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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

Inclusion Properties of Certain Subclasses of Analytic Functions

Maslina Darus and Imran Faisal

Abstract

The purpose of the present paper is to introduce several new classes of analytic functions and investigate various inclusion properties of these classes. Some interesting applications of integral operators are also considered.

key words. Analytic functions, Differential operators, Integral-preserving properties.

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1 Introduction and preliminaries

Let A denote the class of analytic functions f in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ normalized by $f(0) = f'(0) - 1 = 0$. Thus each $f \in A$ has a Taylor series representation

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \cdot \tag{1}$$

For two functions f and g analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z)\,(z \in \mathbb{U}),$$

if there exists a Schwarz function $w(z)$ which (by definition) is analytic in $\mathbb{U}$ with

$$w(0) = 0 \text{ and } |w(z)| < 1,$$

such that

$$f(z) = g(w(z))\ z \in \mathbb{U}.$$

We denote by $\Lambda S(\xi)$, $\Lambda K(\xi)$ and $\Lambda C(\xi,\rho)$ the subclasses of A consisting of all analytic functions which are, respectively, starlike of order ξ, convex of order ξ and close-to-convex of order ρ and type ξ in $\mathbb{U}$ [8, 12].

Let $\mathfrak{N}$ be the class of all functions ϕ which are analytic and univalent in $\mathbb{U}$ and for which $\phi(\mathbb{U})$ is convex with $\phi(0)=1$ and $\Re\phi(z)>0$ $(z\in\mathbb{U})$.

Let $f, g \in A$ where f and g is defined by

$$f(z)=z+\sum_{k=2}^{\infty}a_k z^k,\ \ g(z)=z+\sum_{k=2}^{\infty}b_k z^k\cdot$$

Then the Hadamard product (or convolution) $f*g$ of the functions f and g is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k\cdot$$

Making use of the subordination's principle between analytic functions, we introduce the subclasses $\Lambda S(\xi,\phi)$, $\Lambda K(\xi,\phi)$ and $\Lambda C(\xi,\rho,\phi,\psi)$ of the class A for $0\le\xi,\rho<1$ and $\phi,\psi\in\mathfrak{N}$, which are defined by

$$\Lambda S(\xi,\phi)=\left\{z\in A:\frac{1}{1-\xi}\left(\frac{zf'(z)}{f(z)}-\xi\right)\prec\phi(z),\ (z\in\mathbb{U})\right\},$$

$$\Lambda K(\xi,\phi)=\left\{z\in A:\frac{1}{1-\xi}\left(\left\{1+\frac{zf''(z)}{f'(z)}\right\}-\xi\right)\prec\phi(z),\ (z\in\mathbb{U})\right\},$$

and

$$\Lambda C(\xi,\rho,\phi,\psi)=\left\{z\in A:\exists g\in\Lambda S(\xi,\phi)\wedge\frac{1}{1-\rho}\left(\frac{zf'(z)}{g(z)}-\rho\right)\prec\psi(z),\ (z\in\mathbb{U})\right\}\cdot$$

For a function $f\in A$, authors have introduced the following differential operator in [7] such that

$$\begin{aligned}
D^0 f(z) &= f(z),\\
D^1_\lambda(\alpha,\beta,\mu)f(z) &= \left(\frac{\alpha-\mu+\beta-\lambda}{\alpha+\beta}\right)f(z)+\left(\frac{\mu+\lambda}{\alpha+\beta}\right)zf'(z),\\
D^2_\lambda(\alpha,\beta,\mu)f(z) &= D(D^1_\lambda(\alpha,\beta,\mu)f(z)),\\
&\ \vdots\\
D^n_\lambda(\alpha,\beta,\mu)f(z) &= D(D^{n-1}_\lambda(\alpha,\beta,\mu)f(z))\cdot
\end{aligned}\tag{2}$$

If f is given by (1) then from (2) we have

$$D_\lambda^n(\alpha,\beta,\mu)f(z) = z+\sum_{k=2}^{\infty}\Big(\frac{\alpha+(\mu+\lambda)(k-1)+\beta}{\alpha+\beta}\Big)^n a_k z^k \tag{3}$$

$$(f\in A,\alpha,\beta,\mu,\lambda\geq 0,\alpha+\beta\neq 0,n\in N_o)$$

By specializing the parameters of $D_\lambda^n(\alpha,\beta,\mu)f(z)$ we get the following differential operators. If we substitute

• $\beta=0$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(\frac{\alpha+(\mu+\lambda)(k-1)}{\alpha})^n a_k z^k$
of differential operator given by Darus and Faisal [9].

