UNIQUE FACTORIZATION RINGS WITH ZERO DIVISORS

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I. Introduction. Throughout this paper R will be a commutative ring with identity having total quotient ring $T(R)$. Unique factorization domains (UFD's) play an important role in commutative ring theory. Classically, a UFD is defined to be an integral domain R satisfying the two factorization conditions: (UFD1) every nonzero element of R that is not a unit is a product of irreducible elements and (UFD2) if $0 \neq r_1 \cdots r_n = s_1 \cdots s_m$ where the r_i's and s_j's are irreducible, then $n = m$ and after a suitable reordering r_i and s_i are associates for $i = 1, \cdots, n$. (Here by an irreducible element we mean a nonzero nonunit element r of R with the property that $r = ab$ for $a, b \in R$ implies a or b is a unit. Two elements a and b are associates if $a = ub$ where u is a unit.) If r is an irreducible element in a UFD, then r is a principal prime, i.e., (r) is a prime ideal. Indeed, if $r \mid ab \neq 0$, then after factoring a and b into irreducibles, the condition (UFD2) shows that $r \mid a$ or $r \mid b$. Conversely, in any ring, a regular principal prime is irreducible. Moreover, if $p_1 \cdots p_n = q_1 \cdots q_m$ are two factorizations of a regular element into a product of regular principal primes, then $n = m$ and after a reordering p_i and q_i are associates for $i = 1, \cdots, n$. ($q_1 \cdots q_m = p_1 \cdots p_n \in (p_1)$, so say $q_1 \in (p_1)$. Then $q_1 = ap_1$. But q_1 is irreducible so a is a unit and hence p_1 and q_1 are associates. Cancelling p_1 from both sides of the equation and applying induction yields the conclusion. Actually, more is true. Any factor of a product of regular principal primes is itself a product of regular principal primes. Hence any irreducible factor of a product of regular principal primes is actually a principal prime. Thus we get the "stronger" result that a product of principal primes has a unique decomposition into irreducible elements.) Thus an integral domain is a UFD if and only if it satisfies (UFD3): every nonzero nonunit is a product of principal primes.

In this paper we study generalizations of properties (UFD1) and (UFD2), or (UFD3) to rings with zero divisors. There are many ways in which this can be done. In fact, two such generalizations have already been given by Galovich [11] and Fletcher ([8] and [9]).

Galovich [11] defines a unique factorization ring to be a commutative ring with identity satisfying (UFD1) and (UFD2). He shows that if R is not an integral domain, then every irreducible element is nilpotent, so R is a quasilocal ring. Further, if R has more than one nonassociate irreducible element, then $M^2 = 0$, where M is the unique maximal ideal of R. Thus a ring R is a unique factorization ring (in the sense of Galovich) if and only if R is either a UFD, a quasilocal ring (R,M) with $M^2 = 0$ (here every nonzero nonunit is irreducible), or R is a special principal ideal ring (SPIR) (i.e., R is a principal ideal ring with a single nilpotent prime ideal). Galovich then gives a complete structure theory for rings of the last two types using Cohen's Structure Theorems for complete local rings. Since Galovich gives such a complete characterization of his "unique factorization rings," we will not consider them further.

In the second section we consider the unique factorization rings defined by Fletcher [8]. In fact, throughout this paper by a unique factorization ring (UFR) we mean a unique factorization ring as defined by Fletcher. Fletcher [9] showed that a ring is a UFR if and only if it is a finite direct product of UFD's and SPIR's. We will give an alternate proof of this result via Mori's theorem characterizing rings in which every principal ideal is a product of prime ideals (π-rings) and our work on factorial rings contained in the third section. We also give several new conditions characterizing UFR's and π-rings.

In the third section we define a third generalization of UFD's which we call a factorial ring. A factorial ring is a ring in which UFD1 and UFD2 hold for the regular elements or equivalently, R is factorial if every regular nonunit is a product of principal primes. Examples of factorial rings include direct products of UFD's, total quotient rings, and rank one discrete valuation pairs. Here we use a slight modification of Kennedy's [14] notion of a Krull ring (with zero divisors). We characterize a factorial ring as a Krull ring with trivial divisor class group.

We follow standard terminology from commutative ring theory with the possible exceptions of Fletcher's definitions of irreducible element and associate elements (in Section II). Our general references will be Gilmer [12], Kaplansky [13], and Larsen and McCarthy [15]. Fossum [10] is a general reference concerning Krull domains and Larsen and McCarthy [15] is an excellent reference for valuation pairs and Prüfer rings.

II. Unique Factorization Rings. We begin with a brief review of the following definitions from Fletcher [8] necessary to give Fletcher's definition of a unique factorization ring.

