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On characterization inextensible flows of curves according to Bishop frame in $\mathbb{E}^3$

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Abstract

In this paper, we study inextensible flows of curves in $\mathbb{E}^3$. We research inextensible flows of curves according to Bishop frame in $\mathbb{E}^3$.

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1 Introduction

One of the oldest topics in the calculus of variations is the study of the elastic rod which, according to Daniel Bernoulli's idealization, minimizes total squared curvature among curves of the same length and first order boundary data. The classical term elastica refers to a curve in the plane or $\mathbb{R}^3$ which represents such a rod in equilibrium.

On the other hand, physically, inextensible curve and surface flows give rise to motions in which no strain energy is induced. The swinging motion of a cord of fixed length, for example, or of a piece of paper carried by the wind, can be described by inextensible curve and surface flows. Such motions arise quite naturally in a wide range of physical applications. For example, both Chirikjian and Burdick [4] and Mochiyama et al. [12] study the shape control of hyper-redundant, or snake-like, robots. Inextensible curve and surface flows also arise in the context of many problems in computer vision [11] and computer animation [5].

There have been numerous studies in the literature on plane curve flows, particularly on evolving curves in the direction of their curvature vector field (referred to by various names such as "curve shortening", "flow by curvature", and "heat flow"). Particularly relevant to this article are the methods developed by Gage and Hamilton [7] and Grayson [8] for studying the shrinking of closed plane curves to a circle via the heat equation. In [6] Gage also studies area-preserving evolutions of plane curves.

In this paper, we study inextensible flows of curves in $\mathbb{E}^3$. We research inextensible flows of curves according to Bishop frame in $\mathbb{E}^3$. Necessary and sufficient conditions for an inelastic curve flow are expressed as a partial differential equation involving the curvature.

2 Preliminaries

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The Euclidean 3-space $\mathbb{E}^3$ provided with the standard flat metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbb{E}^3$. Recall that, the norm of an arbitrary vector $a \in \mathbb{E}^3$ is given by $\|a\| = \sqrt{\langle a, a \rangle}$. γ is called a unit speed curve if velocity vector v of γ satisfies $\|a\| = 1$.

Denote by $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ the moving Frenet–Serret frame along the curve γ in the space $\mathbb{E}^3$. For an arbitrary curve γ with first and second curvature, κ and τ in the space $\mathbb{E}^3$, the following Frenet–Serret formulae is given

$$\begin{aligned} \mathbf{T}' &= \kappa \mathbf{N} \\ \mathbf{N}' &= -\kappa \mathbf{T} + \tau \mathbf{B} \\ \mathbf{B}' &= -\tau \mathbf{N}, \end{aligned}$$

where

$$\begin{aligned} \langle \mathbf{T}, \mathbf{T} \rangle &= \langle \mathbf{N}, \mathbf{N} \rangle = \langle \mathbf{B}, \mathbf{B} \rangle = 1, \\ \langle \mathbf{T}, \mathbf{N} \rangle &= \langle \mathbf{T}, \mathbf{B} \rangle = \langle \mathbf{N}, \mathbf{B} \rangle = 0. \end{aligned}$$

Here, curvature functions are defined by $\kappa = \kappa(s) = \|\mathbf{T}'(s)\|$ and $\tau(s) = -\langle \mathbf{N}, \mathbf{B}' \rangle$.

Torsion of the curve γ is given by the aid of the mixed product

$$\tau = \frac{[\gamma', \gamma'', \gamma''']}{\kappa^2}.$$

In the rest of the paper, we suppose everywhere $\kappa \neq 0$ and $\tau \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. One can express parallel transport of an orthonormal frame along a curve simply by parallel transporting each component of the frame. The tangent vector and any convenient arbitrary basis for the remainder of the frame are used. The Bishop frame is expressed as

$$\begin{array}{l} \mathbf{T}' = k_1 \mathbf{M}_1 + k_2 \mathbf{M}_2, \\ \mathbf{M}_1' = -k_1 \mathbf{T}, \\ \mathbf{M}_2' = -k_2 \mathbf{T}. \end{array} \tag{2.1}$$