• $\beta=1$, $\mu=0$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(\frac{\alpha+\lambda(k-1)+1}{\alpha+1})^n a_k z^k$
of differential operator given by Aouf et al. [1].

• $\alpha=1$, $\beta=o$, and $\mu=0$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(1+\lambda(k-1))^n a_k z^k$
of differential operator given by Al-Oboudi [2].

• $\alpha=1$, $\beta=o$, $\mu=0$ and $\lambda=1$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(i)^n a_k z^k$
of Sălăgean's differential operator [3].

• $\alpha=1$, $\beta=1$, $\lambda=1$ and $\mu=0$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(\frac{k+1}{2})^n a_k z^k$
of differential operator given by Uralegaddi and Somanatha [4].

• $\beta=1$, $\lambda=1$ and $\mu=0$, we get $D^n f(z)=z+\sum_{k=2}^{\infty}(\frac{k+\alpha}{\alpha+1})^n a_k z^k$
of differential operator given by Cho and Srivastava [5, 6].

Next, by using the operator $D_\lambda^n(\alpha,\beta,\mu)f(z)$, we introduce the following subclasses of analytic functions for $0\leq\xi,\rho<1$ and $\phi,\psi\in\mathfrak{N}$:

$$\Lambda S_{\alpha,\beta}^{n,\lambda}(\xi,\phi,\mu)=\Bigg\{z\in A: D_\lambda^n(\alpha,\beta,\mu)f(z)\in\Lambda S(\xi,\phi)\Bigg\},$$

$$\Lambda K_{\alpha,\beta}^{n,\lambda}(\xi,\phi,\mu)=\Bigg\{z\in A: D_\lambda^n(\alpha,\beta,\mu)f(z)\in\Lambda K(\xi,\phi)\Bigg\},$$

$$\Lambda C_{\alpha,\beta}^{n,\lambda}(\xi,\rho,\phi,\psi,\mu)=\Bigg\{z\in A: D_\lambda^n(\alpha,\beta,\mu)f(z)\in\Lambda C(\xi,\rho,\phi,\psi)\Bigg\}.$$

We also note that

$$f(z) \in \Lambda K^{n,\lambda}_{\alpha,\beta}(\xi, \phi, \mu) \;\; \Leftrightarrow \;\; -zf'(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}(\xi, \phi, \mu)\cdot \tag{4}$$

In particular, we set

$$\Lambda S^{n,\lambda}_{\alpha,\beta}(\xi; \frac{1+Az}{1+Bz}, \mu) = \Lambda S^{n,\lambda}_{\alpha,\beta}(\xi; A, B, \mu) \; (-1 < B < A \leq 1)$$

and

$$\Lambda K^{n,\lambda}_{\alpha,\beta}(\xi; \frac{1+Az}{1+Bz}, \mu) = \Lambda K^{n,\lambda}_{\alpha,\beta}(\xi; A, B, \mu) \; (-1 < B < A \leq 1)\cdot$$

Next we will investigate various inclusion relationships as well as integral preserving properties for the subclasses of analytic functions newly introduced above.

2 Inclusion Relationships Associated with Operator $D^n_\lambda(\alpha, \beta, \mu)$

First we will state the following lemma which we need for our main results.

Lemma 2.1. [10] Let ϕ be convex univalent in $\mathbb{U}$ with $\phi(0) = 1$ and $\Re\{\kappa\phi(z) + \nu\} > 0 \, (\kappa, \nu \in \mathbb{C})\cdot$ If p is analytic in $\mathbb{U}$ with $p(0) = 1$, then

$$p(z) + \frac{zp'(z)}{\kappa p(z) + \nu} \prec \phi(z) \;\; (z \in \mathbb{U}),$$

implies

$$p(z) \prec \phi(z) \;\; (z \in \mathbb{U})\cdot$$

Lemma 2.2. [11] Let ϕ be convex univalent in $\mathbb{U}$ and ω be analytic in $\mathbb{U}$ with $\Re\{\omega(z)\} \geq 0$. If p is analytic in $\mathbb{U}$ with $p(0) = \phi(0)$, then

$$p(z) + \omega(z)zp'(z) \prec \phi(z) \;\; (z \in \mathbb{U}),$$

implies

$$p(z) \prec \phi(z) \;\; (z \in \mathbb{U})\cdot$$

Theorem 2.3. If $f \in A$ and $\phi \in \mathfrak{N}$ with $\Re\{\phi(z)\} < \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}/1 - \xi$. Then

$$\Lambda S^{n+1,\lambda}_{\alpha,\beta}\left(\xi, \phi, \mu\right) \subset \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi, \phi, \mu\right) \subset \Lambda S^{n-1,\lambda}_{\alpha,\beta}\left(\xi, \phi, \mu\right)\cdot$$