Let R be a commutative ring with 1 and let $r \in R$. A refinement of a factorization $r = a_1 \cdots a_n$ is obtained by factoring one or more of the a_i's. A nonunit r is said to be irreducible if each factorization of r has a refinement containing r. Two elements a and b of R are associates if $(a) = (b)$. The U-class of an element r is defined to be the saturated multiplicatively closed set $U(r) = \{b \in R \mid b(r) = (r)\}$. A U-decomposition of an element r is a factorization $r = (p_1' \cdots p_{k'}')(p_1 \cdots p_k)$ such that (i) the p_i''s and p_i's are irreducible, (ii) $p_i' \in U(p_1 \cdots p_k)$ for $i = 1, \cdots, k'$ and (iii) $p_i \notin U(p_1 \cdots \hat{p}_i \cdots p_k)$ for $i = 1, \cdots, k$. Fletcher shows that if r is a product of irreducible elements, then r has a U-decomposition. Two U-decompositions of $r \in R$, $r = (p_1' \cdots p_{k'}')(p_1 \cdots p_k) = (q_1' \cdots q_{\ell'}')(q_1 \cdots q_\ell)$ are associates if (i) $k = \ell$ and (ii) after a suitable change in the order of the factors in the second half of each U-decomposition (called the relevant factors), we have p_i and q_i are associates for $i = 1, \cdots, k$. A ring R is called a unique factorization ring (UFR) if (UFR1) every nonunit of R has a U-decomposition and (UFR2) any two U-decompositions of a nonunit element of R are associates.

Before proceeding, some remarks about the above definitions are in order. Fletcher [8, page 580] gives an example of two elements that are associate (i.e., $(a) = (b)$), but b does not have the form $a = ub$ where u is a unit. However, if a is a regular element (i.e., a nonzero divisor) and a and b are associates, then $(a) = (b)$, so $a = rb$ and $b = as$ so $a = rb = rsa$. Since a is regular, $rs = 1$, so r is a unit. Thus if a is regular, Fletcher's notion of associates agrees with the usual definition. Also, if a is regular, then $U(a)$ is just the set of units of R, for if $t(a) = (a)$, then ta and a are associates, so $ta = ua$ where u is a unit, so $t = u$ is a unit. Hence if t is a regular element, a U-decomposition of t has the form $t = (\ )(t_1 \cdots t_n)$ where $t_1, \cdots, t_n$ are irreducible elements in the usual sense. In this case to say that any two U-decompositions of t are associates just says that if $t = t_1 \cdots t_n = t_1' \cdots t_m'$ where the t_i's and t_i''s are irreducible, then $n = m$ and after a reordering the t_i's and t_i''s are associates.

Fletcher shows [9, Theorem 19] that a ring is a UFR if and only if it is a finite direct product of UFD's and SPIR's. It follows from Fletcher's structure theorem for UFR's that an element of a UFR is irreducible if and only if it is prime. Hence in a UFR every element (principal ideal) is a product of principal prime elements (ideals). Either of these two properties actually characterizes UFR's. This is an immediate corollary to the result of Mori mentioned in the Introduction. (Mori proves this result in a series of four papers [17]-[20]. A proof of this result may also be found in [12].) Without using Fletcher's characterization of a UFR as a direct product of UFD's and SPIR's, we show that an irreducible element in a UFR is prime. Fletcher's structure theory for UFR's will then follow from Mori's theorem.

<u>Theorem 2.1</u> (Mori). Let R be a commutative ring. Then every principal ideal of R is a product of prime ideals if and only if R is a finite direct product of SPIR's and integral domains with the property that every principal ideal is a product of prime ideals.

Recall that a ring is called a π-ring if every principal ideal is a product of prime ideals. A π-ring that is an integral domain is called a π-domain. Thus in a π-domain every nonzero principal ideal is a product of invertible prime ideals. Many characterizations of π-domains may be found in [3],[4],[6] and [7]. For example, an integral domain R is a π-domain if and only if R is a locally factorial Krull domain or if and only if $R(X)$ is a UFD.

Corollary 2.2. For a commutative ring R the following conditions are equivalent.

(1) Every element (principal ideal) of R is a product of principal prime elements (ideals).

(2) R is a finite direct product of UFD's and SPIR's.

(3) R is a π-ring and every invertible ideal of R is principal.

Lemma 2.3. An element of a UFR is irreducible if and only if it is prime.

