mathbb{R}^n)$ separates the points of $\mathbb{R}^n$. Let G be a group.

A set of appropriate transformations of the form

$$W(g)\; f(x) = Q(x,g)\; f(V(g)x)$$

is called an appropriate representation of the group G on $A(\mathbb{R}^n)$ if,

(i) $W(g_1 g_2) = W(g_1)W(g_2)$, $W(e) = I$, where $g_1, g_2 \in G$, e is the identity element in G and I is the identity transformation on E,

(ii) $V(g_1 g_2)x = V(g_2)\,[V(g_1)x]$, for $g_1, g_2 \in G$ and $x \in \mathbb{R}^n$.

(iii) $Q(x,g)$ is, for every $x \in \mathbb{R}^n$ and $g \in G$, a linear operator on E.

DEFINITION 1.3. Let $W(g)$ and $W^*(g)$ be appropriate representations of a group G acting on the spaces $A(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$ respectively. An operator T from $A(\mathbb{R}^n)$ to $B(\mathbb{R}^n)$ is said to obey an appropriate functional equation if, for every $f(x)$ in $A(\mathbb{R}^n)$, we have that

$$[TW(g)f(x)]\ (u) = W^*(g)\,[Tf(x)]\,(u) \qquad \text{(II.2.1)}$$

We shall, for brevity, sometimes write this equation as

$$TW(g) = W^*(g)T,$$

and in this case we shall also say that T intertwines with the representations $W(g)$ and $W^*(g)$.

The study of linear transformations obeying specific functional equations goes back to M. PLANCHEREL [17]. This author studied the properties of the solutions of the functional equation in the linear operator T,

$$[Tf(\alpha x)](y) = \alpha^{-1}\,[Tf(x)]\ (\alpha^{-1}y),$$

for all $\alpha > 0$, where $f \in L_2(0,\infty)$.

Plancherel called these operators, Watson transforms.

A more systematic study of this subject has been undertaken by H. Kober [11], [12] and J.L.B. Cooper [4], [3]. The first of these authors has studied pairs of functional equations aiming at finding relations between the solutions of each individual equation. These pairs of functional equations include almost all the familiar integral transforms of functions of one variable.

Kober has also characterized concrete examples of transforms that obey one or more functional equations. In particular, characterizations of the Fourier transform appear in [12], theorems 6 and 6'. The first of these is due to J. L. B. Cooper.

Some single functional equations characterize almost uniquely the operator involved. However, in general, pairs of equations are needed, since very often single functional equations are satisfied by more than one transform. For instance, the Weyl fractional integral,

$$[I_\alpha^+ \; f(x)] \; (y) = \Gamma^{-1}(\alpha) \int_{-\infty}^{y} (y-x)^{\alpha-1} f(x)dx$$

where $\Gamma(\alpha)$ stands for the Gamma function, and a convolution transform

$$[Cf(x)] \; (y) = \int_{-\infty}^{\infty} g(x-y) \; f(x)dx,$$

satisfy the functional equation

$$[Tf(x+\alpha)](y) = [Tf(x)] \; (y+\alpha).$$

The problem of linear operators intertwining between representations of the group of translations of the real line is dealt with by J.L.B. Cooper in [4] Under suitable hypothesis these operators fall into four canonical forms, of which the most important are

1) $[Tf(x+h)](y) = [Tf(x)](y+h) \quad , y, h \in \mathbb{R},$

2) $[Tf(x+h)](y) = e^{h \cdot q(u)} \; [Tf(x)] \; (y)$

with $h \in \mathbb{R}$ and y in some interval $E \subset \mathbb{R}$.

The study of necessary conditions for there to exist a non-zero solution of the equation (1) is discussed in [4]. The corresponding problem for higher dimensions is discussed in [5].

DEFINITION 1.4. Let G and H be groups. We shall denote the elements of G as g_i and those of H by h_i. Let $g \to vg$ be a homomorphism of G into the group of automorphisms of H. Define the product of ordered pairs (g,h), $g \in G$, $h \in H$ by the formula

$$(g_1,h_1)(g_2,h_2) = (g_1g_2,\ vg_1(h_2)\cdot h_1).$$

It is easy to verify that the set $G \times H$ of all ordered pairs (g,h) becomes a group with respect to the group operation defined above. This group is called the semidirect product of the groups. The identity element in $G \times H$ is (e,e') where e and e' are the identity elements in G and H respectively. The inverse of the element (g,h) is given by $(g^{-1},\ vg^{-1}(h^{-1}))$.

Every element (g,h) can be factorized as

$$(g,h) = (e,h)\ (g,e') = (g,e')\ (e,vg^{-1}(h)).$$

We shall, for brevity, write $\tilde{g}$ for the pair (g,h); g for the pair (g,e') and h for the pair (e,h), wherever no misunderstanding arises.

Let us consider G to be a group of linear transformations from $\mathbb{R}^n$ onto $\mathbb{R}^n$ and H to be the group of translations of $\mathbb{R}^n$. By considering the natural action of the group G on $\mathbb{R}^n$, namely $g \to vg = g$, we get the semidirect product $G\widetilde{M}(n)$ whose realization as a group of transformations on $\mathbb{R}^n$ is defined by

$$(g,h)x = gx + h.$$

It is easy to verify that this realization conforms with the group structure of $G\widetilde{M}(n)$.

2. THE FORMS OF SOME APPROPRIATE REPRESENTATIONS

In this section we shall find the form of appropriate representations of semidirect products $G\widetilde{M}(n)$ as defined above under rather general hypotheses set upon the group G.

We consider appropriate representations of the group $G\widetilde{M}(n)$ on a space $A(\mathbb{R}^n)$ of functions on $\mathbb{R}^n$ with values in a vector space E, where $A(\mathbb{R}^n)$ separates the points of $\mathbb{R}^n$, of the form (II.2.2) $[W(\tilde{g})f](x) = Q(x,\tilde{g})f(V(\tilde{g})x)$, $x \in \mathbb{R}^n$, $\tilde{g} \in G\widetilde{M}(n)$ with the additional hypothesis that the multiplier $Q(x,\tilde{g})$, and $V(\tilde{g})x$ are continuous in $\tilde{g}$ and x. We assume further that the operator $V(\tilde{g})$ on $\mathbb{R}^n$ is affine, that is

$$V(\tilde{g})x = A(\tilde{g})x + B(\tilde{g}),$$

where $A(\tilde{g})$ is a linear operator on $\mathbb{R}^n$ into $\mathbb{R}^n$ and $B(\tilde{g})$ is an element in $\mathbb{R}^n$, for every $\tilde{g} \in G\widetilde{M}(n)$. This hypothesis stems from the fact that this is the case in many important applications.

The following observation will further restrict the forms of $V(\tilde{g})$. Let us consider the semidirect product $E^+(n)$ of the proper rotation group SO(n) and the group of translation $\tau(n)$ on $\mathbb{R}^n$, that is

$$E^+(n) = SO(n) \times \tau(n).$$

The elements $\tilde{g}$ of $E^+(n)$ are ordered pairs $\tilde{g} = (g,h)$, $g \in SO(n)$, $h \in \tau(n)$; the group action is defined by

$$\tilde{g}x = (g,h)\,x = gx + h.$$

Let $g = (g,0)$ be an element in SO(n). Continuity of $V(\tilde{g})x$ implies that $V(g)x$ must be uniformly bounded over all $g \in SO(n)$ over any bounded region of x. We deduce from the definition of $V(\tilde{g})x$ that

$$V(g^n)x = A^n(g)x + \sum_{i=0}^{n-1} A^i(g)B(g)$$

and this expression is bounded over any bounded region of x provided that $A^n(g)$ is uniformly bounded in n ; whence $\|A(g)\| \leq 1$ for any $g \in SO(n)$, and thus also $\|A^{-1}(g)\| \leq 1$.

We deduce that $A(g)$ is, for every g, an isometry, so that

$$(A(g)(x+y), A(g)(x+y)) = (x+y, x+y),$$

for any g in SO(n) and x,y in $\mathbb{R}^n$. Since $A(g)$ is linear we have that

$$(A(g)x, A(g)y) = (x,y).$$

Therefore $A(g)$ is an element in SO(n) for every g in SO(n).

From condition (ii) in definition 1.2 of the previous section we deduce that

$$A(g_1 g_2) = A(g_2)A(g_1),$$

for all g_1, g_2 in SO(n), that is, the mapping T: SO(n) $\to$ SO(n) defined by

$$T(g) = A(g^{-1})$$

is an endomorphism of SO(n).

There are two cases to be considered: (a) T is an automorphism, (b) the kernel of T is non-trivial. In either case we shall assume that $n \geq 3$.

In case (a) we have that $T(g)$ is an irreducible unitary representation of SO(n). We recall that the natural action of SO(n) on $\mathbb{R}^n$ is irreducible and has signature (1,0,...,0). We claim that this is the signature of $T(g)$, and consequently $T(g)$ is equivalent to SO(n).

In fact, the dimension (or degree), $N(p_1,\dots,p_m)$ of an irreducible representation of SO(2m) with signature $(p_1,\dots,p_m)$ is given by

$$N(p_1,\dots,p_m) = \frac{\prod_{0<j<k\leq n} (\ell_j-\ell_k)(\ell_j+\ell_k)}{\prod_{0<j<k\leq n} (\overset{\circ}{\ell}_j-\overset{\circ}{\ell}_k)(\overset{\circ}{\ell}_j+\overset{\circ}{\ell}_k)},$$

where $\ell_j = p_j + m - j$ and $\overset{\circ}{\ell}_j = m - j$. A similar formula gives the dimensions of the irreducible representations of SO(2m + 1). (See Boerner [2] p.266 or Miller [16] p.362).

We see by simple inspection of this that, if the signature $(p_1,\dots,p_m)$ of the representation $T(g)$ is different from $(1,0,\dots,0)$, then $N(p_1,\dots,p_m) > 2m$. This contradicts the fact that $T(g)$ is a representation of dimension 2m. A similar proof follows if $n = 2m + 1$.

Therefore, the representation $T(g)$ is equivalent to SO(n), that is

$$T(g) = BgB^{-1}$$

for some matrix B. It is straightforward to check that B is an orthogonal matrix. If B is in SO(n) then after a rotation of axes we can take $T(g) = g$. If B is an orthogonal matrix not in SO(n) and n is odd, then $B = -I_n \tilde{g}$ for some fixed element $\tilde{g}$ in SO(n). We see that this case reduces to the one above. If, on the other hand, n is even, then B is of the form $(\varepsilon_r \varepsilon_s \tilde{g}_{rs})$, where $\tilde{g} = (\tilde{g}_{rs})$ in a fixed element in SO(n) and $\varepsilon_r = 1$ for $1 \leq r \leq n - 1$, $\varepsilon_n = -1$. Thus B is a change of coordinates even if not orientation preserving.

Therefore, if $T(g)$ is an automorphism, then under a rotation of axis (proper or improper) we can take $A(g) = g^{-1}$.

In order to examine the case (b), let us begin by recalling that for $n \geqslant 3$, $n \neq 4$, the Lie algebras of the complex groups SO(n) are simple (see Miller [16] p.392), that is, they do not contain any proper ideals. This implies that the Lie algebras of the real groups SO(n), $n \geqslant 3$, $n \neq 4$, are simple (see Hausner and Schwartz [10] Corollary 13 p.99).

A Lie group G is locally simple, that is, G does not contain any proper normal local Lie subgroup, if and only if its Lie algebra is simple, (see Miller [16] th.9. 14 p.394).

From these remarks and the hypothesis that the kernel of $T(g)$ is non-trivial, we deduce that, for $n \geqslant 3$, $n \neq 4$,

$$T(g) = I_n$$

for all g in SO(n).

We shall therefore consider only the cases when $A(g) = g^{-1}$ and $A(g) = I$ respectively.

The forms that $B(g)$ can attain are found in the following:

<u>LEMMA 1</u> Let $V(g)$, $g \in SO(n)$, be the set of operators on $\mathbb{R}^n$ defined by

$$V(g)x = A(g)x + B(g), \; x \in \mathbb{R}^n, \; g \in SO(n),$$

and such that

$$V(g_1g_2) = V(g_2)V(g_1), V(I_n) = I_n.$$

Also, let $V(g)x$ be continuous in g and x .

If either $A(g) = g^{-1}$ or $A(g) = I_n$, for all $g \in SO(n)$, then $B(g) = 0$.

