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CHARACTERIZATION INEXTENSIBLE FLOWS OF DEVELOPABLE SURFACES ASSOCIATED FOCAL CURVE OF SPACELIKE CURVE WITH TIMELIKE BINORMAL IN $\mathbb{E}_1^3$

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Abstract

In this paper, we study inextensible flows of developable surfaces associated with focal curves of spacelike curves with timelike binormal in Minkowski 3-space $\mathbb{E}_1^3$. We show that if flow of this developable surface is inextensible then we characterize this surface in terms of curvatures of spacelike curve in Minkowski 3-space $\mathbb{E}_1^3$.

key words. Developable surface, Minkowski 3-space, inextensible flows.

AMS subject classifications. 31B30, 58E20.

1 Preliminaries

By the 20th century, researchers discovered the bridge between theory of relativity and Lorentzian manifolds in the sense of differential geometry. Since, they adapted the geometrical models to relativistic motion of charged particles. Consequently, the theory of the curves has been one of the most fascinating topic for such modeling process. As it stands, the Frenet-Serret formalism of a relativistic motion describes the dynamics of the charged particles.

The Minkowski 3-space $\mathbb{E}_1^3$ is the Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$g = -dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}_1^3$.

Since g is an indefinite metric, recall that a vector $v \in \mathbb{E}_1^3$ can have one of three Lorentzian causal characters: it can be spacelike if $g(v,v) > 0$ or $v = 0$, timelike if $g(v,v) < 0$ and null (lightlike) if $g(v,v) = 0$ and $v \neq 0$. Similarly, an arbitrary curve $\gamma = \gamma(s)$ in $\mathbb{E}_1^3$ can locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are respectively spacelike, timelike or null (lightlike), if all of its velocity vectors $\gamma'(s)$ are, respectively, spacelike, timelike or null (lightlike), respectively.

Minkowski space is originally from the relativity in Physics. In fact, a timelike curve corresponds to the path of an observer moving at less than the speed of light, a null curves correspond to moving at the speed of light and a spacelike curves to moving faster than light.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}_1^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N} \\ \nabla_{\mathbf{T}}\mathbf{N} &= -\kappa\mathbf{T} + \tau\mathbf{B} \\ \nabla_{\mathbf{T}}\mathbf{B} &= \tau\mathbf{N}, \end{aligned}$$

where

$$\begin{aligned} g(\mathbf{T}, \mathbf{T}) &= 1, \quad g(\mathbf{N}, \mathbf{N}) = 1, \quad g(\mathbf{B}, \mathbf{B}) = -1, \\ g(\mathbf{T}, \mathbf{N}) &= g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{aligned}$$

2 Inextensible Flows of Developable Surfaces Associated with Focal Curve of Spacelike Curve with Timelike Binormal in the $\mathbb{E}_1^3$

For a unit speed curve γ, the curve consisting of the centers of the osculating spheres of γ is called the parametrized focal curve of γ. The hyperplanes normal to γ at a point consist of the set of centers of all spheres tangent to γ at that point. Hence the center of the osculating spheres at that point lies in such a normal plane. Therefore, denoting the focal curve by C_γ, we can write

$$C_\gamma(s) = (\gamma + c_1\mathbf{T} + c_2\mathbf{N})(s), \tag{3.1}$$

where the coefficients c_1, c_2 are smooth functions of the parameter of the curve γ, called the first and second focal curvatures of γ, respectively. Further, the focal curvatures c_1, c_2 are defined by

$$c_1 = \frac{1}{\kappa}, \; c_2 = -\frac{c_1'}{\tau}, \; \kappa \neq 0, \; \tau \neq 0. \tag{3.2}$$

On the other hand, a ruled surface in $\mathbb{E}_1^3$ is (locally) the map $X_{(\gamma,\delta)} : I \times \mathbb{R} \to \mathbb{E}_1^3$ defined by

$$X_{(\gamma,\delta)}(s,u) = \gamma(s) + u\delta(s), \tag{3.3}$$

where $\gamma : I \longrightarrow \mathbb{E}_1^3$, $\delta : I \longrightarrow \mathbb{E}_1^3 \backslash \{0\}$ are smooth mappings and I is an open interval. We call the base curve and the director curve. The straight lines $u \to \gamma(s) + u\delta(s)$ are called rulings of $X_{(\gamma,\delta)}$.