Proof. ($\Leftarrow$) In any ring a principal prime is irreducible. ($\Rightarrow$) Suppose that R is a UFR. Let r be irreducible. First suppose that r is not regular. Then there is an $a \neq 0$ with $ar = 0$. Let $a = (a_1' \cdots a_{n'}')(a_1 \cdots a_n)$ be a U-decomposition of a. If $r \in U(a_1 \cdots a_n)$, then $(a) = (a_1 \cdots a_n) = r(a_1 \cdots a_n) = r(a) = 0$, a contradiction. Hence r is one of the relevant irreducible factors involved in a U-decomposition of 0. Suppose that $ef \in (r)$, so $ef = cr$. Let c have a U-decomposition $c = (c_1' \cdots c_{\ell'}')(c_1 \cdots c_\ell)$. If r is one of the relevant irreducible factors in a U-decomposition of cr, then after taking a

U-decomposition of e and f, multiplying them together and reducing the product to a U-decomposition of ef, we see that r must be a relevant irreducible factor of e or f, so $e \in (r)$ or $f \in (r)$. Thus we can assume that r is not a relevant factor of rc so $r \in U(c_1 \cdots c_\ell)$, i.e., $r(c_1 \cdots c_\ell) = (c_1 \cdots c_\ell)$. Now $(a_1 \cdots a_n)(c_1 \cdots c_\ell) = (0)$. Moreover, since $(c_1 \cdots c_\ell) = r(c_1 \cdots c_\ell)$, in this product we can delete any of the a_i's with $(a_i) = (r)$. But this yields a U-decomposition of 0 without r as a factor. Thus we are reduced to showing that an irreducible element that is regular is prime. Let $ab \in (r)$ where a and b are regular elements. Then $ab = cr$ for some $c \in R$. Factoring a and b into irreducible elements and applying the remarks in the third paragraph of this section concerning U-decompositions of regular elements, we see that r must be an irreducible factor of a or b so $a \in (r)$ or $b \in (r)$. Hence by Corollary 3.5 (in the next section) we see that (r) is prime.

It is interesting to note that in Lemma 2.3 we only used the uniqueness of U-decompositions (UFR2) for 0 and for regular elements. Combining this observation, Lemma 2.3 and Corollary 2.2 yields the following characterizations of UFR's.

<u>Theorem 2.4</u>. For a commutative ring R the following conditions are equivalent.

(1) R is a UFR.

(2) R satisfies (UF1) (i.e., every element of R has a U-decomposition, or equivalently, is a product of irreducible elements) and R satisfies UFR2 for 0 and for regular elements.

(3) Every principal ideal of R is a product of principal prime ideals.

(4) Every element of R is a product of principal primes.

(5) R is a finite direct product of UFD's and SPIR's.

In the introduction we observed that if a regular element r is a product of principal primes, then this is the only decomposition of r into irreducible factors, up to order and associates. Thus any two U-decompositions of r are associates. Suppose that a nonregular element is a product of principal primes. Then since a principal prime is irreducible, r has a U-decomposition. A natural question is whether any two U-decompositions of r are associates. (This is of course the case for a UFR.) The answer is no. Let K be a field and $R = K[[X,Y]]/(X,Y)(X)(Y)$. Then $\overline{X}$ and $\overline{Y}$ (the images of X and Y, respectively, in R) are principal primes, but $(\ )(\overline{X}\,\overline{X}\,\overline{Y}) = (\ )(\overline{X}\,\overline{Y}\,\overline{Y})$ are two nonassociate U-decompositions of $\overline{0}$.

It is well known [13, Theorem 5] that an integral domain is a UFD if and only if every nonzero prime ideal contains a nonzero principal prime ideal. We wish to extend this result to UFR's. We first however give a lemma that may be of independent interest. We will need the following known facts about the ring $R(X)$. For a commutative ring R, the ring $R(X)$ is the ring $R[X]_S$ where S is the multiplicatively closed set $S = \{a_0+a_1X+\cdots+a_nX^n \in R[X] \mid (a_0,\cdots,a_n) = R\}$. We will use the following facts about $R(X)$ which may be found in [2]. There is a one-to-one correspondence between the minimal (maximal) prime ideals of $R(X)$ given by $P \longleftrightarrow PR(X)$. For any ideal A of R, $AR(X) \cap R = A$. An ideal A of R is finitely generated and locally principal if and only if $AR(X)$ is principal. Every finitely generated locally principal ideal of $R(X)$ is principal.

Lemma 2.5. Let R be a commutative ring and let A be an ideal of R. Suppose that every prime ideal P minimal over A has the property that P/A is finitely generated and locally principal. Then there are only finitely many prime ideals minimal over A.