Proof: If $A(g) = I_n$, then the expression

$$V(g^n)x = A^n(g)x + \sum_{i=0}^{n-1} A^i(g)\, B(g)$$

becomes

$$V(g^n)x = x + n\, B(g),$$

so that B(g) must be zero in order that V(g)x be uniformly bounded over any bounded region of x.

Let us now consider the case with $A(g) = g^{-1}$. To begin with we notice that if $\tilde{g}_1 = (g_1,0)$ and $\tilde{g}_2 = (g_2,0)$ then

$$\begin{aligned} V(g_1.g_2)x &= (g_1g_2)^{-1}x + B(g_1g_2) \\ &= V(g_2)V(g_1)x \\ &= V(g_2)[g_1^{-1}x + B(g_1)] \\ &= g_2^{-1}g_1^{-1}x + g_2^{-1}B(g_1) + B(g_2), \end{aligned}$$

so that

$$B(g_1g_2) = g_2^{-1}B(g_1) + B(g_2).$$

It can now be easily proved by induction that

$$B(g_1\ldots g_s) = g_s^{-1}\ldots g_2^{-1}B(g_1) + g_s^{-1}\ldots g_3^{-1}B(g_2) + \ldots + \\ + g_s^{-1}B(g_{s-1}) + B(g_s)\,. \qquad \text{(II.2.2)}$$

Since $B(I_{nxn},0) = B(I_{nxn}) = 0_{nxn}$, then (II.2.2) gives, with $g_1 = g^{-1}$, $g_2 = g$ and $g_s = I_{nxn}$ for $s \geqslant 3$,

$$B(I) = g^{-1}B(g^{-1}) + B(g),$$

$$B(g) = -g^{-1}B(g^{-1}) \qquad \text{(II.2.3)}$$

Every rotation $g \in SO(n)$ can be expressed as a product of rotations $g_{j,k}(\theta)$ of the (x_j, x_k) plane through an angle θ in the (x_j, x_k) plane and such that the rotation that takes place $e_j = (0,\ldots,\underset{j}{1},\ldots,0)$ to the vector $e_k = (0,\ldots,\underset{k}{1},\ldots,0)$ is assumed positive. (See Vilenkin [29] theorem 1, p.438). Let us consider the case when $g_{j,k}(\theta)$ is $g_{1,2}(\theta)$, the rotation of the (x_1, x_2) plane which leaves invariant the subspaces with coordinates $(x_3,\ldots,x_n)$, all other cases reduce to this by a change of axis.

The expression (II.2.2) with

$$g_1,\ldots,g_s = g_{1,2}(0)$$

implies that $B_r(g_{1,2}(\theta))$ is zero for $r \geqslant 2$, where $B_r(g)$, $r = 1,2,\ldots,n$ are the coordinates of the vector $B(g)$, otherwise $B(g)$ would not be bounded. Thus, the problem is reduced to that of examining the forms of

$$B(g_{1,2}(\theta)) = (B_1(g_{1,2}(\theta)), B_2(g_{1,2}(\theta))).$$

In terms of the complex plane $g_{1,2}(\theta)$ is the rotation θ that takes $z \in \mathbb{C}$ into $e^{i\theta}Z$, $V(\theta)$ takes Z into $e^{-i\theta}Z + B(\theta)$.

Setting $g_1 = e^{i\alpha}$, $g_2 = e^{i\beta}$ in (II.2.2) we obtain

$$B(\alpha+\beta) = e^{-i\alpha}B(\beta) + B(\alpha) = e^{-i\beta}B(\alpha) + B(\beta),$$

so that

$$\frac{B(\beta)}{e^{-i\beta}-1} = \frac{B(\alpha)}{e^{-i\alpha}-1} = K,$$

for some constant K and $\alpha, \beta \neq 2m\pi$, $m = 0, 1, \ldots$ The expression

$$B(\alpha) = K(e^{-i\alpha}-1)$$

holds for any α, since $B(2m\Pi) = 0$ for $m = 0, 1, \ldots$ We now deduce from (II.2.3) above that

$$K(e^{-i\alpha}-1) = -K\, e^{i\alpha}(e^{i\alpha}-1).$$

This equality is true if and only if $K = 0$; hence $B(g_{1,2}(0)) = 0$.

Finally, we deduce from (II.2.2) that $B(g)$ is zero for any $g \in SO(n)$. The Lemma is now proved.

The appropriate representations of the Euclidean groups on some linear spaces of functions are important for the applications and so are those of groups that contain Euclidean groups as subgroups; which we, now, are about to study.

Therefore, we shall restrict ourselves to examining appropriate representations of groups $G\widetilde{M}(n) = G \times \tau(n)$ in the following cases:

$$V(g) = g^{-1},\ g \in G \tag{II.2.4}$$

$$V(g) = I_{n\times n},\ g \in G \tag{II.2.5}$$

We begin with examining the case (II.2.4). In the first place we prove a result concerning the multiplier $Q(x,\tilde{g})$, $x \in \mathbb{R}^n$, $\tilde{g} \in G\widetilde{M}(n)$ and draw some functional relations involving the operator V.

LEMMA 2. Let $W(\tilde{g})f(x) = Q(x,\tilde{g})f(V(\tilde{g})x)$ be an appropriate representation of $G\widetilde{M}(n)$ on a space $A(\mathbb{R}^n)$ where

(i) $A(\mathbb{R}^n)$ is a linear space of functions defined on $\mathbb{R}^n$ into a linear space E, which separates the points of $\mathbb{R}^n$.

(ii) $Q(x,\tilde{g})$ and $V(\tilde{g})x$ are continuous in x and $\tilde{g}$, with $V(\tilde{g})x = A(\tilde{g})x + B(\tilde{g})$ and $V(g)x = V(g,0)x = g^{-1}x$.

Then,

(a) For any x in $\mathbb{R}^n$ and $\tilde{g} \in \tilde{GM}(n)$, $Q(x,\tilde{g})$ is invertible, $Q(x,(e,0))$ is the identity operator on E for all x and

$$Q(x,\tilde{g})^{-1} = Q(V(\tilde{g})x, \tilde{g}^{-1})$$

(b) For any $h \in \tau(n)$ and any $g \in G$ we have that

$$V(h)gx = gV(g^{-1}h)x$$

for all x in $\mathbb{R}^n$.

Proof: The operator $Q(x,(e,0))$ coincides with $W(e,0)$; hence $Q(x,e)$ is the identity operator on E for every x.

By definition we have that

$$\begin{aligned} W(\tilde{g}_1\tilde{g}_2)f(x) &= Q(x,\tilde{g}_1\tilde{g}_2) \ f(V(\tilde{g}_1\tilde{g}_2)x) \\ &= W(\tilde{g}_1) \ [W(\tilde{g}_2)f(\alpha)] \\ &= W(\tilde{g}_1) \ [Q(x,\tilde{g}_2) \ f(V(\tilde{g}_2)x)] \\ &= Q(x,\tilde{g}_1) \ Q(V(\tilde{g}_1)x,\tilde{g}_2) \ f(V(\tilde{g}_2)V(\tilde{g}_1)x), \end{aligned}$$

so that

$$Q(x,\tilde{g}_1\tilde{g}_2) = Q(x,\tilde{g}_1) \ Q(V(\tilde{g}_1)x, \tilde{g}_2)$$

holds for every x in $\mathbb{R}^n$ and g_1,g_2 in $\tilde{GM}(n)$. This expression with $\tilde{g}_1 = \tilde{g}$, $\tilde{g}_2 = \tilde{g}^{-1}$ gives

$$Q(x,\tilde{g}) \ Q(V(\tilde{g})x,\tilde{g}^{-1}) = Q(x,(e,0)),$$

so that $Q(x,\tilde{g})$ is invertible for any $x \in \mathbb{R}^n$ and $\tilde{g} \in \tilde{GM}(n)$ and

$$Q(x,\tilde{g})^{-1} = Q(V(\tilde{g})x,\tilde{g}^{-1}).$$

Also, setting $x = 0$ in the same expression, we see that

$$Q(0,\tilde{g}_1\tilde{g}_2) = Q(0,\tilde{g}_1)\, Q(0,\tilde{g}_2),$$

this is to say, $Q(0,\tilde{g})$ is a representation of $G\tilde{M}(n)$ on E as $\tilde{g}$ ranges over $G\tilde{M}(n)$.

Since every element $\tilde{g} \in G\tilde{M}(n)$ can be factorized as

$$\tilde{g} = (e,h)(g,0) = (g,0)\,(e,g^{-1}h),$$

it follows from $V(\tilde{g}_1\tilde{g}_2) = V(\tilde{g}_2)\, V(\tilde{g}_1)$ that

$$V(\tilde{g}) = V(g^{-1}h)\, V(g)$$

$$= V(g^{-1}h)g^{-1}$$

and that

$$V(\tilde{g}) = V(g)\, V(h)$$

$$= g^{-1}V(h),$$

thus

$$g^{-1}[V(h)x] = V(g^{-1}h)[g^{-1}x]$$

for every $\tilde{g} \in G\tilde{M}(n)$ and all $x \in \mathbb{R}^n$. This in turn can be written as

$$V(h)\, gx = gV(g^{-1}h)x\ . \tag{II.2.6}$$

LEMMA 3. Let the hypothesis of Lemma 2 hold.

Then,

(a) for every g in G and h in $\mathbb{R}^n$ we have that

$$B(gh) = gB(h)$$

and

$$A(h)g = gA(g^{-1}h),$$

(b) $$V(h)\,B(k) = V(k)\,B(h) = B(h+k)$$

for all h, k in $\mathbb{R}^n$,

(c) for every set $\{h_i\}_{i=1}^m$ of vectors in $\mathbb{R}^n$ we have that

$$A\left(\sum_{i=1}^{m} h_i\right) = \prod_{i=1}^{n} A(h_i)$$

Proof: The proof follows from the expression II.2.6 found in the previous lemma, as we show:

$$\begin{aligned} V(h)gx &= A(h)gx + B(h) \\ &= gV(g^{-1}h)x \\ &= gA(g^{-1}h)x + gB(g^{-1}h). \end{aligned}$$

Hence

$$gB(g^{-1}h) = B(h)$$

and

$$A(h)g = gA(g^{-1}h).$$

Also, $$V(h)0 = A(h)0 + B(h) = B(h),$$

$$\begin{aligned} V(h_1 + h_2)0 &= B(h_1 + h_2) \\ &= V(h_1)\,V(h_2)0 \\ &= V(h_1)\,B(h_2) \end{aligned}$$

$$= A(h_1)\ B(h_2) + B(h_1)$$

$$= A(h_2)\ B(h_1) + B(h_2);$$

therefore,

$$\begin{aligned} V(h_1 + h_2)x &= A(h_1 + h_2)x + B(h_1 + h_2) \\ &= V(h_1)\ [A(h_2)x + B(h_1)] \\ &= A(h_1)A(h_2)x + A(h_1)B(h_2) + B(h_1) \\ &= A(h_1)A(h_2)x + V(h_1)B(h_2) \\ &= A(h_1)A(h_2)x + B(h_1 + h_2); \end{aligned}$$

hence $\qquad A(h_1)\ A(h_2) = A(h_1 + h_2) = A(h_2)\ A(h_1).$

It now follows by induction that

$$A(\sum_{i=1}^{m} h_i) = \prod_{i=1}^{n} A(h_i).$$

We have thus seen that the operators A(h) and B(h) appearing in $V(\tilde{g})$ satisfy the functional equations

$$\begin{cases} \text{(a)} & B(gh) = g\ B(h) \\ \text{(b)} & A(\sum_{i=1}^{m} h_i) = \prod_{i=1}^{m} A(h_i) \\ \text{(c)} & A(h_2)\ B(h_1) + B(h_2) = B(h_1 + h_2) \\ \text{(d)} & A(h)g = gA(g^{-1}h) \end{cases} \qquad \text{(II.2.7)}$$

The forms that A(h) and B(h) admit will be found under hypotheses placed upon the group G, which we deem to be general enough. In particular, some of the most important groups which appear in theoretical physics fulfill these hypotheses.

THEOREM 2.1. Let G be a group of transformations acting irreducibly on $\mathbb{R}^n$, $n \geq 4$. Let us denote the orthogonal complement of any set X in $\mathbb{R}^n$ by $X^\perp$. If, for each vector e_k in an orthogonal basis $\{e_i\}_{i=1}^n$, the set

$$G_k = \{g \in G \; ; \; ge_k = e_k\}$$

acts irreducibly on $\{e_k\}^\perp$, then the only solutions to the system of functional equations (II.2.7) are

$$A(h) = I_{n\times n} \quad , \quad B(h) = Ch$$

for all $h \in \mathbb{R}^n$, where C is an arbitrary real constant and $I_{n\times n}$ is the n-th rank identity matrix.

