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Frenet equations of biharmonic curves in terms of exponential maps in the special 3-dimensional Kenmotsu manifold

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Abstract

In this article, we study matrix representation of biharmonic curves in 3-dimensional Kenmotsu manifold. We characterize Frenet frame of the biharmonic curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

key words. Kenmotsu manifold, biharmonic curve, Matrix representation

AMS subject classifications. 53C41, 53A10.

1 Introduction

In vector calculus, the Frenet–Serret formulas describe the kinematic properties of a particle which moves along a continuous, differentiable curve in three-dimensional Euclidean space $\mathbb{R}^3$. More specifically, the formulas describe the derivatives of the so-called tangent, normal, and binormal unit vectors in terms of each other. The formulas are named after the two French mathematicians who independently discovered them: Jean Frédéric Frenet, in his thesis of 1847, and Joseph Alfred Serret in 1851. Vector notation and linear algebra currently used to write these formulas was not yet in use at the time of their discovery.

The Frenet–Serret formulas admit a kinematic interpretation. Imagine that an observer moves along the curve in time, using the attached frame at each point as her coordinate system. The Frenet–Serret formulas mean that this coordinate system is constantly rotating as an observer moves along the curve. Hence, this coordinate system is always non-inertial. The angular momentum of the observer's coordinate system is proportional to the Darboux vector of the frame.

Let (N,h) and (M,g) be Riemannian manifolds. A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where the section $\mathcal{T}(\phi) := \boldsymbol{trace} \nabla^{\phi} d\phi$ is the tension field of ϕ.

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \boldsymbol{trace} R(\mathcal{T}(\phi), d\phi) \, d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Obviously, every harmonic map is biharmonic. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this article, we study matrix representation of biharmonic curves in 3-dimensional Kenmotsu manifold. We characterize Frenet frame of the biharmonic curves in terms of their curvature and torsion in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

2 Preliminaries

We consider the special 3-dimensional manifold

$$\mathbb{K} = \left\{ (x,y,z) \in \mathbb{R}^3 : (x,y,z) \neq (0,0,0) \right\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1 = z\frac{\partial}{\partial x}, \quad \mathbf{e}_2 = z\frac{\partial}{\partial y}, \quad \mathbf{e}_3 = -z\frac{\partial}{\partial z} \tag{2.1}$$

are linearly independent at each point of $\mathbb{K}$. Let g be the Riemannian metric defined by

$$\begin{aligned} g(\mathbf{e}_1, \mathbf{e}_1) &= g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1, \\ g(\mathbf{e}_1, \mathbf{e}_2) &= g(\mathbf{e}_2, \mathbf{e}_3) = g(\mathbf{e}_1, \mathbf{e}_3) = 0. \end{aligned}$$

The characterising properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2.$$

Let η be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(M)$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = -\mathbf{e}_2,\ \phi(\mathbf{e}_2) = \mathbf{e}_1,\ \phi(\mathbf{e}_3) = 0.$$

Then using the linearity of and g we have

$$\eta(\mathbf{e}_3) = 1,$$

$$\phi^2(Z) = -Z + \ \eta(Z)\mathbf{e}_3,$$

$$g(\phi Z, \phi W) = g(Z, W) - \eta(Z)\eta(W),$$

for any $Z, W \in \chi(\mathbb{K})$. Thus for $\mathbf{e}_3 = \xi$, (ϕ, ξ, η, g) defines an almost contact metric structure on $\mathbb{K}$, [3,9].

3 Biharmonic Curves in the 3-Dimensional Kenmotsu Manifold

Let γ be a curve on the 3-dimensional Kenmotsu manifold parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{3.1}$$

where κ is the curvature of γ and τ its torsion and

$$\begin{aligned} &g(\mathbf{T}, \mathbf{T}) = 1,\ g(\mathbf{N}, \mathbf{N}) = 1,\ g(\mathbf{B}, \mathbf{B}) = 1, \\ &g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned} \tag{3.2}$$

Now, we consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$.

Theorem 3.2. *If γ is a biharmonic curve in 3-dimensional Kenmotsu manifold, then γ is a helix.*

4 New Approach for Biharmonic Curves in $\mathbb{K}$

A map

$$\exp : R \times \mathbb{K}_3^3 \to GL(3, \mathbb{R}) \subset \mathbb{K}_3^3, \ \ (t, \mathcal{A}) \to \exp(t, \mathcal{A}) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \mathcal{A}^k$$

is called exponential map in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

Definition 4.1. $\langle \mathcal{A}, \mathcal{B} \rangle_{\mathbb{K}} = \mathfrak{trace}\left(\mathcal{A}\mathcal{B}^T\right)$ *is called an inner product for* $\mathcal{A}, \mathcal{B} \in \mathbb{K}_3^3$.

Firstly, let us calculate the arbitrary parameter t according to the arclength parameter s. It is well known that

$$s = \int_0^t \left\|\gamma'(t)\right\|_{\mathbb{K}} dt, \tag{4.1}$$

where

$$\gamma'(t) = \mathcal{A}\gamma. \tag{4.2}$$

The norm of Equation (4.1), we obtain

$$\|\mathcal{A}\gamma\|_{\mathbb{K}} = \sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)},$$

where $\gamma\gamma^T = I$.