Proof. By passing to R/A, we may take $A = 0$. Hence we are assuming that every minimal prime ideal of R is finitely generated and locally principal. We need that R has only finitely many minimal primes. Pass to the ring $R(X)$. By the remarks of the above paragraph the minimal prime ideals of $R(X)$ are principal. It suffices to show that $R(X)$ has only finitely many minimal primes. Thus we may assume that the minimal prime ideals of R are principal. Let S be the multiplicatively closed subset of R generated by the minimal principal prime elements. It suffices to show that $0 \in S$. For if $0 = p_1 \cdots p_n$ where the p_i's are minimal principal primes, then any minimal prime contains a p_i and hence has the form (p_i). Suppose that $0 \notin S$. Then we can enlarge (0) to a prime ideal P maximal with respect to $P \cap S \neq \phi$. But since P contains a minimal prime ideal, this is absurd.

Theorem 2.6. A ring R is a UFR if and only if

(a) every rank zero prime ideal is principal, and

(b) every prime ideal of rank > 0 contains a principal prime ideal of rank > 0.

Proof. ($\Rightarrow$) This is evident from the fact that a UFR is a finite direct product of UFD's and SPIR's. ($\Leftarrow$) Suppose that R satisfies (a) and (b). By the previous lemma, R has only a finite number of minimal primes, say $(p_1),\cdots,(p_n)$. Let M be a maximal ideal of R. If rank $M > 0$, then M contains a nonminimal principal prime (p). Since $(p) \supsetneq (p_i)$ for some i, we have $(p_i) = (p)(p_i)$. In R_M, we have $(p_i)_M = (p)_M(p_i)_M$, so by Nakayama's Lemma $(p_i)_M = (0)_M$, so R_M is an integral domain. Hence each maximal ideal M contains a unique minimal prime ideal (p_i) and either $M = (p_i)$ or $(0)_M = (p_i)_M$. Hence either (p_i) is both a maximal and minimal prime ideal or $(p_i) = (p_i)^2$. The (p_i)'s are also comaximal. Let $(0) = (p_1)^{s_1}\cdots(p_n)^{s_n}$ where either $s_i = 1$ or $s_i > 1$ and (p_i) is a maximal ideal. (Here we have used the fact that $\sqrt{(0)} = (p_1)\cap\cdots\cap(p_n)$.) Then $R \cong R/(p_1)^{s_1} \times \cdots \times R/(p_n)^{s_n}$. If $s_i > 1$, then $R/(p_i)^{s_i}$ is a SPIR and if $s_i = 1$, then every nonzero prime ideal of $R/(p_i)$ contains a nonzero principal prime ideal by (b) and hence $R/(p_i)$ is a UFD.

Using the techniques of the above proof, it is not hard to show that a ring R is a π-ring if and only if (a) every minimal prime ideal of R is finitely generated and locally principal and (b) every prime ideal of rank > 0 contains a finitely generated locally principal prime ideal of rank > 0.

Our last result of this section considers the question of when $R[X]$, $R[[X]]$, or $R(X)$ is a UFR.

<u>Theorem 2.7</u>. (1) $R[X]$ is a UFR if and only if R is a finite direct product of UFD's.

(2) If $R[[X]]$ is a UFR, then R is a finite direct product of UFD's.

(3) $R(X)$ is a UFR if and only if R is a π-ring.

Proof. (1) Suppose that $R = D_1 \times \cdots \times D_n$ is a direct product of UFD's. Then each $D_i[X]$ is a UFD and hence $R[X] = D_1[X] \times \cdots \times D_n[X]$ is a direct product of UFD's and hence is a UFR. Conversely, suppose that $R[X]$ is a UFR. Let $P_1, \cdots, P_n$ be the minimal prime ideals of R. Then $P_1R[X], \cdots, P_nR[X]$ are the minimal prime ideals of $R[X]$. Now since no $P_iR[X]$ can be maximal, it follows as in the proof of Theorem 2.6 that $R[X] = R[X]/P_1R[X] \times \cdots \times R[X]/P_nR[X] \cong (R/P_1)[X] \times \cdots \times (R/P_n)[X]$ and each $(R/P_i)[X]$ is a UFD. Hence each R/P_i is a UFD and $R \cong R/P_1 \times \cdots \times R/P_n$.

(2) The proof is similar to (1). Since a power series ring over a UFD need not be a UFD, a power series ring over a UFR need not be a UFR.