Proof: For the proof we consider the matrix realization of all operators involved, induced by the orthogonal basis $\{e_i\}_{i=1}^n$.

Given a fixed index k, we deduce from the expression

$$A(h)g = gA(g^{-1}h), \qquad \text{(II.2.7) (d)},$$

by replacing $h = \lambda e_k$ and $g \in G_k$, that

$$A(\lambda e_k)g = gA(\lambda e_k),$$

so that

$$A'(\lambda e_k)g' = g' A'(\lambda e_k),$$

where $A'(\lambda e_k)$ and g' are the matrices obtained from $A(\lambda e_k)$ and g by removing the row and column k. On account of this expression and the hypothesis that G_k acts irreducibly on $\{e_k\}^\perp$ we deduce that for each k there is a real function $\mu'_k(\lambda)$ such that

$$A'(\lambda e_k) = \mu'_k(\lambda) \; I_{(n-1)x(k-1)} .$$

In turn, this and the hypothesis that $n \geqslant 4$ imply that $A(\lambda e_n)$ itself is diagonal.

Therefore, $A(x) = \prod_{i=1}^{n} A(x_i e_i)$ is diagonal for every $x \in \mathbb{R}^n$.

We now deduce from (II.2.7) (d) that

$$A(gh) = A(h)$$

for every $g \in G$ and every $h \in \mathbb{R}^n$; whence A(h) is, for every $h \in \mathbb{R}^n$, a multiple of the identity; that is

$$A(h) = a(h) \; I_{nxn} ,$$

where a(h) is some real continuous function.

$$\text{Therefore,} \qquad A(h) = \prod_{i=1}^{n} a(h_i e_i) \; I_{nxn}$$
$$= e^{\sum_{i=1}^{n} h_i \alpha_i} \; I_{nxn},$$

for every $h \in \mathbb{R}^n$, where α_i, i = 1, 2, ..., n are constants.

If the constants α_i, i = 1,2,...,n are not all zero, then we deduce from the expression for A(h) above and the fact of being invariant under the action of G that the subspace of $\mathbb{R}^n$ whose elements satisfy the equation $\sum_{i=1}^{n} \alpha_i h_i = 0$, $h \in \mathbb{R}^n$ is invariant under G. This, however, contradicts the definition of G.

Hence, $A(h) = \underset{n x n}{I}$ for all $h \in \mathbb{R}^n$.

We now deduce from (II.2.7) (c) that

$$B(h_1 + h_2) = B(h_1) + B(h_2)$$

for $h_1, h_2 \in \mathbb{R}^n$, and since B is continuous it must be linear. That is, there is some square matrix B_0 of rank n such that

$$B(h) = B_0 \, . \, h.$$

Thus, the expression $B(gh) = gB(h)$ becomes

$$B_0 g = g B_0,$$

whence $B_0 = C \underset{n x n}{I}$ and $B(h) = Ch$ for all $h \in \mathbb{R}^n$ where C is some real constant. The theorem is now proved.

We now pass on to examine the forms of the appropriate representation $W(\tilde{g})$ in the case when $V(\tilde{g})x = g^{-1}x + Cg^{-1}h$. First of all, we see that by a change in the scale $V(\tilde{g})$ can be written as

$$V(\tilde{g})x = g^{-1}x - g^{-1}h;$$

whenever $C \neq 0$. Let us now see how this expression affects the representation $W(\tilde{g})$.

In the proof of Lemma 2 we saw that

$$Q(x, \tilde{g}_1\tilde{g}_2) = Q(x, \tilde{g}_1) \; Q(V(\tilde{g}_1)x, \; \tilde{g}_2)$$

for all $x \in \mathbb{R}^n$ and all $\tilde{g}_1, \tilde{g}_2 \in \widetilde{GM}(n)$, then setting $x = 0$, $\tilde{g}_1 = (I, -h_1)$ $\tilde{g}_2 = (I, h_2)$ in this expression we obtain

$$Q(0, \, h_2 - h_1) = Q(0, \, -h_1) \; Q(h_1, h_2)$$

and, by virtue of $Q(x,\tilde{g})$ being invertible for every $x \in \mathbb{R}^n$ and every $\tilde{g} \in G\tilde{M}(n)$ we have that

$$Q(h_1,h_2) = Q(0,-h_1)^{-1}Q(0,h_2-h_1).$$

This factorization of $Q(x,h)$ allows us to simplify the action of $W(h)$ as follows.

Let us consider the space of all functions of the form

$$\hat{f}(x) = Q(0,-x)\, f\,(x)$$

where $f(x) \in A(\mathbb{R}^n)$. Let us define $\hat{W}(h)$ on the space of functions $\hat{f}(x)$ by

$$\begin{aligned}
\hat{W}(h)\,\hat{f}(x) &= Q(0,-x)\, W(h)[\,Q^{-1}(0,-x)\,\hat{f}\,(x)]\\
&= Q(0,-x)\,Q(x,h)\, f\,(x-h)\\
&= Q(0,-x)\,Q^{-1}(0,-x)\;Q(0,h-x)\, f(x-h)\\
&= \hat{f}(x-h)\\
&= \tau(h)\,\hat{f}\,(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
\hat{W}(\tilde{g})\,\hat{f}(x) &= \hat{W}(h)\,W(g)\,\hat{f}(x)\\
&= \tau(h)\,[\,Q(x,g)\,\hat{f}(g^{-1}x)]\\
&= Q(x-h,g)\,\hat{f}\,(g^{-1}x-g^{-1}h).
\end{aligned}$$

Similarly,

$$\hat{W}(\tilde{g})\,\hat{f}(x) = W(g)\,\tau\,(g^{-1}h)\hat{f}(x)$$

$$= W(g)\,\hat{f}\,(g^{-1}x - g^{-1}h)$$

$$= Q(x,g)\,\hat{f}\,(g^{-1}x - g^{-1}h).$$

Putting these results together we see that

$$Q(x-h,g) = Q(x,g),$$

that is $Q(x,g)$ is independent of x. Thus $Q(x,g) = Q(g)$ is some representation of G on the space E of values of $\hat{f}(x)$.

We see thus that the appropriate representation $W(\tilde{g})\,f(x) = Q(x,\tilde{g})\,f\,(g^{-1}x - g^{-1}h)$ can be reduced to the form

$$W(\tilde{g})\,f\,(x) = Q(g)\,f\,(g^{-1}x - g^{-1}h) \qquad \text{(II.2.8)}$$

where $Q(g)$ is some representation of G on E.

REMARKS

(1) In the next chapter we shall study appropriate functional equations where the group $G\tilde{M}(n)$ will be taken to be the Euclidean and Poincaré groups. In this respect we point out that the rotation groups SO(n), $n \geq 4$, and the Lorentz group satisfy the hypotheses of theorem 1 above. It can also be noticed that the proof of this theorem relies heavily on the condition that $n \geq 4$. Furthermore, in the case of SO(2) the system (II.2.7) admits solutions different from those given by theorem 1.

To prove this statement we begin by noticing that the system of functional equations

$$\begin{cases} A(gh)g = gA(h) \\ A(h_1+h_2) = A(h_1)A(h_2) \end{cases}$$

where $g \in SO(2)$, $h_1, h_2 \in \mathbb{R}^2$ and $A(0) = \underset{2\times 2}{I}$ admits the unique solution $A(h) = I$ for all h.

For, setting $g = \underset{2\times 2}{-I}$ in the first of these equations we obtain

$$A(-h) = A(h).$$

In turn the second equation gives

$$\begin{aligned} A(2h) &= A(h)A(h) \\ &= A(h)A(-h) \\ &= A(0) \\ &= \underset{2\times 2}{I}, \end{aligned}$$

for all $h \in \mathbb{R}^2$. We deduce, as in the proof of theorem 1, that B(h) is linear and since $gB(h) = B(gh)$, then B(h) must be of the form

$$B(h) = \begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} h$$

where α and β are real constants.

(2) In the case of SO(3) we proceed as follows. Let us define the function F(h,x,y) by

$$F(h,x,y) = (A(h)x,y),$$

where h, $x \in \mathbb{R}^3$, then

$$\begin{aligned} F(gh,gx,gy) &= (A(gh)gx,gy) \\ &= (gA(h)x,gy) \\ &= (A(h)x,y) \\ &= F(h,x,y). \end{aligned}$$

It can be easily deduced from theorem 1.7.1 that $F(h,x,y)$ must have the form

$$F(h,x,y)=a(\|h\|)(x,y) + b(\|h\|)[h,x,y] + c(\|h\|)(x,h)(y,h)$$

where, as in section 7 of Chapter I, $[h,x,y]$ stands for the determinant whose entries are the coordinates of the vectors h,x,y.

Since $F(-h,h,y) = F(h,h,y)$, then $A(-h)h = A(h)h$; so that

$$A.(h)\ A(-h)h = A(h)A(h)h,$$

$$A(0)h = A(2h)h$$

$$2h = A(2h)2h$$

This is to say $A(h) = h$, for all $h \in \mathbb{R}^3$ set $h_1 \perp h$, $h \perp y$, then

$$(A(h_1+h)h,y) = a(\|h_1+h_2\|)(h,y) + b(\|h_1+h\|,y)$$

$$+ c(\|h_1+h\|)(h_2+h,h)(h_1+h,y)$$

$$= b(\|h_1+h\|)[h_1,h,y] + c(\|h_1+h\|)$$

$$+ c(\|h_1+h\|)(h,h)(h_1,y)$$

$$= (A(h_1)h,y)$$

$$= b(\|h_1\|)[h_1,h,y].$$

Thus, $b(\|h_1+h\|)[h_1,h,y] + c(\|h_1+h\|)(h,h)(h_1,y)$

$$= b(\|h_1\|)[h_1,h,y]. \quad (1)$$

By taking $h_1 \perp y$ in this formula we get $b(\|h_1+h\|)[h_1,h,y] = b(\|h_1\|)[h_1, h,y]$; so that $b(\|h_1+h\|) = b(\|h_1\|)$ and by passing to the limit as h_1 tends to zero we obtain

$$b(\|h\|) = b(\|0\|)$$

for all $h \in \mathbb{R}^3$.

From formula (1) above and the fact that $b(\|h\|)$ is constant we get $c(\|h\|) = 0$. Finally,

$$\begin{aligned} F(h,h,y) &= (A(h)h,y) \\ &= a(\|h\|)\,(h,y) \\ &= (h,y), \end{aligned}$$

so that $((a(\|h\|)-1)\,h,y) = 0$, for all $y \varepsilon \mathbb{R}^3$, that is $a(\|h\|) = 1$, for all $h \varepsilon \mathbb{R}^3$ and consequently $A(h) = I$, any $h \varepsilon \mathbb{R}^3$.

Therefore, $A(h) = I$ and this in turn implies that $B(h) = ch$, for all h in $\mathbb{R}^3$, with c some real constant. A similar result follows by solving the same functional equation when $g \varepsilon SO(2)$, $h \varepsilon \mathbb{R}^2$.

(3) The solutions of the single functional equation

$$B(gh) = gB(h)$$

where g ranges over a group G of linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $h \in \mathbb{R}^n$ can be quite varied. For instance, let us assume that there is some real function of h, say $F(h)$, which is invariant under G, that is $F(gh) = F(h)$ for all $g \in G$ and all $h \in \mathbb{R}^n$. Then the function defined on $\mathbb{R}^n$ with values on $\mathbb{R}^n$ defined by

$$B(h) = F(h).h$$

satisfies $B(gh) = gB(h)$ for all $g \in G$ and all $h \in \mathbb{R}^n$.

The following Lemma provides a solution to a system of functional equations in which the equation $B(gh) = gB(h)$ appears, with $g \in SO(n)$, $n \geq 3$. We shall find application to this Lemma later on.

LEMMA 4. Let $F(x) = (F_1(x),\ldots,F_n(x))$ be a function defined on $\mathbb{R}^n$, $n \geq 3$ with values in $\mathbb{R}^n$ such that,

(a) $F(\theta x) = \theta F(x)$, (b) $\sum_{r=1}^{n} x_r F_r(x) = 0$

for all $\theta \in SO(n)$ and all $x \in \mathbb{R}^n$.

Then $F(x) = 0$ for all $x \in \mathbb{R}^n$.