Substituting above equation in (4.1), we have

$$s = \sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}t$$

Lemma 4.2. *Let $\mathcal{A}$ be a be an anti-symmetric matrix and $n \in \mathbb{N}$. Then,*

i) *If n is odd, $\mathcal{A}^n$ is an anti-symmetric matrix.*

ii) *If n is even, $\mathcal{A}^n$ is a symmetric matrix.*

iii) *The trace of an anti-symmetric matrix is zero.*

The first, second and third derivatives of γ are given as follows:

$$\begin{aligned} \gamma'(s) &= \frac{\mathcal{A}\gamma}{\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}}, \\ \gamma''(s) &= \frac{\mathcal{A}^2\gamma}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^2}, \\ \gamma'''(s) &= \frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^3}. \end{aligned} \tag{4.3}$$

5 Matrix Representation of Biharmonic Curves in terms of Exponential Maps in the $\mathbb{K}$

Using above sections we obtain following results in $\mathbb{K}$.

Theorem 5.1. [12], *Let $\gamma : I \longrightarrow \mathbb{K}$ be a unit speed non-geodesic biharmonic curve in special 3-dimensional Kenmotsu manifold $\mathbb{K}$. Then,*

$$\begin{aligned} \mathcal{A}\gamma \quad &= \quad \sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}(C_1 e^{-\cos\varphi s}\sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma), \\ &\qquad C_1 e^{-\cos\varphi s}\sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma), -C_1 e^{-\cos\varphi s}\cos\varphi). \end{aligned}$$

$$\begin{aligned} \mathcal{A}^2\gamma = \sqrt{\boldsymbol{trace}\,(\mathcal{A}^4)}\tfrac{\sin\varphi}{\kappa}(&\left(\tfrac{\kappa}{\sin^2\varphi}\cos(\tfrac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\sin(\tfrac{\kappa}{\sin^2\varphi}s+\sigma)\right)(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s}), \\ &\left(-\tfrac{\kappa}{\sin^2\varphi}\sin(\tfrac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\cos(\tfrac{\kappa}{\sin^2\varphi}s+\sigma)\right)(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s}), \\ &(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})), \end{aligned} \tag{5.1}$$

$$
\begin{aligned}
\mathcal{A}^3\gamma &= \frac{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^3}{\kappa}\eth\sin\varphi(C_1(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})e^{-\cos\varphi s}\sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\\
&+C_1e^{-\cos\varphi s}\cos\varphi(-\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\\
&+\cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma))(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s}))\\
&-\frac{\boldsymbol{trace}\,(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)}(C_1e^{-\cos\varphi s}\sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma),\\
&\frac{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^3}{\kappa}\eth\sin\varphi[-C_1e^{-\cos\varphi s}\sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})\\
&-C_1e^{-\cos\varphi s}\cos\varphi(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\\
&+\cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma))(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})]\\
&-\frac{\boldsymbol{trace}\,(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)}C_1e^{-\cos\varphi s}\sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma),\\
&\frac{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^3}{\kappa}\eth\sin\varphi[C_1e^{-\cos\varphi s}\sin\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)(-\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\\
&+\cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma))(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})\\
&-(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)+\cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma))\\
&(q_1e^{\sin\varphi s}+q_2e^{-\sin\varphi s})C_1e^{-\cos\varphi s}\sin\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma))]+\frac{\boldsymbol{trace}\,(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)}C_1e^{-\cos\varphi s}\cos\varphi).
\end{aligned}
$$

where σ, q_1, q_2 *are constants of integration and* $\eth=[\frac{\boldsymbol{trace}(\mathcal{A}^6)}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^6}-\frac{\left(\boldsymbol{trace}(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^5}]^{\frac{1}{2}}$.

In the light of Theorem 4.1, we also give the following theorems :

Theorem 4.2. *Let* $\gamma : I \longrightarrow \mathbb{P}$ *be a unit speed non-geodesic biharmonic curve in special 3-dimensional Kenmotsu manifold* $\mathbb{K}$.. *Then the new Frenet equations of this curve are*

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathbf{T} = -\frac{\mathcal{A}^2\gamma}{\boldsymbol{trace}\,(\mathcal{A}^2)},\\
&\nabla_{\mathbf{T}}\mathbf{N}=\frac{\mathcal{A}^3\gamma}{\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\sqrt{\boldsymbol{trace}\,(\mathcal{A}^4)}},\\
&\nabla_{\mathbf{T}}\mathbf{B} = [\frac{\boldsymbol{trace}\,(\mathcal{A}^6)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^6} - \frac{\left(\boldsymbol{trace}\,(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^4\gamma}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^4}\\
&\qquad +\frac{\boldsymbol{trace}\,(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^6}\mathcal{A}^2\gamma].
\end{aligned} \tag{5.2}$$

Proof. We assume that γ is a unit speed non-geodesic biharmonic curve in special 3-dimensional Kenmotsu manifold $\mathbb{K}$.