(3) Suppose that R is a π-ring. Then $R = R_1 \times \cdots \times R_n$ where each R_i is either a SPIR or π-domain. If R_i is a SPIR, then it follows from [1, Theorem 8] that $R_i(X)$ is a SPIR. If R_i is a π-domain, it follows from [7, Theorem 2] that $R_i(X)$ is a UFD. Hence $R(X) = R_1(X) \times \cdots \times R_n(X)$ is a direct product of SPIR's and UFD's and so is a UFR. Conversely, suppose that $R(X)$ is a UFR. Let $P_1, \cdots, P_n$ be the minimal prime ideals of R. Then $P_1R(X), \cdots, P_nR(X)$ are the minimal prime ideals of $R(X)$. Hence $P_1R(X), \cdots, P_nR(X)$ are comaximal, so we have $(0) = P_1R(X)^{s_1} \cdots P_nR(X)^{s_n} = P_1^{s_1}R(X) \cap \cdots \cap P_n^{s}R(X)$, so contracting back to R gives $(0) = P_1^{s_1} \cap \cdots \cap P_n^{s_n}$ and $P_1, \cdots, P_n$ are still comaximal. Hence $R \cong R/P_1^{s_1} \times \cdots \times R/P_n^{s_n}$. Now $(R/P_1^{s_1})(X) \cong R(X)/P_i^{s_i}R(X)$ is either SPIR or UFD (according to whether $s_i > 1$ or $s_i = 1$). It follows from [1, Theorem 8] or [7, Theorem 2] that $R/P_i^{s_i}$ is a SPIR or π-domain, respectively. Hence R is a finite direct product of SPIR's and

π-domains, so R is a π-ring.

Another way to state (2) of Theorem 2.7 is that R[X] is not a UFR if R is a SPIR. Since every finitely generated locally principal ideal of R(X) is principal, R(X) is a UFR if and only if it is a π-ring. The proof of (1) of Theorem 2.7 may be modified to yield the result that R[X] is a π-ring if and only if R is a finite direct product of π-domains.

III. Krull Rings and Factorial Rings. Kennedy [14] has extended the notion of a Krull domain to commutative rings with identity which may contain zero divisors. We begin with some necessary definitions. Let R be a commutative ring with identity having total quotient ring T(R). We do not assume that $R \neq T(R)$. Let F(R) denote the collection of regular fractional ideals of R, so $F(R) = \{R\}$ if $R = T(R)$. For $A,B \in F(R)$, the set $[A:B] = \{x \in T(R) \mid xB \subseteq A\}$ is again an element of F(R). We will usually denote $[R:[R:A]]$ $(A \in F(R))$ by A_v and we will say that A is divisorial if $A = A_v$. As in the domain case, A_v is the intersection of the (necessarily regular) principal ideals containing it. The set D(R) of divisorial ideals of R becomes a semigroup under v-multiplication (for $A,B \in D(R)$, define $A*B = (AB)_v$) with R as identity. As in the domain case, Kennedy shows that D(R) is a group if and only if R is completely integrally closed. Let I(R) be the subgroup of D(R) consisting of invertible ideals and let P(R) be the subgroup of I(R) consisting of regular principal ideals. The quotient group $C(R) = I(R)/P(R)$, called the class group of R, is a subgroup of the divisor class semigroup $Cl(R) = D(R)/P(R)$. Thus R is completely integrally closed if and only if $Cl(R)$ is a group.

We now give the definition of a Krull ring. Our definition differs from that given by Kennedy in that we allow a total quotient ring to be a Krull ring. (This is analogous to allowing a field to be a Krull domain.) Allowing a total quotient ring to be a Krull ring appears to make the statement of certain theorems more uniform; for example, then a direct product of rings is a Krull ring if and only if each direct factor is a Krull ring.

A ring R is a Krull ring if either (1) $R = T(R)$ or (2) there exists a family $\{(V_\alpha, P_\alpha)\}$ of discrete rank one valuation pairs of $T(R)$ such that (I) $R = \bigcap V_\alpha$ and (II) $v_\alpha(a) = 0$ for all but a finite number of α for each regular $a \in T(R)$ and each P_α is a regular ideal of V_α.

Krull rings can be defined without recourse to valuation pairs. ([15] is an excellent reference for valuation pairs.) For Kennedy's definition, this is done in [14] and [16].

Proposition 3.1. R is a Krull ring if and only if R is completely integrally closed and R satisfies the maximum condition on divisorial ideals.

Proof. First, suppose that $R = T(R)$. Then by (our) definition R is a Krull ring, R is completely integrally closed and R satisfies the maximum condition on divisorial ideals (indeed, R is the only divisorial ideal). Thus we may suppose that $R \neq T(R)$. The implication ($\Rightarrow$) appears in [14, Proposition 2.2] while the implication ($\Leftarrow$) appears in [16, Theorem 5].

Proposition 3.2. Let $R_1, \cdots, R_n$ be commutative rings with identity. Then $R_1 \times \cdots \times R_n$ is a Krull ring if and only if each R_i is a Krull ring.

