Revista Notas de Matemática
Vol.7(2), No. 311, 2011, pp. 119-127
http://www.saber.ula.ve/notasdematematica
Pre-prints
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Linear operator defined by lambda function for certain analytic functions

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Abstract

For analytic function f in the open unit disc U, a linear operator defined by lambda function is introduced. The object of the present paper is to discuss some properties for $I_{\mu,s}f(z)$ belonging to some classes by applying Jack's lemma.

key words. Jack's lemma, linear operator, analytic function, starlike functions.

AMS subject classifications. 30C45

1 Introduction and preliminaries

Let A denote the class of all analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $U = \{z \in C : |z| < 1\}$.

Let us recall lambda function (4) defined by

$$\lambda(z,s) = \sum_{n=0}^{\infty} \frac{z^n}{(2n+1)^s}$$

$$(z \in U; s \in \ C,\ when,\ |z| < 1;\ \Re(s) > 1,\ when,\ |z| = 1),$$

and let $\lambda^{(-1)}(z,s)$ be defined such that

$$\lambda(z,s) * \lambda^{(-1)}(z,s) = \frac{1}{(1-z)^{\mu+1}}, \quad \mu > -1.$$

We now define $(z\lambda^{(-1)}(z,s))$ as the following:

$$(z\lambda(z,s)) * (z\lambda^{(-1)}(z,s)) = \frac{z}{(1-z)^{\mu+1}}$$

$$= z + \sum_{n=2}^{\infty} \frac{(\mu+1)_{n-1}}{(n-1)!} z^n, \quad \mu > -1$$

and obtain the following linear operator:

$$I_{\mu,s}f(z) = (z\lambda^{(-1)}(z,s)) * f(z)$$

where $f \in A$, $z \in U$, and

$$(z\lambda^{(-1)}(z,s)) \ = z + \sum_{n=2}^{\infty} \frac{(\mu+1)_{n-1}(2n-1)^s}{(n-1)!} z^n.$$

A simple computation, gives us

$$I_{\mu,s}f(z) = z + \sum_{n=2}^{\infty} \frac{(\mu+1)_{n-1}(2n-1)^s}{(n-1)!} a_n z^n \,. \tag{2}$$

where $(\mu)_n$ is the Pochhammer symbol defined by

$$(\mu)_n = \frac{\Gamma(\mu+n)}{\Gamma(\mu)} = \{ \ 1, n = 0\mu(\mu+1)...(\mu+n-1), \quad n = \{1,2,3,...\}.$$

In the following definition, we introduce a new class of analytic functions containing a linear operator defined by lambda function of Eq. (2).

Definition 1.1 Let a function $f \in A$, then $f \in S_{\mu,s}$ if and only if

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} > \alpha, \quad z \in U, \quad 0 \le \alpha < 1. \tag{3}$$

Let f and g be analytic in U. Then f is said to be subordinate to g if there exists an analytic function w satisfying $w(0) = 0$ and $|w(z)| < 1$, such that$f(z) = g(w(z))$, $z \in U$. We denote this subordination as $f(z) \prec g(z)$ or $(f \prec g)$, $z \in U$.

The basic idea in proving our result is the following lemma due to Jack (1) (also, due to Miller and Mocanu (2)).

Lemma 1.1 *(1) Let $\omega(z)$ be analytic in U with $\omega(0) = 0$. Then if $|\omega(z)|$ attains its maximum value on the circle $|z| = r$ at a point $z_0 \in U$, then we have $z_0\omega'(z_0) = \ k\omega(z_0)$, where $k \ge 1$ is a real number.*

2 Main Results

In the present paper, we follow similar works done by Shiraishi and Owa (5) and Ochiai et al. (3), we derive the following result:

Theorem 2.1 *If $f \in A$ satisfies*

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} < \frac{\alpha-3}{2(\alpha-1)}, \quad z \in U$$

for some $\alpha\,(-1 < \alpha \leq 0)$, *then*

$$\frac{I_{\mu,s}f(z)}{z} \prec \frac{1+\alpha z}{1-z}, \quad z \in U.$$

This implies that

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\} > \frac{1-\alpha}{2}.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{I_{\mu,s}f(z)}{z} = \frac{1-\alpha\omega(z)}{1-\omega(z)}, \quad (\omega(z) \neq 1).$$

Clearly, $\omega(z)$ is analytic in U and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in U. Since

$$\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)} = \frac{-\alpha z\omega'(z)}{1-\alpha\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1,$$

we see that

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} = \Re\{\frac{-\alpha z\omega'(z)}{1-\alpha\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} + 1\}$$
$$< \frac{\alpha-3}{2(\alpha-1)} \quad (z \in U)$$

for $-1 < \alpha \leq 0$. If there exists a point $z_0 \in U$ such that

$$\max_{|z| \leq |z_0|} |\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 1.1, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \geq 1$. Thus we have

$$\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)} = \frac{-\alpha z_0\omega'(z_0)}{1-\alpha\omega(z_0)} + \frac{z_0\omega'(z_0)}{1-\omega(z_0)} + 1$$
$$= 1 + \frac{k}{1-e^{i\theta}} - \frac{k}{1-\alpha e^{i\theta}}.$$