Proof: Let us write $e_r = (0, \ldots, \underset{r}{1}, \ldots 0)$. We deduce from (b) that $F_r(\lambda e_r) = 0$ for all $\lambda \in \mathbb{R}$ and $r = 1, 2, \ldots, n$.

If, on the other hand, $\theta \in SO(n)$ leaves e_1 fixed, that is, $\theta e_1 = e_1$ then it follows from (a) that

$$F(\lambda e_1) = {}^{\theta}F(\lambda e_1),$$

so that

$$\begin{pmatrix} F_2(e_1) \\ \vdots \\ F_n(e_1) \end{pmatrix} = \theta' \begin{pmatrix} (F_2(e_1) \\ \vdots \\ F_n(e_1) \end{pmatrix}$$

where θ' is the rotation in $SO(n-1)$ obtained from θ by removing the row and column one.

Hence $F_r(\lambda e_1) = 0$. $2 \leq r \leq n$ and, since $F(\lambda e_1) = 0$, we see that $F(\lambda e_1) = 0$ for all $\lambda \in \mathbb{R}$.

Finally, given $x \in \mathbb{R}^n$ there are $\theta \in SO(n)$ and $\lambda \in \mathbb{R}$ such that

$$\theta \lambda e_1 = x,$$

so that

$$F(x) = \theta F(\lambda e_1) = 0.$$

REMARK: The example with $F(x) = (-x_2, x_1)$ shows that Lemma 4 is not true for $\mathbb{R}^2$.

We leave here the study of the case with $V(g) = g^{-1}$ and pass on to examine the case when $V(g) = I_{nxn}$.

The hypotheses that we shall place upon the group G in this case are to be different from those laid down in theorem 1. Nevertheless, some of the groups which are of outstanding importance in mathematics and in physics fulfill the hypotheses of theorem 1 and those of theorem 2 below.

We begin with finding the relations linking the operators A(h) and B(h). In this respect let us recall that every element $\tilde{g} = (g,h) \in G\tilde{M}(n)$ can be factorized as

$$\tilde{g} = (e,h)\ (g,0) = (g,0)\ (e,g^{-1}h),$$

then

$$\begin{aligned} V(\tilde{g})x &= V(g)V(h)x \\ &= V(h)x \\ &= A(h)x + B(h), \\ V(\tilde{g})x &= V(g^{-1}h)V(g)x \\ &= V(g^{-1}h)x \\ &= A(g^{-1}h)x + B(g^{-1}h) \end{aligned}$$

so that

$$A(g^{-1}h) = A(h),\ B(h) = B(g^{-1}h),$$

or equivalently

$$A(gh) = A(h),\quad B(h) = B(gh),$$

for every $g \in G$ and every $h \in \mathbb{R}^n$.

The expressions,

$$A(h_1 + h_2) = A(h_1)A(h_2)$$

and

$$A(h_1)B(h_2) + B(h_1) = B(h_1 + h_2),$$

where $h_i \in \mathbb{R}^n$, $i = 1, 2$, follow here, too.

Therefore, the operators $A(h)$ and $B(h)$ appearing in the operator $V(\tilde{g})$ satisfy the system of functional equations

$$\begin{cases} \text{(a)} \quad A(gh) = A(h) \\ \text{(b)} \quad A(0) = I_{nxn} \\ \text{(c)} \quad B(gh) = B(h) \\ \text{(d)} \quad A(h_1 + h_2) = A(h_1)A(h_2) \\ \text{(e)} \quad A(h_1)\, B\, (h_1) + B(h_2) = B(h_1 + h_2). \end{cases} \qquad \text{(II.2.9)}$$

THEOREM 2.2.

Let G be a group of linear transformations acting irreducibly on $\mathbb{R}^n$. If for some non-zero vector v in $\mathbb{R}^n$ there is an element g in G such that

$$gv = -v,$$

then the system of simultaneous functional equations (II.2.9) admits the unique solution

$$A(h) = I_{nxn} \, , \, B(h) = 0$$

for all h in $\mathbb{R}^n$.

Proof: With g and v as in the hypotheses, we deduce from the equation (II.2.9) (a) that

$$A(\lambda v) = A(-\lambda v)$$

for any $\lambda \in \mathbb{R}$.

Let H be the set in $\mathbb{R}^n$ of those vectors such that if $h \in H$ and $\lambda \in \mathbb{R}$, then there is some constant k_λ such that

$$A(\lambda h) = A(k_\lambda v).$$

The set H is manifestly not empty. It is also an invariant linear subspace of $\mathbb{R}^n$.

In fact, let h_1, $h_2 \in H$, then

$$\begin{aligned} A(\lambda h_1 + \mu h_2) &= A(\lambda h_1)\, A\, (\mu h_2) \\ &= A(k_\lambda v)\, A\, (k_\mu v) \\ &= A(k_{\lambda+\mu} v), \end{aligned}$$

for any real constants λ, μ; whence $H = \mathbb{R}^n$.

We also notice that, if $A(h) = A(\lambda v)$, then $A(-h) = A(-\lambda v)$, so that

$$A(h) = A(-h)$$

for any $h \in \mathbb{R}^n$.

Therefore,

$$A(0) = A(h)A(-h) = A(2h),$$

so that

$$A(h) = I_{nxn} ,$$

for all $h \in \mathbb{R}^n$.

This fact and the equation (II.2.9) (e) imply that B(h) is linear in h.

Let $(b_{ij})_{n\times n}$ be the matrix of B in some basis in $\mathbb{R}^n$. The equation (II.2.9) (e) can be written as $\sum_j b_{kj}g_{ji} = b_{ki}$, $1 \leq k, i \leq n$, where $g = (g_{ij}) \in G$. If B is non zero, then there is some index j such that the subspace of $\mathbf{R}^n$ whose elements, $h = (h_1,\ldots h_n)$, satisfy the expression $\sum_i b_{ji}h_i = 0$ is obviously invariant. This, however, contradicts the hypothesis that G leaves no subspace invariant.

Hence $B(h) = 0$ for all $h \in \mathbf{R}^n$.

Particular instances of groups that obey the hypothesis of theorem 2.2 are the rotation groups and the Lorentz group.

REMARK:

Generally speaking if G does not act irreducibly on $\mathbf{R}^n$ then the system (II.2.9) may admit several solutions. For instance, suppose that every element in some nontrivial subspace X of R^n is left fixed under the action of G and that the restriction of all the elements of G to the orthogonal complement of X is invariant. Then the orthogonal projection $\Pi_X(x)$ of $\mathbf{R}^n$ on X satisfies

$$\Pi_X(gx) = \Pi_X(x),\ g \in G,\ x \in \mathbf{R}^n.$$

Therefore, $A(h) = I_{n\times n}$ and $B(h) = \Pi_X(h)$ is yet another solution of (II.2.9).

To end this section we wish to point out that, on the hypotheses of theorem 2.2, the appropriate representation

$$W(\tilde{g})\, f\,(x) = Q(x_i\tilde{g})\ f\ (V(\tilde{g})x),$$

with $V(g)x = V(g,v)x = x$ attains the form

$$W(\tilde{g})\ f\ (x) = Q(x,\tilde{g})\ f\ (x).$$

3. THE FORMS OF LINEAR TRANSLATION INVARIANT OPERATORS

As we saw in the previous section, the appropriate representation of $\tilde{GM}(n) = G \times \tau(n)$

$$W(\tilde{g})\ f\ (x) = Q(x,\tilde{g})\ f\ (V(\tilde{g})x),$$

with $V(\tilde{g})x = g^{-1}x - g^{-1}h$ can be reduced under suitable hypotheses to the canonical form (II.2.8).

$$W(\tilde{g})\ f\ (x) = Q(g)\ f\ (g^{-1}x - g^{-1}h),$$

where $Q(g)$ is some representation of G on the range space of the functions in the space $A(\mathbb{R}^n)$.

The condition that an operator T intertwines between appropriate representations of type (II.2.8) becomes

$$\{T[Q(g)\ f\ (g^{-1}x-g^{-1}h)]\}\ (u) =$$

$$= Q^*(g)\{T[f(x)]\}\ (g^{-1}u-g^{-1}h) \qquad (II.3.1)$$

An operator T satisfies (II.3.1) if and only if T satisfies the system of simultaneous functional equations

$$\begin{cases} [T\ f(x-h)]\ (u) = [T\ f(x)]\ (u-h) & \text{(a)} \\ [TQ(g)\ f\ (g^{-1}x)]\ (u) = Q^*(g)[Tf(x)]\ (g^{-1}u) & \text{(b)} \end{cases} \qquad (II.3.2)$$

We shall therefore study first the equation (II.3.2) (a) under suitable hypotheses and afterwards we shall specialise to the solutions of (a) that also satisfy (II.3.2) (b) in the specific case when G is SO(n) and the group of Lorentz. The latter case will be discussed in the next chapter.

For brevity we shall sometimes write the equation (II.3.2) (a) as

$$T\tau(h) = \tau(h)\, T,$$

and say that T is translation invariant.

Translation invariant operators are an important class of operators which occurs in Fourier analysis and in the theory of Partial Differential Equations among other branches of mathematics. They are also important in Theoretical Physics and considerable attention has been paid to their study.

In particular, Schwartz [18] p.197, has proved that if $T: \mathcal{D} \rightarrow \mathcal{D}'$ is translation invariant, where $\mathcal{D}$ is the vector space of all infinitely differentiable complex functions with compact support in $\mathbb{R}^n$, and $\mathcal{D}'$ is the vector space of distributions on $\mathcal{D}$, then there exists a distribution S such that

$$T(\phi) = S * \phi,$$

for all $\phi \in \mathcal{D}$. Here, as usual, $S * \phi$ denotes the convolution product. It is also proved in the same treatise that T also commutes with derivations and that these two properties of T are equivalent (see Schwartz [18] p.197). A rather more detailed account of these results is presented in EDWARDS [7] p.332.

Also, L. HÖRMANDER [23] has given what appears to be an independent proof of these results of Schwartz's and has applied it to finding the forms of linear translation invariant operators acting on Lp spaces of

Lebesgue summable functions.

This author, in the same paper, examines the problem of finding necessary conditions for the existence of translation invariant operators. We shall comment on these conditions later on.

The more general problem of operators intertwining between appropriate representations of the group of translations is discussed by J. L. B. COOPER in [4] for the real line and in [5] for higher dimensions. The latter paper deals, in particular, with the problem of linear bounded translation invariant operators acting on Lp $(\mathbb{R}^n,\mu)$ spaces, where μ is a Radon measure, thereby generalizing the aforesaid result of Hörmander's.

The study of translation invariant operators on Lp $(\mathbb{R}^n,\mu)$ spaces is not only a sound problem in itself, but the case of spaces with "weights" appears quite naturally, as we saw in section II.2 when we reduced the general form of appropriate representations of the group of translations to its canonical form

$$W(h)\ f\ (x) = f(x - h).$$

The results found in Cooper [7] concerning the forms of translation invariant operators have been further improved by the same author by weakening the conditions imposed on the function spaces upon which the operators act. These developments are the content of this section.

1) RADON MEASURES

Let us denote, as usual, by $C^o{}_c(\mathbb{R}^n)$ the space of all real valued continuous functions with compact support in $\mathbb{R}^n$, endowed with the topology of the uniform convergence on compact sets. A positive Radon measure μ is a linear functional μ on $C^o{}_c(\mathbb{R}^n)$ such that

$$\mu(\phi) = < \mu,\phi > \geqslant 0$$

if $\phi(x) \geqslant 0$ for all x.

Radon measures are characterized as follows. A continuous linear functional on $C^o_c(\mathbb{R}^n)$ is a positive Radon measure if and only if for every compact set K in $\mathbb{R}^n$ there is some positive constant m_K such that

$$< \mu,\phi > \leqslant m_K \|\phi\|_\infty$$

for every ϕ whose support is contained in K.

In fact, the condition is obviously sufficient. On the other hand, if K is a compact set in $\mathbb{R}^n$ we can always find a function ψ in $C^o_c(\mathbb{R}^n)$ such that $\psi(x) = 1$ if $x \in K$ and $\psi(x) \geqslant 0$ for all x, then

$$\phi(x) \leqslant \|\phi\|_\infty \psi(x),$$

thus

$$< \mu,\phi > \leqslant \|\phi\|_\infty < \mu,\psi >$$

$$= m_K \|\phi\|_\infty$$

A general Radon measure is a sum

$$\mu = (\mu_1 - \mu_2) + i (\mu_3 - \mu_4)$$

where μ_i, i = 1, 2, 3, 4 are positive Radon measure.