Proof. This follows easily from the characterization of Krull rings given in Proposition 3.1 and the following facts about direct products of rings: $T(R_1 \times \cdots \times R_n) = T(R_1) \times \cdots \times T(R_n)$, $R_1 \times \cdots \times R_n$ is completely integrally closed if and only if each R_i is completely integrally closed, $A_1 \times \cdots \times A_n$ is a divisorial ideal of $R_1 \times \cdots \times R_n$ if and only if each A_i is a divisorial ideal of R_i and $R_1 \times \cdots \times R_n$ satisfies the maximum condition on divisorial ideals if and only if each R_i does.

Under Kennedy's definition the above proposition would not be true. For Kennedy [14, Proposition 2.5] remarks that if R is a Krull ring and S is a total quotient ring, then $R \times S$ is a Krull ring, but S is not a Krull ring under Kennedy's definition.

Proposition 3.3. Let $R_1, \cdots, R_n$ be commutative rings with identity. Then $C(R_1 \times \cdots \times R_n) = C(R_1) \times \cdots \times C(R_n)$ and $Cl(R_1 \times \cdots \times R_n) = Cl(R_1) \times \cdots \times Cl(R_n)$.

Proof. Both equalities follow from the facts that the ideal $A_1 \times \cdots \times A_n$ of $R_1 \times \cdots \times R_n$ (each A_i is an ideal of R_i) is a regular principal ideal (respectively invertibel or divisorial) if and only if each A_i is a regular principal ideal of R_i (respectively invertible or divisorial).

Recall that R is called a UFR if R is a finite direct product of UFD's and SPIR's. Clearly a UFR is a Krull ring with trivial divisor class group. Kennedy remarked this [14, Proposition 3.1] and raises the question of whether a Krull ring with trivial divisor class group is a UFR. Matsuda [16] answered this question in the negative as follows. Let D be a UFD and let S be a total quotient ring that is not a UFR, then $D \times S$ is a Krull ring, not a UFR, but $Cl(D \times S) = Cl(D) \times Cl(S) = 0$.

We next give another natural extension of UFD's to commutative rings with identity that may contain zero divisors. We will show that this class of rings is just the Krull rings with trivial divisor class group. To avoid confusion with our previously given generalizations of UFD's, we call these rings factorial rings. In what follows, a nonunit element a of a ring R is <u>irreducible</u> if for any factorization $a = bc$ in R either b or c is a unit in R. Two elements a and b of R are <u>associates</u> if $a = \lambda b$ where λ is a unit of R. If a and b are both regular, then a and b are associates if and only if $(a) = (b)$.

A commutative ring R with identity is called a <u>factorial ring</u> if (I) every regular nonunit element of R is a product of irreducible (regular) elements, and (II) if $r_1 r_2 \cdots r_n = s_1 \cdots s_m$ are two factorizations of a regular nonunit of R into irreducible elements, then $n = m$ and after reordering (if necessary) r_i and S_i are associates for $i = 1, \cdots, n$. Thus R is factorial if and only if UFD1 and UFD2 hold for regular elements of R.

Clearly an integral domain is a factorial ring if and only if it is a UFD (i.e., a factorial domain!). A total quotient ring is vacuously a factorial ring. It is easily seen that a direct product of rings

$R_1 \times \cdots \times R_n$ is factorial if and only if each R_i is factorial. Hence a direct product of factorial domains (UFD's) and total quotient rings is a factorial ring. Thus a factorial ring need not be a UFR, but a UFR is a factorial ring. It will follow from Theorem 3.6 that if R is a factorial ring and S is a multiplicatively closed set consisting of regular elements, then R_S is a factorial ring.

Our definition of a factorial ring is analogous to that of Prüfer rings which generalize Prüfer domains. Recall that an integral domain is called a Prüfer domain if every finitely generated nonzero ideal is invertible. Prüfer domains are also characterized among integral domains as those integral domains whose lattice of ideals is distributive. A ring R is called a Prüfer ring if every finitely generated regular ideal is invertible. A ring R is called an arithmetical ring if the lattice of ideals of R is distributive. Our generalization of UFD's to factorial rings is similar to the generalization of Prüfer domains to Prüfer rings just as the generalization of UFD's to UFR's is similar to the generalization of Prüfer domains to arithmetical rings. (An excellent treatment of Prüfer rings may be found in [15].)

Our next theorem gives several equivalent conditions for a commutative ring with identity ring to be a factorial ring. In the case of an integral domain R, these conditions are all well known to be equivalent to R being a UFD. We first need a lemma concerning divisorial ideals.