If follows that

$$\Re\{\frac{1}{1-\omega(z_0)}\} = \Re\{\frac{1}{1-e^{i\theta}}\} = \frac{1}{2}$$

and

$$\Re\{\frac{1}{1-\alpha\omega(z_0)}\} = \Re\{\frac{1}{1-\alpha e^{i\theta}}\} = \frac{1}{2} - \frac{1-\alpha^2}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

Therefore, we have

$$\Re\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\} = 1 - \frac{k(\alpha^2-1)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

This implies that, for $-1 < \alpha \leq 0$,

$$\Re\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\} \geq 1 + \frac{(1-\alpha^2)}{2(\alpha-1)^2}$$

$$= \frac{\alpha-3}{2(\alpha-1)}.$$

This contradicts the condition in the theorem. Then, there is no $z_0 \in U$ such that $|\omega(z_0)| = 1$ for all $z \in U$, that is

$$\frac{I_{\mu,s}f(z)}{z} \prec \frac{1+\alpha z}{1-z}, \qquad z \in U.$$

Furthermore, since

$$\omega(z) = \frac{\frac{I_{\mu,s}f(z)}{z} - 1}{\frac{I_{\mu,s}f(z)}{z} - \alpha}, \qquad z \in U$$

and $|\omega(z)| < 1$ $(z \in U)$, we conclude that

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\} > \frac{1-\alpha}{2}.$$

Taking $\alpha = 0$ in the theorem, we have the following corollary:

Corollary 2.1 *If $f \in A$ satisfies*

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} > \frac{3}{2}, \qquad z \in U,$$

then

$$\frac{I_{\mu,s}f(z)}{z} \prec \frac{1}{1-z}, \qquad z \in U$$

and

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\} > \frac{1}{2}, \qquad z \in U.$$

Theorem 2.2 *If $f \in A$ satisfies*

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} > \frac{3\alpha-1}{2(\alpha-1)}, \qquad z \in U,$$

for some $\alpha\,(-1 < \alpha \le 0)$, *then*

$$\frac{z}{I_{\mu,s}f(z)} \prec \frac{1+z}{1-z}, \qquad z \in U,$$

and

$$\left|\frac{I_{\mu,s}f(z)}{z} - \frac{1}{1-\alpha}\right| < \frac{1}{1-\alpha}, \quad z \in U.$$

This implies that

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\} > 0, \qquad z \in U.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{z}{I_{\mu,s}f(z)} = \frac{1-\alpha\omega(z)}{1-\omega(z)}, \qquad \omega(z) \neq 1\,. \tag{4}$$

Then, we have that $\omega(z)$ is analytic in U and $\omega(0) = 0$. We want to prove that $|\omega(z)| < 1$ in U.

Differentiating Eq. (4) , we obtain

$$\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)} = \frac{-z\omega'(z)}{1-\omega(z)} + \frac{\alpha z\omega'(z)}{1-\alpha\omega(z)} + 1$$

and, hence

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} = \Re\{\frac{-z\omega'(z)}{1-\omega(z)} + \frac{\alpha z\omega'(z)}{1-\alpha\omega(z)} + 1\}$$
$$> \frac{3\alpha-1}{2(\alpha-1)}, \qquad z \in U,$$

for $(-1 < \alpha \le 0)$. If there exists a point $(z_0 \in U)$ such that Lemma 1.1, gives us that $\omega(z_0) = e^{i\theta}$ and $z_0\omega'(z_0) = k\omega(z_0), k \ge 1$. Thus we have

$$\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)} = \frac{-z_0\omega'(z_0)}{1-\omega(z_0)} + \frac{\alpha z_0\omega'(z_0)}{1-\alpha\omega(z_0)} + 1$$
$$= 1 - \frac{k}{1-e^{i\theta}} + \frac{k}{1-\alpha e^{i\theta}}.$$

Therefore, we have

$$\Re\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\} = 1 + \frac{k(\alpha^2-1)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

This implies that, for $-1 < \alpha \leq 0$,

$$\Re\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\} = 1 - \frac{k(1-\alpha^2)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

$$\leq \frac{3\alpha-1}{2(\alpha-1)}.$$

This contradicts the condition in the theorem. Hence, there is no $z_0 \in U$ such that $|\omega(z_0)| = 1$ for all $z \in U$, that is

$$\frac{z}{I_{\mu,s}f(z)} \prec \frac{1+z}{1-z}, \qquad z \in U.$$

Furthermore, since

$$\omega(z) = \frac{1 - \frac{I_{\mu,s}f(z)}{z}}{1 - \alpha\frac{I_{\mu,s}f(z)}{z}}, \qquad z \in U$$

and $|\omega(z)| < 1$ $(z \in U)$, we conclude that

$$\left|\frac{I_{\mu,s}f(z)}{z} - \frac{1}{1-\alpha}\right| < \frac{1}{1-\alpha}, \quad z \in U$$

which implies that

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\} > 0, \qquad z \in U,$$

we complete the proof of the theorem. By setting $\alpha = 0$ in Theorem 2.2, we readily obtain the following:

Corollary 2.2 *If $f \in A$ satisfies*

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} > \frac{1}{2}, \quad z \in U,$$

then

$$\frac{z}{I_{\mu,s}f(z)} \prec \frac{1+z}{1-z}, \qquad z \in U$$

and

$$\left|\frac{I_{\mu,s}f(z)}{z} - 1\right| < 1, \quad z \in U.$$

Theorem 2.3 *If $f \in A$ satisfies*

$$\Re\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\} < \frac{\alpha(2-\gamma)-(2+\gamma)}{2(\alpha-1)}, \quad z \in U,$$

for some $\alpha\,(-1<\alpha\leq 0)$ *and* $0<\gamma\leq 1$, *then*

$$\Big(\frac{I_{\mu,s}f(z)}{z}\Big)^{\frac{1}{\gamma}} \prec \frac{1+\alpha z}{1-z}\,, \qquad z\in U$$

This implies that

$$\Re\Big\{\frac{I_{\mu,s}f(z)}{z}\Big\}^{\frac{1}{\gamma}} > \frac{1-\alpha}{2}, \qquad z\in U.$$

Proof. Let us define the function $\omega(z)$ by

$$\frac{I_{\mu,s}f(z)}{z} = \Big(\frac{1-\alpha\omega(z)}{1-\omega(z)}\Big)^{\gamma}, \qquad \omega(z)\neq 1.$$

Clearly, $\omega(z)$ is analytic in U and $\omega(0)=0$. We want to prove that $|\omega(z)|<1$ in U. Since

$$\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)} = \gamma\Big(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\alpha z\omega'(z)}{1-\alpha\omega(z)}\Big)+1$$

we see that

$$\Re\Big\{\frac{z(I_{\mu,s}f(z))'}{I_{\mu,s}f(z)}\Big\} = \Re\Big\{\gamma\Big(\frac{z\omega'(z)}{1-\omega(z)} - \frac{\alpha z\omega'(z)}{1-\alpha\omega(z)}\Big)+1\Big\}$$
$$< \frac{\alpha(2-\gamma)-(2+\gamma)}{2(\alpha-1)}, \quad z\in U$$

for $\alpha\,(-1<\alpha\leq 0)$ and $0<\gamma\leq 1$. If there exists a point $(z_0\in U)$ such that

$$\max_{|z|\leq|z_0|}|\omega(z)| = |\omega(z_0)| = 1,$$

then Lemma 1.1, gives us that $\omega(z_0)=e^{i\theta}$ and $z_0\omega'(z_0)=\ k\omega(z_0), k\geq 1$. Thus we have

$$\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)} = \gamma\Big(\frac{z_0\omega'(z_0)}{1-\omega(z_0)} - \frac{\alpha z_0\omega'(z_0)}{1-\alpha\omega(z_0)}\Big)+1$$
$$= 1+\frac{k}{1-e^{i\theta}} - \frac{k}{1-\alpha e^{i\theta}}$$

Therefore, we have

$$\Re\Big\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\Big\} = 1+\frac{\gamma k(1-\alpha^2)}{2(1+\alpha^2-2\alpha\cos\theta)}.$$

This implies that, for $\alpha\,(-1<\alpha\leq 0)$ and $0<\gamma\leq 1$

$$\Re\Big\{\frac{z_0(I_{\mu,s}f(z_0))'}{I_{\mu,s}f(z_0)}\Big\} \geq \frac{\alpha(2-\gamma)-(2+\gamma)}{2(\alpha-1)}.$$

This contradicts the condition in the theorem. Hence, there is no $z_0 \in U$ such that $|\omega(z_0)| = 1$ for all $z \in U$, that is

$$(\frac{I_{\mu,s}f(z)}{z})^{\frac{1}{\gamma}} \prec \frac{1-\alpha z}{1-z}, \qquad z \in U.$$

Furthermore, since

$$\omega(z) = \frac{(\frac{I_{\mu,s}f(z)}{z})^{\frac{1}{\gamma}} - 1}{(\frac{I_{\mu,s}f(z)}{z})^{\frac{1}{\gamma}} - \alpha},$$

and $|\omega(z)| < 1$ $(z \in U)$, we conclude that

$$\Re\{\frac{I_{\mu,s}f(z)}{z}\}^{\frac{1}{\gamma}} > \frac{1-\alpha}{2}, \qquad z \in U,$$

we complete the proof of the theorem.

Acknowledgment

The work presented here was fully supported by UKM-ST-06-FRGS0244-2010.

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