Definition 3.2

A Schwartz space $\mathcal{J}$ is a space of infinitely differentiable functions in which $\mathcal{D} = C^\infty_c$ is continuously embedded, on which D_i is a continuous operator for any i and for any ϕ,

$$\frac{\tau(-tei)\phi - \phi}{t} \rightarrow D_i\phi$$

in the topology of $\mathcal{J}$ as $t \rightarrow 0$.

Definition 3.3

We shall say that a space $A(\mathbb{R}^n)$ of functions (or equivalence classes of functions) on $\mathbb{R}^n$ has the α-property *or is* an α-space if there is Schwartz space embedded in A.

Let $B(\mathbb{R}^n)$ be any linear topological function space of functions on $\mathbb{R}^n$. If for every $f \in B$,

$$\frac{\tau(-tei)\ f - f}{t}$$

tends to a limit in the topology of B as t tends to zero, we write this limit as $D_i f$. We say that f is in $\mathcal{E}^\infty(B)$ if $D^\alpha f$ exists in B for every sequence α of nonnegative integers.

Examples of α-spaces are the $Lp(\mathbb{R}^n,\mu)$ spaces where $p \geq 1$ and μ is a positive Radon measure.

Definition 3.4

The space B, as defined above, is said to obey the β-condition or to be a β-space if there is an a in $\mathbb{R}^n$ and an integer m such that for each function class in $\mathcal{E}^\infty(\mathbf{B})$ there is a member $\hat{f}$ of the class f such that $\hat{f}$ is continuous in a neighbourhood of a and $\hat{f}(a) \rightarrow 0$ as $D^\alpha f \rightarrow 0$ whenever $|\alpha| \leq m$.

If B is a normed space then B is a β-space if and only if there is a constant c such that

$$|\ \hat{f}\ (a)\ | \leq c \sum_{|\alpha| \leq m} \| D^\alpha f \|$$

REMARK: It has been proved by J. L. B. Cooper that Lp spaces are β-spaces for suitable Radon measures. Else where we prove a similar result for Orlicz spaces.

THEOREM 3.1 (J. L. B. Cooper). Let A be an α-space with the Schwartz-space $\mathcal{J}$ continuously embedded in A; let B be a β-space. Let T be a continuous linear map of A to B such that for each h and each ϕ in

$$(T\tau(h)\phi(x))(u) = \tau(h)\,[T\phi(x)]\ (u).$$

Then there is a distribution t in $\mathcal{J}'$ such that for all u in $\mathbf{R}^n$ and ϕ in $\mathcal{J}$

$$(T\phi(x))\ (u) = < t, \tau\,(u)\tilde{\phi} > = (t * \phi)\ (u)$$

where $\tilde{\phi}(x) = \phi(-x)$ for all x.

Proof. We can assume without loss of generality that the point a in the definition of the β-condition is 0.

For any coordinate vector e_i and real t,

$$\frac{\tau(-tei)\phi - \phi}{t} \rightarrow D_i\phi \quad \text{as } t \rightarrow 0$$

in the topology of $A(R^n)$, because it does so in $\mathcal{D}$ and $\mathcal{J}$ and these spaces are continuously embedded in A. Since

$$T\,[\ \frac{(\tau(-tei) - I)}{t}\ \phi\ (x)]\ (u) =$$

$$= [\ \frac{\tau(-tei) - I}{t}\ ]\ (T\phi(x))(u),$$

then the right hand side of this expression converges and so that $D_iT\phi$ exists in the sense of B.

It now follows by iteration that

$$TD^{\alpha}\phi = D^{\alpha}T\phi$$

for any finite sequence of interges α; whence $T\phi$ is in $\mathcal{E}^{\infty}(B)$ and so by hypothesis $(\hat{T}\phi(x))(u)$ is continuous in some neighbourhood of 0, and such that $\hat{T}\phi(x)(0) \to 0$ as $\phi \to 0$, thus there is an element s in ' such that

$$(\hat{T}\phi(x))(0) = < S(x), \phi(x) >$$

$$= < \tilde{S}(x), \tilde{\phi}(x) >$$

$$= < t(x), \tilde{\phi}(x) > .$$

Since $\tau(-u)\ (\hat{T}\phi(x))(0)$ is in the class of $\tau(-u)\ T\phi$ = $T\tau(-u)\phi$ for any u and is continuous in a neighbourhood of 0, then it follows that $(\hat{T}\Phi(x))(u)$ is everywhere continuous; thus, by identifying functions with their equivalence class we see that

$$[T\phi(x)](u) = \tau(-u)\ [T\phi(x)](0)$$

$$= T\ [\tau(-u)\phi(x)]\ (0)$$

$$= < t(x),\ \tau(u)\tilde{\phi}(x) >$$

$$= (t * \phi)(u) .$$

If $\mathcal{J}$ is dense in A then we can extend T to all A by continuity.

CHAPTER III

OPERATORS INTERTWINING WITH REPRESENTATIONS OF THE EUCLIDEAN AND POINCARE GROUPS. HOMOTHETIC INVARIANCE. DIFFERENTIAL OPERATORS

INTRODUCTION

In this chapter we study the solutions of the functional equation

$$[T\ f(x-h)]\,(u) \;=\; [Tf(x)]\,(u-h)$$

that also satisfy the equation

$$[TQ(g)f(g^{-1}x)]\ (u) = Q^*(g)\ [Tf(x)]\,(g^{-1}u).$$

We examine separately the cases when $Q(g)$ and $Q^*(g)$ are single valued representations of the rotation groups and the proper Lorentz group.

We also study appropriate representations of the group of homothetics of $\mathbb{R}^n$, on suitable function spaces. We thereafter examine the solutions of the appropriate functional equation

$$[T\ Q(g)f(g^{-1}x-g^{-1}h)]\,(u) =$$

$$= Q^*(g)[Tf(x)]\,(g^{-1}u-g^{-1}h),$$

that satisfy the functional equation

$$[Tf(\lambda x)]\,(u) = \lambda^{\mu}\ [Tf(x)]\,(\lambda u),$$

that is the additional condition of homothetic invariance.

Let us recall that the Euclidean group $E^+(n)$ is defined to be the semidirect product (Def. 1.4, Ch.II) $SO(n) \times \tau(n)$, of the special orthogonal group SO(n) and the group $\tau(n)$ of translations on $\mathbb{R}^n$.

By the Poincaré group we mean the semidirect product $L^1(4) \times \tau(4)$, where $L^1(4)$ is the proper Lorentz group.

1. REDUCTION OF THE FUNCTIONAL EQUATION
$TQ(g)f(g^{-1}x - g^{-1}h)(u) = Q^*(g)\, Tf(x)\,(g^{-1}u - g^{-1}h)$

Let $A(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$ be linear function spaces whose elements take values in linear spaces E and F respectively. In this chapter G will mean either the rotation groups SO(n) or the group of Lorentz. We shall, however, make a clear distinction whenever necessary.

Let $T: A(\mathbb{R}^n) \to B(\mathbb{R}^n)$ be a linear operator, then T can be expressed as an array of transformations (T_{ij}) by the formula

$$(Tf)_i = \sum_j T_{ij} f_j$$

If $A(\mathbb{R}^n)$ and $B(\mathbb{R}^n)$ satisfy the hypothesis of theorem (II.3.1), that is, $A(\mathbb{R}^n)$ is an α-space in which a Schwartz space $\mathcal{J}$ is continuously embedded and $B(\mathbb{R}^n)$ is a β-space, and if T satisfies the equation

$$Tf(x-h)(u) = Tf(x)(u)$$

for every f in $\mathcal{J}$, then we have from theorem (II.3.1) that

$$T_{ij}f_j(x)(u) = (t_{ij} * f_j)(u)$$
$$= \int t_{ij}(u-x) f_j(x) dx.$$

This is to say, T can be expressed as an array of distributions (t_{ij}) in $\mathcal{J}$.

Let us now assume that T also satisfies the equation (II.3.2)(b), that is,

$$[TQ(g)f(g^{-1}x)](u) = Q^*(g)[Tf(x)](g^{-1}u);$$

then by writing

$$(Q(g)v)_i = \sum_j Q_{ij}(g)v_j$$

and

$$(Q^*(g)w)_i = \sum_s Q^*_{rs}(g)w_s,$$

with v in E and w in F, the condition (II.3.2) (b) can be written as

$$\sum_j \sum_i T_{ri}\ Q_{ij}(g)\ f_j(g^{-1}x)(u) =$$

$$= \sum_i \sum_s Q^*_{rs}(g)\ T_{si}\ f_i(x)\ (g^{-1}u)\ . \qquad \text{(III.1.1)}$$

Now, take f in such that its components are $\delta_{kj} f_k$ with k fixed and replace it in (III.1.1), we get

$$\sum_i T_{ri}\ Q_{ik}\ (g)\ f_k(g^{-1}x)\ (u)$$

$$= \sum_s Q^*_{rs}(g)\ T_{sk}\ f_k(x)\ (g^{-1}u)$$

so that

$$\sum_i Q_{ik}(g) \int t_{ri}\ (u-x)\ f_k(g^{-1}x)\ dx =$$

$$= \sum_s Q^*_{rs}(g) \int t_{sk}\ (g^{-1}u-x) f_k(x) dx\ .$$

By means of a change of variable the latter expression becomes

$$\sum_i Q_{ik}(g) \int t_{ri}(g(u-x))f_k(x)dx =$$

$$= \sum_s Q^*_{rs}(g) \int t_{sk}(n-x)\, f_k(x)dx \,,$$

whence

$$\sum_i Q_{ik}(g) \quad t_{ri}(gx) = \sum_s Q^*_{rs}(g)\, t_{sk}(x)$$

We have thus proved.

THEOREM 1.1. Let $A(\mathbb{R}^n)$ be an α-space of functions defined on $\mathbb{R}^n$ with values in a vector space E in which a Schwartz space $\mathcal{J}$ is continuously embedded. Let T be a translation invariant continuous operator from $A(\mathbb{R}^n)$ to a β-space $B(\mathbb{R}^n)$ satisfying (III.1.1). Then T can be expressed as an array of distributions (t_{ij}) in $\mathcal{J}'$ such that for any g in G

$$\sum_i Q_{ik}(g)\, tri\, (gx) = \sum_s Q^*_{rs}(g)\, t_{sk}(x) \qquad \text{(III.1.2)}$$

We recall that here G is either SO(n) or, in the case of $\mathbb{R}^4$, G can also be the Lorentz group.

2. OPERATORS INTERTWINING WITH REPRESENTATIONS OF THE EUCLIDEAN GROUP $E^+(n)$

In this section we examine the functional equation (III.1.2) on the assumption that G is the rotation group SO(n). Consequently $Q(g)$ and $Q^*(g)$ becomes representations of SO(n) on E and F respectively. We assume further that $Q(g)$ and $Q^*(g)$ are single valued representations.

If $Q(g)$ and $Q^*(g)$ are not irreducible, then, as we saw in Chapter II, the spaces E and F split into direct sums of invariant irreducible subspaces.

Let $E_i \subset E$ be one such invariant irreducible subspace, and let $f_{i_1},\ldots f_{i_p}$ be the coordinate functions of $f \in A(\mathbb{R}^n)$ corresponding to E_i. Then, the restriction T^i of T to E_i take the function $f_i = (f_{i_1},\ldots,f_{i_p})$ into, say, the invariant irreducible spaces F_j and F_k in F. The operator $\Pi_{F_j}\ T^i$ defined by

$$(\Pi_{F_j}\ T^i)\ (f_i(x))(u) = \Pi_{F_j}\ (T^i\ f_i(x))(u)$$

where Π_{F_j} is the projection of F on F_i, intertwines with the irreducible components $Q^i(g)$ and $Q^{*j}(g)$ of $Q(g)$ and $Q^*(g)$ on E_i and F_j respectively.

Therefore, we can restrict ourselves to examine the case when both $Q(g)$ and $Q^*(g)$ are irreducible. Every irreducible single valued representation of the rotation group is equivalent to an irreducible component of some tensor representation (see Theorem I.5.2), that is, we can suppose that the indices i,j in (III..1.2) are multiindices

$$i = i_1 \ldots i_M, \quad j = j_1 \ldots j_N$$

with

$$1 \leqslant i_r, i_s \leqslant n\ .$$

The matrices $Q(g)$ and $Q^*(g)$ are Kronecker products of the matrix g of the rotation,

$$Q_{ij}(g) = g_{ij} = g_{i_1 j_1} \ldots g_{i_M j_M} \qquad \text{(III.2.1)}$$

with a similar formula for $Q^*_{rs}(g)$.