From the proof of above Theorem we obtain

$$\mathbf{T} = \frac{\mathcal{A}\gamma}{\sqrt{-\mathfrak{trace}\,(\mathcal{A}^2)}} \tag{5.3}$$

So, by differentiating of the formula (5.3), we get

$$\nabla_{\mathbf{T}}\mathbf{T} = -\frac{\mathcal{A}^2\gamma}{\boldsymbol{trace}\,(\mathcal{A}^2)}.$$

Also, we have the principal normal of the curve

$$\mathbf{N} = \frac{\mathcal{A}^2\gamma}{\sqrt{\boldsymbol{trace}\,(\mathcal{A}^4)}}. \tag{5.4}$$

Differentiating of the formula (5.4), we get

$$\nabla_{\mathbf{T}}\mathbf{N} = \frac{\mathcal{A}^2\gamma'}{\sqrt{\boldsymbol{trace}\,(\mathcal{A}^4)}}.$$

Using (5.3) in above equation, we have

$$\nabla_{\mathbf{T}}\mathbf{N} =\frac{\mathcal{A}^3\gamma}{\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\sqrt{\mathfrak{trace}\,(\mathcal{A}^4)}}.$$

Finally, the same above method we will find $\nabla_{\mathbf{T}}\mathbf{B}$. We have the binormal of the curve

$$\mathbf{B} =[\frac{\boldsymbol{trace}\,(\mathcal{A}^6)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^6} - \frac{\left(\boldsymbol{trace}\,(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^3} + \frac{\boldsymbol{trace}\,(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}\,(\mathcal{A}^2)}\right)^5}\mathcal{A}\gamma]. \tag{5.5}$$

Also, by differentiating of the formula (5.5), we get

$$\nabla_{\mathbf{T}}\mathbf{B} = [\frac{\boldsymbol{trace}\left(\mathcal{A}^6\right)}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^6} - \frac{\left(\boldsymbol{trace}\left(\mathcal{A}^4\right)\right)^2}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^5}]^{-\frac{1}{2}}[\frac{\mathcal{A}^3\gamma'}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^3} + \frac{\boldsymbol{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\gamma'].$$

Since, we have

$$\nabla_{\mathbf{T}}\mathbf{B} = \wp[\frac{\mathcal{A}^3}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^3}\frac{\mathcal{A}\gamma}{\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}} + \frac{\boldsymbol{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^5}\mathcal{A}\frac{\mathcal{A}\gamma}{\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}}],$$

where $\wp = [\frac{\boldsymbol{trace}(\mathcal{A}^6)}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^6} - \frac{\left(\boldsymbol{trace}(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^5}]^{-\frac{1}{2}}$

So we immediately arrive at

$$\nabla_{\mathbf{T}}\mathbf{B} = \wp[\frac{\mathcal{A}^4\gamma}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^4} + \frac{\boldsymbol{trace}\left(\mathcal{A}^4\right)}{\left(\sqrt{-\boldsymbol{trace}\left(\mathcal{A}^2\right)}\right)^6}\mathcal{A}^2\gamma]. \tag{5.6}$$

This completes the proof of the theorem.

Corollary 5.3. *Let $\gamma : I \longrightarrow \mathbb{P}$ be a unit speed non-geodesic biharmonic curve in special 3-dimensional Kenmotsu manifold $\mathbb{K}$. Then*

$$\begin{aligned}\mathcal{A}^4\gamma = -[&\frac{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^4}{\sqrt{\boldsymbol{trace}(\mathcal{A}^4)}}\frac{\tau}{\wp} - \frac{\boldsymbol{trace}(\mathcal{A}^4)}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^2}]\sqrt{\boldsymbol{trace}\left(\mathcal{A}^4\right)}\frac{\sin\varphi}{\kappa}\\ &(\left(\frac{\kappa}{\sin^2\varphi}\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma) + \cos\varphi\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)(q_1e^{\sin\varphi s} + q_2e^{-\sin\varphi s}),\\ &\left(-\frac{\kappa}{\sin^2\varphi}\sin(\frac{\kappa}{\sin^2\varphi}s+\sigma) + \cos\varphi\cos(\frac{\kappa}{\sin^2\varphi}s+\sigma)\right)(q_1e^{\sin\varphi s} + q_2e^{-\sin\varphi s}),\\ &(q_1e^{\sin\varphi s} + q_2e^{-\sin\varphi s})),\end{aligned} \tag{5.7}$$

where σ, q_1, q_2 are constants of integration and $\eth = [\frac{\boldsymbol{trace}(\mathcal{A}^6)}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^6} - \frac{\left(\boldsymbol{trace}(\mathcal{A}^4)\right)^2}{\left(\sqrt{-\boldsymbol{trace}(\mathcal{A}^2)}\right)^5}]^{\frac{1}{2}}$.

Proof. In view of (5.6) and Theorem 5.1 we have (5.7). This proves the corollary.

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