Proof Let $f(z) \in \Lambda S^{n+1,\lambda}_{\alpha,\beta}(\xi,\phi,\mu)$ and set

$$p(z) = \frac{1}{1-\xi}\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)f(z)} - \xi\right), \tag{5}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Simultaneously applying (3) and (5) we get

$$\frac{\alpha+\beta}{\mu+\lambda}\left(\frac{D^{n+1}_\lambda(\alpha,\beta,\mu)f(z)}{D^n_\lambda(\alpha,\beta,\mu)f(z)}\right) = -\left(1 - \frac{\alpha+\beta}{\mu+\lambda}\right) + \left(\xi + (1-\xi)p(z)\right) \tag{6}$$

By a simple calculation with (5) and (6), we obtain

$$\frac{1}{1-\xi}\left(\frac{-z(D^{n+1}_\lambda(\alpha,\beta,\mu)f(z))'}{D^{n+1}_\lambda(\alpha,\beta,\mu)f(z)} - \xi\right) = p(z) + \frac{zp'(z)}{(1-\xi)p(z) + \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}} \ (z \in \mathbb{U}) \tag{7}$$

Since $\Re\{\phi(z)\} < \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}/1-\xi$ implies $\Re\{(1-\xi)p(z) + \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}\} > 0 \ (z \in \mathbb{U})$

Applying Lemma 2.1 to (7), it follows that $f(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}(\xi,\phi,\mu)\cdot$

Theorem 2.4. If $f \in A$ and $\phi \in \mathfrak{N}$ with $\Re\{\phi(z)\} < \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}/1-\xi$. Then

$$\Lambda K^{n+1,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \subset \Lambda K^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \subset \Lambda K^{n-1,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)\cdot$$

Proof. Applying (4) and Theorem 2.3 we conclude that

$$f \in \Lambda K^{n+1,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \Rightarrow -zf'\Lambda S^{n+1,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \subset \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \Rightarrow$$

$$-zf' \in \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right) \Leftrightarrow f \in \Lambda K^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)\cdot$$

Which proves Theorem 2.4.

Taking

$$\phi(z) = \frac{1+Az}{1+Bz}, \ (-1 < B < A \le 1)$$

in Theorems 2.3 and 2.4, we have the following corollary:

Corollary:2.5. Let $(1+A)/(1+B) < \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}/1-\xi$ for $-1 < B < A \le 1$, then

$$\Lambda S^{n+1,\lambda}_{\alpha,\beta}\left(\xi,A,B,\mu\right) \subset \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,A,B,\mu\right) \subset \Lambda S^{n-1,\lambda}_{\alpha,\beta}\left(\xi,A,B,\mu\right)\cdot$$

$$\Lambda K_{\alpha,\beta}^{n+1,\lambda}\left(\xi, A, B, \mu\right) \subset \Sigma_1 K_{\alpha,\beta}^{n,\lambda}\left(\xi, A, B, \mu\right) \subset \Lambda K_{\alpha,\beta}^{n-1,\lambda}\left(\xi, A, B, \mu\right)\cdot$$

Theorem 2.6. If $f \in A$ and $\phi, \psi \in \mathfrak{N}$, $0 \le \xi, \rho < 1$ with $\Re\{\phi(z)\} < \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}/1 - \xi$. Then

$$\Lambda C_{\alpha,\beta}^{n+1,\lambda}(\xi, \rho, \phi, \psi, \mu) \subset \Lambda C_{\alpha,\beta}^{n,\lambda}(\xi, \rho, \phi, \psi, \mu) \subset \Lambda C_{\alpha,\beta}^{n-1,\lambda}(\xi, \rho, \phi, \psi, \mu)\cdot$$

Proof. To prove the inclusion, let $f(z) \in \Lambda C_{\alpha,\beta}^{n+1,\lambda}(\xi, \rho, \phi, \psi, \mu)$ then by definition there exist a function $g(z) \in \Lambda S_{\alpha,\beta}^{n+1,\lambda}\left(\xi, \phi, \mu\right)$ such that

$$\frac{1}{1-\rho}\left(\frac{z(D_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{D_\lambda^{n+1}(\alpha,\beta,\mu)g(z)} - \rho\right) \prec \psi(z) \ (z \in \mathbb{U})\cdot$$