Replacing (III.2.1) in (III.1.2):

$$\sum_i Q_{ik}(g)\ t_{ri}\ (gx) = \sum_s Q_{rs}{}^*\ (g)\ t_{sk}(x)$$

we obtain

$$\sum_{i_1 \ldots i_M=1}^{n} g_{i_1 \ldots i_M, k_1 \ldots k_M} t_{r_1 \ldots r_N, i_1 \ldots i_M} (gx) =$$

$$= \sum_{s_1 \ldots s_N=1}^{n} g_{r_1 \ldots r_N, s_1 \ldots s_N} t_{s_1 \ldots s_N, k_1 \ldots k_M} (x) \tag{III.2.2}$$

Here we have written $g_{i_1 \ldots i_M, k_1 \ldots k_M}$ for $g_{i_1 k_1} g_{i_2 k_2} \ldots g_{i_M k_M}$.

Sometimes we shall, for brevity, write (III.2.2) as

$$\sum_i g_{ik} t_{ri} (gx) = \sum_s g_{rs} t_{sk}(x) .$$

Notice that (III.1.2) can be written as

$$t_{rp}(gx) = \sum_k \sum_s g_{pk} g_{rs} t_{sk}(x).$$

A similar formula can be obtained for (III.2.2).

If $Q(g) = Q^*(g) = g$, $g \in SO(n)$, $n \geqslant 4$, then the functional equation (III.2.2) can be solved by standard methods.

In fact, it can be easily checked that $T(x) = (t_{ij}(x))$ satisfies the functional equations

$$\begin{cases} T(gx)g = gT(x) \\ \sum_{r=1}^{n} x_r T_{rk}(x) = 0, \quad k = 1,2,\ldots,n, \end{cases}$$

then proceeding as in the proof of Lemma 4, Chapter II, we can prove that

$$T(x) = a_1(\|x\|) \underset{n\times n}{I} + a_2(\|x\|) \underset{n\times n}{(x_r x_k)}$$

where $a_i(\|x\|)$, $i = 1,2$ are functions (or distributions) which depend only on the norm $\|x\|$ and are consequently invariant under the action of SO(n).

However, trying to apply this method to tensors of higher degree is entirely hopeless. We shall instead rely on a result of Chapter I Sec.4.

Before we proceed any further, let us introduce some abbreviations in order to simplify the formulas that are to appear.

(a) The tensor $t_{r_1 \ldots r_N,\, i_1 \ldots i_M}(x)$ will be written as $t_{i_1 \ldots i_P}(x)$, $P = M + N$.

(b) By $x_{i_1 \ldots i_P}$ we shall mean $x_{i_1} \ldots x_{i_P}$.

(c) In accordance with (a) above we shall sometimes write

$\delta_{i_1 \ldots i_P,\, i_{P+1} \ldots i_{2P}} = \delta_{i_1 \ldots i_P}$.

(d) Let $A_n = \{a_1,\ldots,a_n\}$ be a set of n distinct objects and let us write $C_n = \{1,\ldots,n\}$. We write $A_n^{\ r} (r \leq n)$ for the set of all subsets of A_n that contain r distinct elements. We shall write the elements of $A_n^{\ r}$ as $a_{i_1} \ldots a_{i_r}$, and sometimes, in accordance with (b) above, simply as $a_{i_1 \ldots i_r}$. The set $C_n^{\ r}$ is defined similarly.

Given an element $c = j_1 \ldots j_r$ in $C_n^{\ r}$, the correspondence

$$j_1 \ldots j_r \in C_n^{\ r} \to a_{j_1} \ldots a_{j_r}$$

is well defined. This will also be written as

$$c(a_1 \ldots a_n) = a_{j_1} \ldots a_{j_r}.$$

(e) Given an element $c = j_1 \ldots j_r$ in C_n^r, we write c' for the complement of c in C_n.

(f) Let σ be a permutation of the r natural numbers $j_1,\ldots,j_r$ and let $c = j_1 \ldots j_r$ be an element in C_n^r, then we write

$$\sigma(c(a_1 \ldots a_n)) = a_{\sigma(j_1)} \ldots a_{\sigma(j_n)} \; .$$

We now go on to find the forms of the tensors $t_{r_1 \ldots r_N, i_1 \ldots i_M}(x)$ appearing in (III.2.2)

Let $y^1, \ldots, y^p$ be p vectors in $\mathbb{R}^n$. It is a simple matter to verify that the form $G(x, y^1,\ldots y^p) = \sum_{i_1 \ldots i_p = 1}^{n} y^1_{i_1} \ldots y^p_{i_p} \; t_{i_1 \ldots i_p}(x)$ is invariant when $x, y^1, \ldots, y^p$ are transformed into gx, $gy^1, \ldots, gy^p$, $g \in SO(n)$, that is

$$G(gx, gy^1, \ldots, gy^p) = G(x, y^1 \ldots y^p).$$

It follows from theorem (I.7.1) that G is the sum of a function of the inner products, say $F((x,y^r), (y^k,y^s))$, and expressions of the form $[u^1, \ldots, u^n] f(x, y^1, \ldots, y^p)$ where the vectors $u^1, \ldots, u^n$ are chosen among the elements of the set $\{x, y^1, \ldots, y^p\}$ and the functions $f(x, y^1, \ldots, y^p)$ depend on the inner products (x, y^k), (y^r, y^s) alone. We recall that the terms in determinants occur only when $P \geqslant n - 1$.

Since $G(x, y^1, \ldots, y^p)$ is linear in every y^k, we deduce that the norms $\|y^k\|$ cannot occur. We have also that

$$G(x, \lambda y^1, \ldots, \lambda y^p) = \lambda^P \, G(x, y^1, \ldots, y^p);$$

whence $F((x,y^r), (y^k,y^s))$ is an homogeneous polynomial in the inner products and no vector y^k appears in the same term in two different inner products. An entirely similar argument holds for the functions $f((x,y^r), (y^k,y^s))$.

We notice further that, since G is homogeneous of degree one in every individual y^k, if a vector y^k appears in the determinant of an expression of the form $[u^1\ldots u^r]f((x,y^r),(y^k,y^s))$, then it cannot appear in the factor $f((x,y^r)(y^k,y^s)$.

Let us now examine the contributions to $G(x,y^1,\ldots,y^p)$ arising from the terms that contain inner products only, that is, from the form $F((x,y^k),(y^r,y^s))$.

According to the remarks made above as to the forms of $F((x,y^k),(y^r,y^s))$ we see that this is the sum of expressions of the following forms. We begin with the expression

$a_1(\|x\|)\ (x,y^1)\ldots(x,y^p)$ where $a_1(\|x\|)$ is a function (or distribution) which depends only on the norm $\|x\|$. The following terms are obtained by replacing pairs of inner products (x,y^r), (x,y^k) in the previous term by the inner product (y^r,y^k). We carry on with this process until we get terms with at most one inner product (x,y^r), say, or none at all. This depends on whether p is odd or even.

We thus conclude that the contribution of the form $F((x,y^r),(y^k,y^s))$ to $t_{i_1\ldots i_p}(x)$, which we write $F_{i_1\ldots i_p}(x)$, can be written with the help of the previous abbreviations as $F_{i_1\ldots i_p}(x) =$

$$= \sum_{k=0}^{K} \sum_{c\in C_p} x_{P-2k}^{c(i_1\ldots i_p)} \sum_{\sigma\in S_{2k}} a_{k_{1\sigma}}(\|x\|)\delta_{\sigma(c'(i_1\ldots i_p))} \qquad \text{(III.2.3)}$$

where $K = P/2$, if P is even, and $K = \frac{P-1}{2}$, if p is odd, here S_{2k} is the set of permutations of the set of the elements appearing in $c'(i_1\ldots i_p)$.

We deduce that the non zero terms in $F_{i_1 \ldots i_p}(x)$, apart from the term $a_0(\|x\|)x_{i_1 \ldots i_p}$, are those where the subindices in $\delta\sigma_{(c'(i_1 \ldots i_p))}$ appear repeated in pairs.

The contribution to $t_{i_1 \ldots i_p}(x)$ of all the terms in $G(x, y^1 \ldots y^p)$ whose determinants contain x is found similarly and it can be written as

$$\sum_{i_0} \sum_{c \in C_p^{n-1}} \varepsilon_{i_0\, c(i_1 \ldots i_p)}\, i_0\, F_{c'(i_1 \ldots i_p)}(x) \qquad \text{(III.2.4)}$$

where $F_{c'(i_1 \ldots i_p)}(x)$ as is in (III.2.3).

These terms can appear if and only if $p > n - 1$.

Similarly, the contributions of the terms where determinants containing only vectors y^k appear can be written as

$$\sum_{c \in C_p^n} \varepsilon_{c(i_1 \ldots i_p)}\, F_{c'(i_1 \ldots i_p)}(x) \qquad \text{(III.2.5)}$$

Putting all these results together we see that $t_{i_1 \ldots i_p}(x)$ is a linear combination of expressions of type (III.2.3), (III.2.4) and (III.2.5).

3. OPERATORS INTERTWINING WITH REPRESENTATIONS OF THE POINCARE GROUP

The study of the equation

$$\sum_i Q_{ik}(g)\; t_{ri}(gx) = \sum_S Q^*_{rs}(g) t_{sk}(x) \qquad \text{(III.1.2)}$$

when $g \in L^1(4)$ is exactly as the case when $g \in SO(n)$, as dealt with in the previous sections. More precisely, we take $Q(g)$ and $Q^*(g)$ to be single valued irreducible representations of the proper Lorentz group.

According to theorem (I.6.1), any irreducible, single valued, finite dimensional representation of the proper Lorentz group $L^1(4)$, is equivalent to an irreducible component of some tensor representations.

Under these hypotheses, the expression (III.1.2) becomes

$$\sum_{i_1 \ldots i_n = 1}^{4} g_{i_1 \ldots i_M,\, k_1 \ldots k_M}\; t_{r_1 \ldots r_M,\, i_1 \ldots i_M}(gx) =$$

$$\sum_{S_1 \ldots S_N = 1}^{4} g_{r_1 \ldots r_N,\, s_1 \ldots s_N}\; t_{s_1 \ldots s_N,\, k_1 \ldots k_M}(x) \qquad \text{(III.3.1)}$$

where $g = \underset{4\times4}{(g_{ij})}$ is a proper Lorentz transformation and $g_{i_1 \ldots i_m,\, k_1 \ldots k_m} = g_{i_1 k_1} \ldots g_{i_m \ldots k_m}$.

Let $q = \underset{4\times4}{(q_{kr})}$ be the inverse to $g = \underset{4\times4}{(g_{ij})}$. We recall that

$$\sum_{j=1}^{4} q_{kr} g_{ri} = S^{ki};\; \sum_j g_{ij} q_{jk} = S^{ik}$$

where S^{ki} is the metric tensor defined by

$$S^{ki} = \begin{cases} 1 & \text{if} \quad k = i = \leqslant 3 \\ -1 & \text{if} \quad k = i = \;\; 4 \\ 0 & \text{otherwise} \end{cases}$$

The expresssion (III.3.1) can now be written as

$$t_{i_1 \ldots i_N, j_1 \ldots j_M}(gx) = \sum_{r_1 \ldots r_N} \sum_{k_1 \ldots k_M} q_{k_1 \ldots k_M, j_1 \ldots j_M} \times$$

$$\times\ g_{i_1 \ldots i_N, r_1 \ldots r_N}\ t_{r_1 \ldots r_N, k_1 \ldots k_M}(x) \qquad \text{(III.3.2)};$$

we see that $t_{i_1 \ldots i_N, j_1 \ldots j_M}(x)$ is contravariant in the subindices $j_1 \ldots j_M$ and covariant in the subindices $i_1 \ldots i_N$.

In order to simplify the notation, we shall write contravariant vectors as $Z^1, \ldots, Z^N$ and covariant vectors as $y^1, \ldots, y^M$.

It can be checked that the form $G(x, Z^1, \ldots Z^N, y^1, \ldots, y^M) =$

$$= \sum_{i_1 \ldots i_N} \sum_{j_1 \ldots j_M} Z^1_{i_1} \ldots Z^N_{i_N}\, y^1_{j_1} \ldots y^M_{j_M}\, t_{i_1 \ldots i_N, j_1 \ldots j_M}(x),$$

is invariant when x, $Z^1, \ldots, Z^N$, $y^1, \ldots, y^M$ are transformed into $gx, gZ^1, \ldots, gZ^N$, $gy^1, \ldots, gy^M$, where g stands for Lorentz transformations.