<u>Lemma 3.4</u>. Let R be a commutative ring with identity and let A and B be divisorial ideals of R. If A and B contain the same regular elements, then A = B.

Proof. Notice that since A is divisional, for a regular element $r \in T(R)$ we have $[R:A] \subseteq (r) \Leftrightarrow r^{-1}[R:A] \subseteq R \Leftrightarrow r^{-1} \in [R:[R:A]] = A$. Hence $[R:A]$ and $[R:B]$ are divisorial ideals contained in the same set of regular principal ideals. Since a divisorial ideal is the intersection of the principal ideals containing it, $[R:A] = [R:B]$. Hence $A = [R:[R:A]] = [R:[R:B]] = B$.

Corollary 3.5. Let A be a divisorial ideal in a commutative ring R. Then A is a prime ideal if and only if whenever the product of two regular elements lies in A one of them lies in A.

Proof. Let $ab \in A$. Suppose that $a \notin A$ and $b \notin A$. Then $(a,A)_v \supsetneqq A$ and $(b,A)_v \supsetneqq A$. Hence by Lemma 3.4 there exist regular elements $a_1 \in (a,A)_v - A$ and $b_1 \in (b,A)_v - A$. Then $a_1b_1 \in (a,A)_v(b,A)_v \subseteq ((a,A)(b,A))_v \subseteq A_v = A$. Since a_1 and b_1 are regular, by hypothesis $a_1 \in A$ or $b_1 \in A$. This contradiction show that A is prime.

The next theorem, the main result of this section, gives several characterizations of factorial rings.

Theorem 3.6. For a commutative ring R with identity the following conditions are equivalent.

(1) R is factorial.

(2) R is a Krull ring with $Cl(R) = 0$, i.e., every divisorial ideal of R is principal.

(3) R is a Krull ring and every maximal divisorial ideal is principal.

(4) R has the maximum condition on regular principal ideals and the intersection of two regular principal ideals is principal.

(5) R is a Krull ring and the intersection of two regular principal ideals is principal.

(6) Every regular prime ideal of R contains a regular principal prime ideal.

(7) Every regular element (principal ideal) is a product of regular prinicpal prime elements (ideals).

Proof. Since the above seven conditions hold for a total quotient ring, we may assume that $R \neq T(R)$. Kennedy [14, Proposition 3.1] gives the following implications: (3)$\Leftrightarrow$(5)$\Leftrightarrow$(2)$\Leftrightarrow$(4). The equivalence of (6) and (7) is similar to the domain case which is given in [13, Theorem 5]. (7)$\Rightarrow$(1): A prime element is irreducible and the decompositions of a regular element into a product of (regular) primes is unique up to order of the factors and associates.
(3)$\Rightarrow$(7): Since R has ACC on regular principal ideals, every principal ideal is a product of irreducible principal ideals. Hence it suffices to show that an irreducible principal ideal is prime. Let r be irreducible. Then clearly (r) is a maximal divisorial ideal. Suppose that $ab \in (r)$ but $a \notin (r)$ and $b \notin (r)$. Then $(a,r) \supsetneq (r)$ and $(b,r) \supsetneq (r)$, so $(a,r)_v = (b,r)_v = R$. Hence $R = ((a,r)_v(b,r)_v)_v = ((a,r)(b,r))_v \subseteq (r)_v = (r)$. This contradiction shows that (r) is prime. (1)$\Rightarrow$(4): Clearly a factorial ring has ACC on regular principal ideals. Let r and s be two regular elements, we need that

$(r) \cap (s)$ is principal. Let t be the product of the irreducible factors dividing both r and s. Let $r = r't$ and $s = s't$ where s' and t' have no common irreducible factors. Since $(r) \cap (s) = (r't) \cap (s't) = t((r') \cap (s'))$, it suffices to show that $(r') \cap (s') = (r's')$. Thus we can assume that r and s have no irreducible factors in common. Now $(r) \cap (s) \supseteq (rs)$. Suppose that $u \in (r) \cap (s)$ is regular. Then r/u and s/u, so by the uniqueness of the factorization into irreducible elements u must have all the irreducible factors of both r and s, so since r and s have no common irreducible factors, rs/u so $u \in (rs)$. Thus $(r) \cap (s)$ and (rs) are two divisorial ideals containing the same regular elements, so by Lemma 3.4, $(r) \cap (s) = (rs)$. $(4) \Rightarrow (5)$: We show that every divisorial ideal is principal. Then $D(R)$ is a group, so R is completely integrally closed and R has ACC on divisorial ideals, so R is a Krull ring. Let $A \subsetneq R$ be a divisorial ideal If (x) is a regular (fractional) principal ideal, then $R \cap (x)$ being the intersection of two regular principal ideals is principal. Hence, $A = \cap\{(s) \mid s$ is regular with $A \subseteq (s) \subsetneq R\}$. If r is irreducible, (r) is a maximal divisorial ideal. Since every irreducible principal ideal is a maximal divisorial ideal, the proof of $(3) \Rightarrow (7)$ gives that every regular element of R is a product of prime elements. Let $t \in A$ be regular. Then $(s) \supseteq A$ implies $(s) \supseteq (t)$ so s is just a product of a some of the primes occurring in t. Hence there are only finitely many distinct regular principal ideals containing A, so A is a finite intersections of principal ideals and hence is principal.