We suppose that

$$p(z) = \frac{1}{1-\rho}\left(\frac{z(D_\lambda^{n}(\alpha,\beta,\mu)f(z))'}{D_\lambda^{n}(\alpha,\beta,\mu)g(z)} - \rho\right) \tag{8}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using the following equation

$$(\frac{\alpha+\beta}{\mu+\lambda})(D_\lambda^{n+1}(\alpha,\beta,\mu)f(z)) = z(D_\lambda^{n}(\alpha,\beta,\mu)f(z))' - (1 - \frac{\alpha+\beta}{\mu+\lambda})(D_\lambda^{n}(\alpha,\beta,\mu)f(z)),$$

we get

$$\frac{1}{1-\rho}\left(\frac{z(D_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{D_\lambda^{n+1}(\alpha,\beta,\mu)g(z)} - \rho\right) =$$

$$\frac{1}{1-\rho}\left(\frac{\frac{z(D_\lambda^{n}(\alpha,\beta,\mu)zf'(z))'}{D_\lambda^{n}(\alpha,\beta,\mu)g(z)} + (\frac{\alpha+\beta}{\mu+\lambda} - 1)(\frac{z(D_\lambda^{n}(\alpha,\beta,\mu)f(z))'}{D_\lambda^{n}(\alpha,\beta,\mu)g(z)})}{\frac{z(D_\lambda^{n}(\alpha,\beta,\mu)g(z))'}{D_\lambda^{n}(\alpha,\beta,\mu)g(z)} + (\frac{\alpha+\beta}{\mu+\lambda} - 1)} - \rho\right)\cdot \tag{9}$$

Since $g(z) \in \Lambda S_{\alpha,\beta}^{n+1,\lambda}\left(\xi, \phi, \mu\right) \subset \Lambda S_{\alpha,\beta}^{n,\lambda}\left(\xi, \phi, \mu\right)$, by using Theorem 2.3, we set

$$q(z) = \frac{1}{1-\xi}\left(\frac{z(D_\lambda^{n}(\alpha,\beta,\mu)g(z))'}{D_\lambda^{n}(\alpha,\beta,\mu)g(z)} - \xi\right)\cdot \tag{10}$$

Then, by virtue of (8), (9) and (10), we get

$$\frac{1}{1-\rho}\left(\frac{z(D_\lambda^{n+1}(\alpha,\beta,\mu)f(z))'}{D_\lambda^{n+1}(\alpha,\beta,\mu)g(z)} - \rho\right) = p(z) + \frac{zp'(z)}{(1-\xi)q(z) + \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}}\cdot \tag{11}$$

Since $\xi > 0$ and $q \prec \phi$ in $\mathbb{U}$ with assumption for $\phi, \psi \in \mathfrak{N}$ and $\Re\{(1-\xi)q(z) + \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}\} > 0\cdot$ Hence, by taking

$$\omega(z) = \frac{1}{(1-\xi)q(z) + \xi - 1 + \frac{\alpha+\beta}{\mu+\lambda}}$$

and and applying Lemma 2.2, we can show that $p \prec \psi$ in $\mathbb{U}$, so that $f(z) \in \Lambda C^{n,\lambda}_{\alpha,\beta}(\xi, \rho, \phi, \psi, \mu)\cdot$ Hence proved.

3 Integral-Preserving Properties

In this section, we present several integral-preserving properties for the subclass of analytic function defined above. We first recall a familiar integral operator $L_c(f)$ defined by

$$L_c(f) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)dt, \ (c > -1; f \in A)\cdot \tag{12}$$

which satisfies the following relationship:

$$z(\Theta^n_{1,\lambda}(\alpha,\beta,\mu)L_c f(z))' = (c+1)(\Theta^n_{1,\lambda}(\alpha,\beta,\mu)f(z)) - (c)(\Theta^n_{1,\lambda}(\alpha,\beta,\mu)L_c f(z))\cdot \tag{13}$$

Theorem:3.1. Let $c > -1$ and $\phi \in \mathfrak{N}$ with $\Re\{\phi(z)\} < c + \xi/1 - \xi$. If $f(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\Bigg(\xi, \phi, \mu\Bigg)$ then $L_c(f) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\Bigg(\xi, \phi, \mu\Bigg)\cdot$