Definition 3.1. *A smooth surface $X_{(\gamma,\delta)}$ is called a developable surface if its Gaussian curvature K vanishes everywhere on the surface.*

Definition 3.2. *Let $\gamma : I \longrightarrow \mathbb{E}_1^3$ be a unit speed curve. We define the following developable surface*

$$X_{(C_\gamma,\gamma')}(s,u) = C_\gamma(s) + u\gamma'(s), \tag{3.4}$$

where $C_\gamma(s)$ is focal curve.

Definition 3.3. *(see [8])* A surface evolution $X(s,u,t)$ and its flow $\dfrac{\partial X}{\partial t}$ are said to be inextensible if its first fundamental form $\{E, F, G\}$ satisfies

$$\frac{\partial E}{\partial t} = \frac{\partial F}{\partial t} = \frac{\partial G}{\partial t} = 0. \tag{3.5}$$

This definition states that the surface $X(s,u,t)$ is, for all time t, the isometric image of the original surface $X(s,u,t_0)$ defined at some initial time t_0. For a developable surface, $X(s,u,t)$ can be physically pictured as the parametrization of a waving flag. For a given surface that is rigid, there exists no nontrivial inextensible evolution.

Definition 3.4. We can define the following one-parameter family of developable ruled surface

$$X(s,u,t) = C_\gamma(s,t) + u\gamma'(s,t). \tag{3.6}$$

Theorem 3.5. *Let X is the developable surface associated with focal curve in $\mathbb{E}_1^3$, then $\dfrac{\partial X}{\partial t}$ is inextensible then*

$$\left[2\frac{\partial c_1}{\partial s}+u\kappa\right]\left[2\frac{\partial^2 c_1}{\partial s\partial t}+u\frac{\partial \kappa}{\partial t}\right]=\left[\left(\frac{\tau}{\kappa}\right)+\frac{\partial c_2}{\partial s}\right]\left[\frac{\partial}{\partial t}\left(\frac{\tau}{\kappa}\right)+\frac{\partial^2 c_2}{\partial s\partial t}\right]. \tag{3.7}$$

Proof. Assume that $X(s,u,t)$ be a one-parameter family of developable surface. We show that $\dfrac{\partial X}{\partial t}$ is inextensible.

$$\begin{aligned} X_s(s,u,t) &= \frac{\partial C_\gamma(s,t)}{\partial s}+u\frac{\partial \gamma'(s,t)}{\partial s} \\ &= \left(2\frac{\partial c_1}{\partial s}+u\kappa\right)\mathbf{N}+\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)\mathbf{B} \\ X_u(s,u,t) &= \gamma'(s,t). \end{aligned} \tag{3.8}$$

If we compute first fundamental form $\{E,F,G\}$, we have

$$\begin{aligned} E &= \langle X_s,X_s\rangle=\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2-\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2, \\ F &= \langle X_s,X_u\rangle=0, \\ G &= \langle X_u,X_u\rangle=1. \end{aligned} \tag{3.9}$$

Moreover, from above equations, it results that

$$\begin{aligned} \frac{\partial E}{\partial t} &= \left[2\frac{\partial c_1}{\partial s}+u\kappa\right]\frac{\partial}{\partial t}\left[2\frac{\partial c_1}{\partial s}+u\kappa\right] \\ &\quad -\left[\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right]\frac{\partial}{\partial t}\left[\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right]=0, \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \frac{\partial F}{\partial t} &= 0, \\ \frac{\partial G}{\partial t} &= 0. \end{aligned} \tag{3.11}$$

Then, taking into account (3.10), we have (3.7). Thus, we complete the proof of the theorem.

Corollary 3.6. If γ is a spacelike general helix with timelike binormal in $\mathbb{E}_1^3$, then the flow of $\dfrac{\partial X}{\partial t}$ is inextensible.

$$\left[2\frac{\partial c_1}{\partial s}+u\frac{\tau}{\rho}\right]\left[2\frac{\partial^2 c_1}{\partial s\partial t}+\frac{u}{\rho}\frac{\partial \tau}{\partial t}\right]-\left[\rho+\frac{\partial c_2}{\partial s}\right]\frac{\partial^2 c_2}{\partial s\partial t}=0, \tag{3.12}$$

where $\rho = \frac{\tau}{\kappa}$.