Here, we shall call the set $\{\mathbf{T}, \mathbf{M}_1, \mathbf{M}_1\}$ as Bishop trihedra and k_1 and k_2 as Bishop curvatures. The relation matrix may be expressed as

$$\begin{array}{l} \mathbf{T} = \mathbf{T}, \\ \mathbf{N} = \cos\theta\,(s)\,\mathbf{M}_1 + \sin\theta\,(s)\,\mathbf{M}_2, \\ \mathbf{B} = -\sin\theta\,(s)\,\mathbf{M}_1 + \cos\theta\,(s)\,\mathbf{M}_2, \end{array}$$

where $\theta\,(s) = \arctan \frac{k_2}{k_1}$, $\tau(s) = \theta'\,(s)$ and $\kappa(s) = \sqrt{k_1^2 + k_2^2}$. Here, Bishop curvatures are defined by

$$\begin{array}{rcl} k_1 & = & \kappa(s) \cos\theta\,(s)\,, \\ k_2 & = & \kappa(s) \sin\theta\,(s)\,. \end{array}$$

On the other hand,

$$\begin{array}{rcl} \mathbf{T} & = & \mathbf{T}, \\ \mathbf{M}_1 & = & \cos\theta\,(s)\,\mathbf{N} - \sin\theta\,(s)\,\mathbf{B}, \\ \mathbf{M}_2 & = & \sin\theta\,(s)\,\mathbf{N} + \cos\theta\,(s)\,\mathbf{B}. \end{array}$$

3 Inextensible Flows of Curves According to Bishop Frame in $\mathbb{E}^3$

Throughout this article, we assume that $F : [0, l] \times [0, \omega] \to \mathbb{E}^3$ is a one parameter family of smooth curves in Euclidean space $\mathbb{E}^3$, where l is the arclength of the initial curve. Let u be the curve parametrization variable, $0 \leq u \leq l$.

The arclength of $\digamma$ is given by

$$s(u) =_0^u \left|\frac{\partial \digamma}{\partial u}\right| du, \tag{3.1}$$

where

$$\left|\frac{\partial \digamma}{\partial u}\right| = \left|\left\langle \frac{\partial \digamma}{\partial u}, \frac{\partial \digamma}{\partial u} \right\rangle\right|^{\frac{1}{2}}. \tag{3.2}$$

The operator $\frac{\partial}{\partial s}$ is given in terms of u by

$$\frac{\partial}{\partial s} = \frac{1}{\nu}\frac{\partial}{\partial u},$$

where $v = \left|\frac{\partial \digamma}{\partial u}\right|$.The arclength parameter is $ds = vdu$.

Any flow of $\digamma$ can be represented as

$$\frac{\partial \digamma}{\partial u} = f\mathbf{T} + g\mathbf{N} + h\mathbf{B}. \tag{3.3}$$

Letting the arclength variation be

$$s(u,t) =_0^u vdu.$$

In the Euclidean space the requirement that the curve not be subject to any elongation or compression can be expressed by the condition

$$\frac{\partial}{\partial t}s(u,t) =_0^u \frac{\partial v}{\partial t}du = 0, \tag{3.4}$$

for all $u \in [0,l]$.

Definition 3.1. A curve evolution $\digamma(u,t)$ and its flow $\frac{\partial \digamma}{\partial t}$ in $\mathbb{E}^3$ are said to be inextensible if

$$\frac{\partial}{\partial t}\left|\frac{\partial \digamma}{\partial u}\right| = 0.$$

Lemma 3.2. Let $\frac{\partial \digamma}{\partial u} = f\mathbf{T} + g\mathbf{M}_1 + h\mathbf{M}_2$ be a smooth flow of the curve $\digamma$. The flow is inextensible if and only if

$$\frac{\partial v}{\partial t} = \left(\frac{\partial f}{\partial u} - gvk_1 - hvk_2\right). \tag{3.5}$$