Proof. Let $f(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\Bigg(\xi, \phi, \mu\Bigg)$ and set

$$p(z) = \frac{1}{1-\xi}\Bigg(\frac{z(D^n_\lambda(\alpha,\beta,\mu)L_c f(z))'}{D^n_\lambda(\alpha,\beta,\mu)L_c f(z)} - \xi\Bigg)\cdot \tag{14}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (13) and (14) we have

$$(c+1)\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))}{D^n_\lambda(\alpha,\beta,\mu)L_c f(z)} = (c) + \xi + (1-\xi)p(z)\cdot \tag{15}$$

Then, by virtue of (13), (14) and (15), we get

$$\frac{1}{1-\xi}\Bigg(\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)f(z)} - \xi\Bigg) = p(z) + \frac{zp'(z)}{(1-\xi)p(z) + c + \xi}\cdot \tag{16}$$

Applying Lemma 2.1 to (16), we conclude that $L_c(f) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)\cdot$ Similarly applying (4) and Theorem 3.1, we have the following result:

Theorem:3.2. Let $c > -1$ and $\phi \in \mathfrak{N}$ with $\Re\{\phi(z)\} < c + \xi/1-\xi$. If $f(z) \in \Lambda K^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)$ then $L_c(f) \in \Lambda K^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)\cdot$

Theorem:3.3. Let $c > -1$, $\phi,\psi \in \mathfrak{N}$ and $0 \leq \xi,\rho < 1$ with $\Re\{\phi(z)\} < c + \xi/1-\xi$. If $f(z) \in \Lambda C^{n,\lambda}_{\alpha,\beta}(\xi,\rho,\phi,\psi,\mu)$ then $L_c(f) \in \Lambda C^{n,\lambda}_{\alpha,\beta}(\xi,\rho,\phi,\psi,\mu)\cdot$

Proof. Let $f(z) \in \Lambda C^{n,\lambda}_{\alpha,\beta}(\xi,\rho,\phi,\psi,\mu)$ so by definition there exist a function $g(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)$ such that

$$\frac{1}{1-\rho}\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)g(z)}-\rho\right) \prec \psi(z)\ (z \in \mathbb{U})\cdot$$

We set

$$p(z) = \frac{1}{1-\rho}\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)L_cf(z))'}{D^n_\lambda(\alpha,\beta,\mu)L_cg(z)}-\rho\right) \tag{17}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Using (13) we get

$$\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)g(z)}\right) = \frac{\frac{z(D^n_\lambda(\alpha,\beta,\mu)L_{c,p}(zf'(z))'}{D^n_\lambda(\alpha,\beta,\mu)L_{c,p}g(z)} + (c)\frac{(D^n_\lambda(\alpha,\beta,\mu)L_c(zf'(z))}{D^n_\lambda(\alpha,\beta,\mu)L_{c,p}g(z)}}{\frac{z(D^n_\lambda(\alpha,\beta,\mu)L_{c,p}(g(z))'}{D^n_\lambda(\alpha,\beta,\mu)L_{c,p}g(z)}+c}\cdot \tag{18}$$

Since $g(z) \in \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)$ implies $L_cg \in \Lambda S^{n,\lambda}_{\alpha,\beta}\left(\xi,\phi,\mu\right)$, by using Theorem 3.1, we have

$$q(z) = \frac{1}{1-\xi}\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)L_cg(z))'}{D^n_\lambda(\alpha,\beta,\mu)L_cg(z)}-\xi\right). \tag{19}$$

Then, by virtue of (17), (18) and (19), we get

$$\frac{1}{1-\rho}\left(\frac{z(D^n_\lambda(\alpha,\beta,\mu)f(z))'}{D^n_\lambda(\alpha,\beta,\mu)g(z)}-\rho\right) = p(z) + \frac{zp'(z)}{(1-\xi)q(z)+\xi+c}\cdot$$

Hence by using Lemma 2.2, we can show that $p \prec \psi$ in $\mathbb{U}$, so that $L_c(f) \in \Lambda C^{n,\lambda}_{\alpha,\beta}(\xi,\rho,\phi,\psi,\mu)\cdot$ Hence proved.

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Maslina Darus
School of Mathematical Sciences,
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail: maslina@ukm.my(corresponding author)

Imran Faisal
School of Mathematical Sciences,
Faculty of Science and Technology
Universiti Kebangsaan Malaysia
Bangi 43600 Selangor D. Ehsan, Malaysia
e-mail: faisalmath@gmail.com