Theorem 3.7. *Let X is the developable surface associated with focal curve in E_1^3. If flow of this developable surface is inextensible then this surface is minimal if and only if*

$$\left(2\frac{\partial^2 c_1}{\partial s^2} + u\frac{\partial \kappa}{\partial s} + \tau\left(f + \frac{\partial c_2}{\partial s}\right)\right)\left(\frac{\tau}{\kappa} + \frac{\partial c_2}{\partial s}\right) \tag{3.13}$$
$$+\left[\left(2\frac{\partial c_1}{\partial s} + u\kappa\right)\tau + \left(\frac{\partial f}{\partial s} + \frac{\partial^2 c_2}{\partial s^2}\right)\right]\left(2\frac{\partial c_1}{\partial s} + u\kappa\right) = 0,$$

where

$$f = \left(2\frac{\partial c_1}{\partial s} + u\kappa\right), \quad g = \left(\frac{\tau}{\kappa} + \frac{\partial c_2}{\partial s}\right).$$

Proof. Assume that $X(s,u,t) = C_\gamma(s,t) + u\gamma'(s,t)$ be a one-parameter family of developable ruled surface.

Firstly, we suppose

$$f = \left(2\frac{\partial c_1}{\partial s} + u\kappa\right), \quad g = \left(\frac{\tau}{\kappa} + \frac{\partial c_2}{\partial s}\right). \tag{3.14}$$

Then, we use above equations and the system (3.8), we obtain

$$\begin{aligned} X_{ss}(s,u,t) &= -\kappa f t + \left(\frac{\partial f}{\partial s} + \frac{\partial c_2}{\partial s}\right)n_1 + \left(f\tau + \frac{\partial g}{\partial s}\right)n_2, \\ X_{su}(s,u,t) &= \kappa n_1, \\ X_{uu}(s,u,t) &= 0. \end{aligned}$$

On the other hand, the normal of surface is

$$\vec{\mathbf{n}} = \frac{X_s \times X_u}{\|X_s \times X_u\|}.$$

If we use above equations, then components of second fundamental form of developable surface are

$$
\begin{aligned}
h_{11} &= \frac{\left(2\frac{\partial^2 c_1}{\partial s^2}+u\frac{\partial \kappa}{\partial s}+\tau\left(f+\frac{\partial c_2}{\partial s}\right)\right)\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)}{\sqrt{-\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2+\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2}} \\
&\quad +\frac{\left[\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)\tau+\left(\frac{\partial f}{\partial s}+\frac{\partial^2 c_2}{\partial s^2}\right)\right]\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)}{\sqrt{-\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2+\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2}}, \\
h_{12} &= \frac{\tau+\kappa\frac{\partial^2 c_2}{\partial s^2}}{\sqrt{-\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2+\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2}}, \\
h_{22} &= 0.
\end{aligned}
$$

Also, components of metric

$$
\begin{aligned}
g_{11} &= \left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2-\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2, \\
g_{12} &= 0, \\
g_{22} &= 1.
\end{aligned}
$$

So, the mean curvature of one-parameter family of developable surface $X(s,u,t)$ is

$$
\begin{aligned}
H &= \mp\frac{\left(2\frac{\partial^2 c_1}{\partial s^2}+u\frac{\partial \kappa}{\partial s}+\tau\left(f+\frac{\partial c_2}{\partial s}\right)\right)\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)}{2\left(\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2-\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2\right)^{\frac{3}{2}}} \\
&\quad \pm\frac{\left[\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)\tau+\left(\frac{\partial f}{\partial s}+\frac{\partial^2 c_2}{\partial s^2}\right)\right]\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)}{2\left(\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right)^2-\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)^2\right)^{\frac{3}{2}}}.
\end{aligned}
$$

This developable surface is minimal if and only if

$$
\begin{aligned}
&\left(2\frac{\partial^2 c_1}{\partial s^2}+u\frac{\partial \kappa}{\partial s}+\tau\left(f+\frac{\partial c_2}{\partial s}\right)\right)\left(\frac{\tau}{\kappa}+\frac{\partial c_2}{\partial s}\right) \\
&+\left[\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)\tau+\left(\frac{\partial f}{\partial s}+\frac{\partial^2 c_2}{\partial s^2}\right)\right]\left(2\frac{\partial c_1}{\partial s}+u\kappa\right)=0,
\end{aligned}
$$

which proves the theorem.

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