We deduce from theorem (I.7.2) that $G(x, Z^1, \ldots, Z^N, y^1, \ldots y^M)$ is a function of the mixed products $[Z^i, y^j] = \sum_k Z^i_k y^j_k$, the forms $\langle x, y^j \rangle = \sum_{ii} S^{ii} x_i y^j_i$, $\langle y^i, y^j \rangle$ and of determinants $[Z^{i_1} \ldots Z^{i_4}]$, $[x_1 y^{j_1} y^{j_2} y^{j_3}]$, $[y^{j_1} \ldots y^{j_4}]$. Also, all the remarks put forward in the previous section about the linearity and homogeneity of the form $G(x_1 y^1, \ldots y^P)$, hold for the form $G(x_1 Z^1, \ldots, Z^N, y^1, \ldots y^M)$.

Therefore, $G(x_1 Z^1, \ldots, Z^N, y^1, \ldots y^M)$ is a linear combination of products of the invariant forms listed above, such that Z^i and every y^i enters in every of these products only once.

Let us begin by finding the contributions to $t_{i_1 \ldots i_N, j_1 \ldots j_M}(x)$ arising from those terms of $G(x_1 Z^1, \ldots Z^N, y^1, \ldots y^M)$ that do not contain any term where mixed products $[Z^i, y^j]$ appear. These contributions can be analysed as products of tensors arising from products of contravariant determinants $[Z^{i_1} \ldots Z^{i_4}]$ and mixed products $[x, Z^i]$ on the one hand, and tensors arising from products of covariant determinants and forms of type $< x, y^j >$, $< y^i, y^j >$ on the other.

Thus, we begin with the term

$$[x, Z^1] \ldots \qquad [x, Z^N]$$

which contributes to some terms of $t_{i_1 \ldots i_N, j_1 \ldots j_M}$ with the factor

$$\overset{0}{H}_{i_1 \ldots i_N}(x) = x_{i_1} \ldots x_{i_N}.$$

The subsequent factors are obtained by replacing in the factor above four mixed products, say $[x, Z^{i_j}]$, $j = 1 \ldots 4$, by the corresponding contravariant determinants

$$[Z^{i_1} \ldots Z^{i_4}] = \sum_{k_1 \ldots k_4} \varepsilon_{k_1 \ldots k_4} Z^{i_1}_{k_1} Z^{i_2}_{k_2} Z^{i_3}_{k_3} Z^{i_4}_{k_4}$$

These contributes the term

$$H^1_{i_1 \ldots i_N}(x) = \sum_{c \in C_N^4} \varepsilon_{c(i_1 \ldots i_4)} x_{c'(i_1 \ldots i_N)}$$

$$= \sum_{c \in C_N^4} \varepsilon_{c(i_1 \ldots i_N)} \overset{0}{H}_{c'(i_1 \ldots i_N)}$$

where c, c' and C_N^4 are as in the previous section.

By recurrence we find the terms that contain 2,3,... symbols $\varepsilon_{i_1 \ldots i_4}$, so that

$$H^k_{i_1 \ldots i_N}(x) = \sum_{c \in C_N^4} \varepsilon_{c(i_1 \ldots i_N)} H^{k-1}_{c'(i_1 \ldots i_N)}.$$

In every of these expressions, factors of type $a(<x,x>)$ may appear. The factors of type $H^k_{i_1 \ldots i_N}(x)$ with $k > 0$ appear if and only if $N = 4k + r$, $0 \leqslant r \leqslant 3$.

The factors, which the products of covariant determinants

$$[y^{j_i}\ldots y^{i_4}] = \sum_{k_1\ldots k_4} \varepsilon^{k_1\ldots k_4}\, y^{j_1}_{k_1}\, y^{j_2}_{k_2}\, y^{j_3}_{k_3}\, y^{j_4}_{k_4}$$

and forms $<x,y^j>$, $<y^i,y^j>$ contribute to the terms of $t_{i_1\ldots i_N, j_1\ldots j_M}(x)$, are obtained in a similar manner as we did in the previous section, with factors of type $a(\|x\|)$ and $\delta_{i_1\ldots i\ ,\ j_1\ldots j}$ replaced by factors of type $a(<x,x>)$ and $S^{i_1\ldots i_r, j_1\ldots j_r}$ respectively.

We thus obtain factors of type

$$F_{j_1\ldots j_M}(x) = \sum_{k=0}^{k} \sum_{c\in C_M^{M-zk}} x_{c(j_1\ldots j_M)} \sum_{\sigma\in S_{2_k}} a_{k_1\sigma}(<x,x>)S^{\sigma(c'(j_1\ldots j_M))},$$

where $K = \frac{M}{2}$ if M is even, and $K = \frac{M-1}{z}$ if K is odd;

$$E^0_{j_1\ldots j_M}(x) = \sum_{j_0} \sum_{c\in C^3_M} \varepsilon^{j_0 c(j_1\ldots j_M)}\, x_{j_0}\, F_{c'(j_1\ldots j_M)}(x),$$

$$E^1_{j_1\ldots j_M}(x) = \sum_{c\in C^4_M} \varepsilon^{c(j_1\ldots j_M)}\, F_{c'(j_1\ldots j_M)}(x)$$

Here we use the same notation as in Section III.2. p. 79, 80

Terms with repeated symbols $\varepsilon^{i_1\ldots i_4}$ are obtained by iteration of terms of type $E^0_{j_1\ldots j_M}$ and $E^1_{j_1\ldots j_M}$, such as

$$\sum_{c\in C^4_N} \varepsilon^{c(j_1\ldots j_M)} E^0_{c'(j_1\ldots j_M)}(x)$$

and so on.

Putting these results together we deduce that the contributions to the terms of $t_{i_1\ldots i_N,\ j_1\ldots j_M}(x)$, arising from terms of $G(x,Z^1,\ldots,Z^N,$

$y^1,\ldots y^M)$ that do not contain mixed products $[Z^i, y^j]$, are linear combinations of products of the following type.

$$H^0_{i_1\ldots i_N}\; F_{j_1\ldots j_M}(x),\quad H^0_{i_1\ldots i_N}\; E^0_{j_1\ldots j_M},\quad H^0_{i_1\ldots i_N}\; E^1_{j_1\ldots j_M}(x),$$

and products of the form $H^0_{i_1\ldots i_N}$ with iterates of factors of type E^0 and E^1, and terms of type

$$H^k_{i_1\ldots i_N}\quad F_{j_1\ldots j_M}(x).$$

The contributions to $t_{i_1\ldots i_N,\, j_1\ldots j_M}(x)$ arising from terms of $G(\xi, Z^1,\ldots, Z^N, y^1,\ldots, y^M)$ where mixed products $[Z^i, y^j]$ appear, is obtained by considering linear combinations of terms of type

$$\delta_{i_{k_1}\ldots i_{k_S}\; j_{r_1}\ldots j_{r_S}}\; G_{i_{k_{S+1}}\ldots\, i_{k_N},\; j_{r_{S+1}}\ldots\, j_{r_M}}$$

where $G_{i_{k_{S+1}}\ldots i_{k_N},\; j_{r_{S+k}}\ldots j_{r_M}}$ is of the type described above.

4. HOMOGENEOUS DISTRIBUTIONS. DIFFERENTIAL OPERATORS.

Let us recall that by $A(\mathbb{R}^n)$ we denote a space of functions on $\mathbb{R}^n$ with values in a linear space E, such that $A(\mathbb{R}^n)$ separates the points of E.

Let $\mathbb{R}^+ \times \mathbb{R}^n$ be the semidirect product of the multiplicative group $\mathbb{R}^+$ of the positive real numbers and $\mathbb{R}^n$. This is to say, $\mathbb{R}^+ \times \mathbb{R}^n$ consists of pairs (λ,h), $\lambda \in \mathbb{R}^+$, $h \in \mathbb{R}^n$ and the multiplication of pairs is defined by

$$(\lambda_1,h_1)(\lambda_2,h_2) = (\lambda_1\lambda_2, \lambda_1 h_2 + h_1).$$

The action of $\mathbb{R}^+ \times \mathbb{R}^n$ as a group of transformations on $\mathbb{R}^n$ is defined by

$$(\lambda,h) : x \to \lambda x + h, \quad x \in \mathbb{R}^n .$$

This definition is consistent with the group operation in $\mathbb{R}^+ \times \mathbb{R}^n$.

We shall sometimes write λ for $(\lambda,0)$ and h for $(1,h)$.

We now pass on to consider appropriate representations of $\mathbb{R}^+ \times \mathbb{R}^n$ on $A(\mathbb{R}^n)$ of the form

$$W(\lambda,h)f(x) = Q(\lambda,h,x)\ f\ (v(\lambda,h)x).$$

On account of the results of Chapter II we define $W(1,h) = W(h)$ by

$$\begin{aligned} W(h)\ f\ (x) &= \tau(h)f(x) \\ &= f(x-h). \end{aligned}$$

Since $(\lambda,h) = (1,h)\ (\lambda,0) = (\lambda,0)\ (1,\lambda^{-1}h)$, then

$$\begin{aligned} W(\lambda,h)\ f\ (x) &= W(h)W(\lambda)f(x) \\ &= W(h)\ Q(\lambda,x)f(V(\lambda)x) \\ &= Q(\lambda,x-h)f(V(\lambda)(x-h) \\ &= W(\lambda)W(\lambda^{-1}h)\ f(x) \\ &= W(\lambda)\ f(x-\lambda^{-1}h) \\ &= Q(\lambda,x)f(V(\lambda)x-\lambda^{-1}h). \end{aligned}$$

Thus,

$$Q(\lambda,x-h)f(V(\lambda)(x-h) = Q(\lambda,x)\ f\ (V(\lambda)x-\lambda^{-1}h).$$

From this expression and the hypothesis that $A(\mathbb{R}^n)$ separates the points of E we deduce that

$$V(\lambda)(x-h) = V(\lambda)x - \lambda^{-1} h ,$$

so that $\quad v(\lambda) = \lambda^{-1}.$

Therefore, $Q(\lambda,x-h) = Q(\lambda,h)$; that is, $Q(\lambda,x)$ is independent of x, and since it is multiplicative we must have that

$$Q(\lambda) = \lambda^{\mu} \qquad \text{(III.4.1)}$$

for some μ .

Hence,

$$w(\lambda)f(x) = \lambda^{\mu} f(\lambda^{-1}x).$$

This expression can be written as

$$w(\lambda)\ f\ (x) = \lambda^{\mu}\ \ f(\lambda x) \qquad \text{(III.4.2)}$$

Let the hypotheses of theorem I Section 1 of the present chapter hold. Let T be an operator as defined in this theorem. We shall now examine the forms of T under the further hypothesis that T obeys

$$[\,Tf(\lambda x)]\,(u\,) = \lambda^{\mu}\,[\,Tf(x)]\,(\lambda u) \qquad \text{(III.4.3)}$$

The expression (III.4.3) holds if and only if for every distribution $t_{\alpha i}$,

$$\int t_{\alpha i}\ (u-x)\phi(\lambda x)dx = \lambda^{\mu} \int t_{\alpha i}(\lambda u-x)\phi(x)dx.$$

This can be written as

$$\int t_{\alpha i}\ (\lambda^{-1}u-x)\phi(\lambda x)dx = \lambda^{\mu} \int t_{\alpha i}(u-x)\phi(x)dx ,$$

$$\frac{1}{\lambda^n}\int t_{\alpha i}(\lambda^{-1}u-\lambda^{-1}x)\phi(x)dx = \lambda^n\int t_{\alpha i}(u-x)\phi(x)dx,$$

so that $t_{\alpha i}(\lambda^{-1}x) = \lambda^{\mu+n}\, t_{\alpha i}(x)$, or equivalently

$$t_{\alpha i}(\lambda x) = \lambda^{-\mu-n}\, t_{\alpha i}(x) \qquad \text{(III.4.4)}$$

We have thus proved

THEOREM 4.1

Let $T = (t_{\alpha i})$ be an operator as defined in theorem (III.1.1). Let the hypotheses of this theorem hold. If T obeys

$$[T\, f(\lambda x)]\ (u) = \lambda^{\mu}\,[Tf(x)]\ (\lambda u) \qquad \text{(III.4.3)},$$

then

$$t_{\alpha i}(\lambda x) = \lambda^{-\mu-n}\, t_{\alpha i}(x).$$

We saw in section III.2 that the study of the appropriate functional equation

$$[TQ(g)f(g^{-1}x)](u) = Q^*(g)\,[Tf(x)]\ (g^{-1}u)$$

where $Q(g)$ and $Q^*(g)$ are single valued representations of SO(n), can be reduced to the case when $Q(g)$ and $Q^*(g)$ are irreducible representations, and in this case $T = (t_{\alpha i})$ obeys the formula

$$\sum_{i_1-i_M=1}^{n} g_{i_1\ldots i_M,\, k_1\ldots k_M}\, t_{r_1\ldots r_N, i_1\ldots i_M}(gx) =$$

$$= \sum_{S_1\ldots S_N=1}^{n} g_{r_1\ldots r_N,\, S_1\ldots S_N}\, t_{S_1\ldots S_N,\, k_1\ldots k_M}(x).$$

Further, in the previous section we saw that the nonzero distribution that transform according to this formula are linear combinations of distributions of the form

$$(t_{i_1 \ldots i_r} \phi(x))(u) = \int a(\|x\|)\, x_{i_1} \ldots x_{i_r} \phi(x-u)dx \qquad \text{(III.4.5)}$$

where $a(\|x\|)$ is a function (or distribution) which depends only on the norm $\|x\|$ and is consequently rotation invariant.