We next give an example of a factorial ring that is not a direct product of UFD's and total quotient rings.

Example 3.7. Let V be a valuation domain with value group the lexicographic direct sum of two copies of Z, i.e., V is a rank 2 discrete valuation ring. The maximal ideal of V is principal, say (p) and $Q = \bigcap_{n=1}^{\infty}(p^n)$ is the other nonzero prime of V. Since V is discrete $Q \neq Q^2$. Then V/Q^2 is a one-dimensional quasi-local ring with $Z(V/Q^2) = Q/Q^2$. Every regular (principal) ideal is of the form (p^n), so V is an indecomposable factorial ring. In terms of valuations, $(V/Q^2,(p)/Q^2)$ is a discrete rank one valuation pair on $T(V/Q^2)$.

If R is an integral domain, then R is Krull (factorial) if and only if $R[X]$ is Krull (factorial). This is no longer true if R is not a domain. For if a R is nilpotent, a/X is integral over $R[X]$. Hence if $R[X]$ is a Krull ring, $R[X]$ must be completely integrally closed which implies that R is reduced. It is however easy to characterize the rings R for which $R[X]$ is factorial. It is interesting to note that we get the same answer as in Theorem 2.7.

Theorem 3.8. For a commutative ring R the following conditions are equivalent.

(1) $R[X]$ is factorial.

(2) $R[X]$ is a UFR.

(3) R is a finite direct product of UFD's.

Proof. Theorem 2.7 gives (3) ⇒ (2). If $R[X]$ is a UFR, then every element of $R[X]$ is a product of principal primes. Hence every regular element of $R[X]$ is a product of principal primes, so by Theorem 3.6, $R[X]$ is factorial. (1) ⇒ (3). By the remarks of the previous paragraph, if $R[X]$ is factorial, then $R[X]$ is a Krull ring and hence R must be reduced. Since X is regular in $R[X]$, X must be a product of principal primes. Hence $(0) = (\bar{X})$ in $R \approx R[X]/(X)$ is a product of principal primes. Thus R has a finite number of minimal primes, each of which is principal, say $(p_1),\ldots,(p_n)$ are the minimal prime ideals of R. Since R is reduced the zero divisors of R are $(p_1) \cup \ldots \cup (p_n)$. Hence $(p_1),\ldots,(p_n)$ are the only nonregular prime ideals of R. Since R is reduced, the only units of $R[X]$ are units of R. Hence elements of R are associate in R if and only if they are associate in $R[X]$. Thus if a regular element of R has two nonassociate factorizations in R, it has two nonassociate factorizations in $R[X]$. But since $R[X]$ is factorial, this is impossible. Moreover, since $R[X]$ has ACC on regular principal ideals, so does R. Hence R is factorial. Hence by Theorem 3.6, every regular prime ideal of R contains a regular principal prime ideal. Hence by Theorem 2.6, R is a UFR.

Actually many other divisibility properties for integral domains can be carried over to rings with zero divisors. Let R be a commutative ring having total quotient ring $T(R)$. The set $P(R)$ of regular fractional principal ideals forms a partially ordered abelian group, with partial order $(a) \leq (b)$ if and only if $(b) \subseteq (a)$. This group, called the group of divisibility of R, may also be characterized as the multiplicative group of units of $T(R)$ mod the units of R. Clearly $P(R) = 0$ if and only if $R = T(R)$. $P(R)$ is lattice ordered if and only if R is a GCD ring, that is, every pair of regular elements of R has a GCD or equivalently the intersection of two regular principal ideals is principal. Thus by Theorem 3.6 a ring R is factorial if and only if R is a GCD ring with ACC on regular principal ideals. Thus a ring R is factorial if and only if $P(R)$ is order isomorphic to a direct sum of copies of Z (with the order $(a_\alpha) \leq (b_\alpha)$ if and only if $a_\alpha \leq b_\alpha$ for each α where $\bigoplus_\alpha Z$ is the given group).

With the same point of view many of the results from [3]-[6] can be extended to rings with zero divisors.

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