Proof. Suppose that $\frac{\partial \digamma}{\partial u}$ be a smooth flow of the curve $\digamma$. Using definition of $\digamma$, we have

$$v^2 = \left\langle \frac{\partial \digamma}{\partial u}, \frac{\partial \digamma}{\partial u} \right\rangle. \tag{3.6}$$

$\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$ commute since and are independent coordinates. So, by differentiating of the formula (3.6), we get

$$2v\frac{\partial v}{\partial t} = \frac{\partial}{\partial t}\left\langle \frac{\partial \digamma}{\partial t}, \frac{\partial \digamma}{\partial t} \right\rangle.$$

On the other hand, changing $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial t}$, we have

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial \digamma}{\partial u}, \frac{\partial}{\partial u}(\frac{\partial \digamma}{\partial u}) \right\rangle.$$

From (3.3), we obtain

$$v\frac{\partial v}{\partial t} = \left\langle \frac{\partial \digamma}{\partial u}, \frac{\partial}{\partial u}\left(f\mathbf{T} + g\mathbf{M}_1 + h\mathbf{M}_2\right) \right\rangle.$$

By the formula of the Bishop, we have

$$\frac{\partial v}{\partial t} = \left\langle \mathbf{T}, \left(\frac{\partial \digamma}{\partial u} - gvk_1 - hvk_2\right)\mathbf{T} + \left(vk_1 f + \frac{\partial g}{\partial u}\right)\mathbf{M}_1 + \left(vk_2 f + \frac{\partial h}{\partial u}\right)\mathbf{M}_2 \right\rangle.$$

Making necessary calculations from above equation, we have (3.5), which proves the lemma.

Theorem 3.3. Let $\frac{\partial \digamma}{\partial u} = f\mathbf{T} + g\mathbf{M}_1 + h\mathbf{M}_2$ be a smooth flow of the curve $\digamma$. The flow is inextensible if and only if

$$\frac{\partial f}{\partial s} = gk_1 + hk_2. \tag{3.7}$$

Proof. Now let $\frac{\partial \digamma}{\partial u}$ be extensible. From (3.4), we have

$$\frac{\partial}{\partial t}s(u,t) =_0^u \frac{\partial v}{\partial t}du =_0^u \left(\frac{\partial f}{\partial u} - gvk_1 - hvk_2\right)du = 0, \tag{3.8}$$

$\forall u \in [0,l]$. Substituting (3.5) in (3.8) complete the proof of the theorem.

We now restrict ourselves to arc length parametrized curves. That is, $v = 1$ and the local coordinate u corresponds to the curve arc length s. We require the following lemma.

Lemma 3.4.

$$\frac{\partial \mathbf{T}}{\partial t}=\left(fk_1+\frac{\partial g}{\partial s}\right)\mathbf{M}_1+\left(fk_2+\frac{\partial h}{\partial s}\right)\mathbf{M}_2, \tag{3.9}$$

$$\frac{\partial \mathbf{M}_1}{\partial t}=-\left(fk_1+\frac{\partial g}{\partial s}\right)\mathbf{T}+\psi\mathbf{M}_2, \tag{3.10}$$

$$\frac{\partial \mathbf{M}_2}{\partial t}=-\left(fk_2+\frac{\partial h}{\partial s}\right)\mathbf{T}-\psi\mathbf{M}_1, \tag{3.11}$$

where $\psi=\left\langle \frac{\partial \mathbf{M}_1}{\partial t},\mathbf{M}_2\right\rangle$.