Since the distribution appearing in (III.4.5) is homogeneous of degree $-\mu-n$ ((III.4.4)), then we must have that

$$a(\|\lambda x\|)\lambda^r = \lambda^{-\mu-n}\, a(\|x\|),$$

so that

$$a(\|\lambda x\|) = \lambda^{-\mu-n-r}\, a(\|x\|) \qquad \text{(III.4.6)}$$

Homogeneous distributions have been extensively studied by I. M. Gel'fand and G.E. Shilov [8] . In particular, these authors have proved ([8] p.80) that an homogeneous distribution $f(x)$ such that

$$f(\lambda x) = \lambda^{-n} f(x)$$

where n is some positive integer, is of the form

$$f(x) = cx^{-n} + C_1\, \delta^{(n-1)}(x).$$

For a definition of the distribution x^{-n} we refer to [10] p.51.

If the degree of homogeneity of $f(x)$ is not a negative integer, then no distribution concentrated at 0 appears in $f(x)$ (see [8] p.81). More precisely, $f(x)$ is a distribution concentrated on the complement of $\{0\}$.

In the present case the distribution $a(\|x\|)$ in (III.4.6) becomes

$$a(\|x\|) = C\,\|x\|_{+}^{-\mu-n-r}\; C\,\delta^{(\mu+n+r-1)}(\|x\|),$$

provided that $-\mu-n-r$ is a negative integer, where $\delta^{(\mu+n+r-1)}(\|x\|)$ is an homogeneous distribution concentrated at $x=0$ and rotation invariant.

Rotation invariant distributions concentrated at the origin have been studied by P. D. Methée [15], who has also studied Lorentz invariant distribution concentrated at the vertex of the $\sum_{i=1}^{n-1} x_i^2 - x_n^2 = 0$ cone. A more detailed account of these results appears in J.Challifour [3], p.144 and 153.

Methée has proved that a rotation invariant distribution $a(\|x\|)$ concentrated at the origin is of the form

$$a(\|x\|) = \sum_{P=0}^{M} a_P\, \Delta_n^{P}\, \delta(x),$$

where $\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} = \sum_{j=1}^{n} D_j^2.$

Let us now examine the operator arising from $\sum_{P=0}^{M} a_P\, \Delta_n^{P}\, \delta(x)$.

To begin with we recall that the distribution $\Delta_n^P \delta(x)$ is homogeneous of degree $-n-2P$. Thus $\sum_{P=0}^{M} \lambda^r x_{i_1}\cdots x_{i_r}\, a_P\, \Delta_n^{P}\, \delta(\lambda x) =$

$$= \sum_{P=0}^{M} x_{i_1}\cdots x_{i_r}\, \lambda^{-n-2P+r}\, a_P \Delta_n^P \delta(x)$$

$$= \lambda^{-n+r} \sum_{P=0}^{M} x_{i_1}\cdots x_{i_r}\, \lambda^{-2P}\, a_P\, \Delta_n^P\, \delta(x)$$

$$= \lambda^{-\mu-n-r} \sum_{P=0}^{M} x_{i_1}\cdots x_{i_r}\, a_P\, \Delta_n^P\, \delta(x),$$

so that

$$\sum_{P=0}^{M} \lambda^{-2P} x_{i_1} \ldots x_{i_r} \; \Delta_n^{P} \; \delta(x) = \lambda^{-\mu-2r} \sum_{P=0}^{M} x_{i_1} \ldots x_{i_r} \; a_P \; \Delta_n^{P} \; \delta(x).$$

and since homogeneous distributions of different degree of homogeneity are linearly independent, we deduce that, in order that $\delta^{(\mu+n+r-1)}(\|x\|)$ be different from zero, we must have that for some $0 \leqslant P \leqslant M$,

$$2p = \mu + 2r \tag{III.4.7}$$

If this is the case, then the distribution

$\sum_{P=0}^{M} x_{i_1} \ldots x_{i_r} \, a_P \, \Delta_n^{P} \, \delta(x)$ reduces to

$$a_{\frac{\mu+2r}{2}} \, x_{i_1} \ldots x_{i_r} \; \Delta_n^{\frac{\mu+2r}{2}} \; \delta(x) \tag{III.4.8}$$

We shall now find a more compact form for this distribution. We recall that the action of (III.4.8) is given by

$$(a_P \, x_{i_1} \ldots x_{i_r} \, \Delta_n^{\frac{\mu+2r}{2}} \; \delta(x)\phi(x))\,(u) =$$

$$= a_P \int x_{i_1} \ldots x_{i_r} \, \Delta_n^{\frac{\mu+2r}{2}} \; \delta(x) \, \phi\,(x-u) dx \quad .$$

The Taylor series of $\phi(x-u)$ can be written as

$$\phi(x-u) = \tilde{\phi}(u-x) = \sum_{n=0}^{n_1} (-j)^n \sum_{|j|=n} \frac{x^j D^j \tilde{\phi}(u)}{j!} + \xi_{n_1}(x,u),$$

thus

$$\int x_{i_1} \ldots x_{i_r} \, \Delta_n^{\frac{\mu+2r}{2}} \; \delta(x)\phi(x-u) \; dx \quad =$$

$$= (-1)^{\mu+2r} \int \delta(x) \; \Delta_n^{\frac{\mu+2r}{2}} \; x^k \{ \sum_{n=0}^{n_1} (-1)^n \sum_{|j|=n} \frac{x^j D^j \tilde{\phi}(u)}{j!} + \xi_{n_1}(x,u) \} \; dx$$

where $x^k = x_{i_1} \ldots x_{i_r} = x_1^{k_1} \ldots x_n^{k_n}$.

For n_1 large enough the term arising from $\xi_{n_1}(x,u)$ is zero, so that the expression above becomes

$$\sum_{n=0}^{n_1} (-1)^n \sum_{|j|=n} \frac{D^j\tilde{\phi}(u)}{j!} \int \delta(x)\, \Delta_n^{\frac{\mu+2r}{2}} x^{k+j}\, dx,$$

and the inner integral can be written as

$$\int \delta(x) \sum_{|P|=\frac{\mu+2r}{2}} \frac{(\frac{\mu+2r}{2})!}{P!}\, D^{2P} x^{k+j}\, dx.$$

This expression is different from zero if and only if $2p_S = i_S + j_S$, $S = 1,2,\ldots n$ and in this case its value is $\frac{(k+j)!}{(\frac{k+j}{2})!}(\frac{\mu+2r}{2})!$, so that

$$\int \Delta_n^{\frac{\mu+2r}{2}} \delta(x) x_{i_1}\cdots x_{i_r}\, \phi(x-u)\, dx =$$

$$= \sum_{n=0}^{n_1} (-1)^n \sum_{|j|=n} \frac{(k+j)!}{(\frac{k+j}{2})!} (\frac{\mu+2r}{2})!\, \frac{D^j\phi(u)}{j!}$$

and since $2|P| = \mu+2r = |k| + |j| = r + |j|$, that is $|j| = \mu + r$, the expression above becomes

$$(-1)^{\mu+r} \sum_{|j|=\mu+r} \frac{(\frac{\mu+2r}{2})!\,(k+j)!}{j!\,(\frac{k+j}{2})!}\, D^j\tilde{\phi}(u)$$

In particular, if $k = 0$, then $r = 0$ and $j = 2\beta$ for some n tuple $\beta = (\beta_1 \cdot\cdot \beta_n)$, thus the expression above becomes

$$\Delta_n^{\frac{\mu}{2}}\, \tilde{\phi}(u).$$

In order to find the solutions of

$$\{T\,[Q(g)f(g^{-1}x-g^{-1}h)]\,\}(u) =$$

$$Q^*(g)\,[T\;\; f(x)]\quad (g^{-1}u-g^{-1}h) \qquad \text{(II.3.1)}$$

with $Q(g)$ and $Q^*(g)$ irreducible single valued representations of the proper Lorentz group, that also satisfy

$$[T\; f(x)]\;(u) = \lambda^{\mu}\;\;[\;T\; f(x)\,]\;(\lambda u) \qquad \text{(III.4.3)},$$

we proceed exactly as we did in the previous case in this section.

In the present case the non-zero terms of T, as found in Section 3, are of the form

$$a(\langle x,x\rangle)x_{i_1}\dots x_{i_r},$$

where $a(\langle x,x\rangle)$ is a Lorentz invariant distribution which is homogeneous of degree $-\mu-4-r$, that is

$$a(\langle\lambda x,\lambda x\rangle) = \lambda^{-\mu-4-r}\;\; a(\langle x,x\rangle),$$

and by an agreement similar to that of the rotation invariant homogeneous distributions dealt with at the beginning of this section, we find that

$$a(\langle x,x\rangle) = |\langle x,x\rangle|^{-\mu-4-r} + \tilde{\delta}^{(\mu+4+r-1)}(\langle x,x\rangle).$$

Gel'fand and Shilov [8] p.251 have proved that $\tilde{\delta}^{(\mu+4+r-1)}(\langle x,x\rangle)$ decomposes into a sum of a distribution concentrated at the vertex of the cone $\langle x,x\rangle = 0$ and a distribution concentrated on the surface of this cone with the vertex removed.

A Lorentz invariant distribution $a(\langle x,x\rangle)$ concentrated at the vertex of the cone $\langle x,x\rangle = 0$ is of the form $\sum_{P=0}^{M} a_P \,\Box^P\delta(x)$ (Challifour [8] p.153) where,

$$\Box = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial x_4^2} = \sum_{i=1}^{4} S^{ii} D^2_i$$

It is now easy to see that the action of the differential operator arising from the distribution $a(\langle x,x \rangle)$ can be formally obtained from that of the previous case, by replacing D_i^2 by $S^{ii}D^2_i$.

Therefore

$$\int x_{i_1} \cdots x_{i_r} \cdot \Box^{\frac{\mu+2r}{2}} \delta(x)\phi(x-u)dx$$

$$= [\sum_{|j|=\mu+r} \frac{(\frac{\mu+2r}{2})!}{(\frac{k+j}{2})!} \frac{(k+j)!}{j!} S^{\frac{j}{2}} D^j] \tilde{\phi}(u)$$

where

$$x_{i_1} \cdots x_{i_r} = x^{k_1} x^{k_2} x^{k_3} x^{k_4} = x^k,$$

$$S^{\frac{j}{2}} = S_{22}^{\frac{j_1}{2}} S_{22}^{\frac{j_2}{2}} S_{33}^{\frac{j_3}{2}} S_{44}^{\frac{j_4}{2}}$$

with the prescription that if $j_i = 0$, then $S^{\frac{j_i}{2}\frac{j_i}{2}} = 1$, we also recall that $j = 2r$ for some n-tuple $r = (r_1 \ldots r_n)$ of non negative integers.

If $k = 0$, then the formula above becomes

$$[\sum_{|j|=\mu} \frac{(\frac{\mu}{2})!}{(\frac{j}{2})!} S^{\frac{j}{2}} D^j] \tilde{\phi}(u) = \Box^{\frac{\mu}{2}} \tilde{\phi}(u).$$

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