Proof. Using definition of $\mathcal{F}$, we have

$$\frac{\partial \mathbf{T}}{\partial t}=\frac{\partial}{\partial t}\frac{\partial \mathcal{F}}{\partial s}=\frac{\partial}{\partial s}(f\mathbf{T}+g\mathbf{M}_1+h\mathbf{M}_2).$$

Using the Bishop equations, we have

$$\frac{\partial \mathbf{T}}{\partial t}=\left(\frac{\partial f}{\partial s}-k_1g-hk_2\right)\mathbf{T}+\left(k_1f+\frac{\partial g}{\partial s}\right)\mathbf{M}_1+\left(k_2f+\frac{\partial h}{\partial s}\right)\mathbf{M}_2. \tag{3.12}$$

Substituting (3.7) in (3.12), we get

$$\frac{\partial \mathbf{T}}{\partial t}=\left(fk_1+\frac{\partial g}{\partial s}\right)\mathbf{M}_1+\left(fk_2+\frac{\partial h}{\partial s}\right)\mathbf{M}_2.$$

Now differentiate the Bishop frame by t :

$$\begin{aligned}
0 &= \frac{\partial}{\partial t}\langle \mathbf{T},\mathbf{M}_1\rangle=\left\langle \frac{\partial \mathbf{T}}{\partial t},\mathbf{M}_1\right\rangle+\left\langle \mathbf{T},\frac{\partial \mathbf{M}_1}{\partial t}\right\rangle=fk_1+\frac{\partial g}{\partial s}+\left\langle \mathbf{T},\frac{\partial \mathbf{M}_1}{\partial t}\right\rangle,\\
0 &= \frac{\partial}{\partial t}\langle \mathbf{T},\mathbf{M}_2\rangle=\left\langle \frac{\partial \mathbf{T}}{\partial t},\mathbf{M}_2\right\rangle+\left\langle \mathbf{T},\frac{\partial \mathbf{M}_2}{\partial t}\right\rangle=fk_2+\frac{\partial h}{\partial s}+\left\langle \mathbf{T},\frac{\partial \mathbf{M}_2}{\partial t}\right\rangle,\\
0 &= \frac{\partial}{\partial t}\langle \mathbf{M}_1,\mathbf{M}_2\rangle=\left\langle \frac{\partial \mathbf{M}_1}{\partial t},\mathbf{M}_2\right\rangle+\left\langle M_1,\frac{\partial \mathbf{M}_2}{\partial t}\right\rangle=\psi+\left\langle \mathbf{M}_1,\frac{\partial \mathbf{M}_2}{\partial t}\right\rangle.
\end{aligned}$$

From the above and using $\left\langle \frac{\partial \mathbf{M}_1}{\partial t},\mathbf{M}_1\right\rangle=\left\langle \frac{\partial \mathbf{M}_2}{\partial t},\mathbf{M}_2\right\rangle=0$, we obtain

$$\begin{aligned}
\frac{\partial \mathbf{M}_1}{\partial t} &= -\left(fk_1+\frac{\partial g}{\partial s}\right)\mathbf{T}+\psi\mathbf{M}_2,\\
\frac{\partial \mathbf{M}_2}{\partial t} &= -\left(fk_2+\frac{\partial h}{\partial s}\right)\mathbf{T}-\psi\mathbf{M}_1,
\end{aligned}$$

where $\psi = \left\langle \frac{\partial \mathbf{M}_1}{\partial t}, \mathbf{M}_2 \right\rangle$.

The following theorem states the conditions on the curvature and torsion for the curve flow $\mathcal{F}(s,t)$ to be inextensible.

Theorem 3.5. Suppose the curve $\frac{\partial \mathcal{F}}{\partial u} = f\mathbf{T} + g\mathbf{M}_1 + h\mathbf{M}_2$ is inextensible. Then, the following system of partial differential equations holds:

$$\frac{\partial k_2}{\partial t} = \left(\frac{\partial}{\partial s}(fk_2) + \frac{\partial^2 h}{\partial s^2} \right) + \psi k_1. \tag{3.13}$$

Proof. Using (3.9), we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial}{\partial s}\left[\left(fk_1 + \frac{\partial g}{\partial s} \right)\mathbf{M}_1 + \left(fk_2 + \frac{\partial h}{\partial s} \right)\mathbf{M}_2 \right] \\
&= \left(\frac{\partial}{\partial s}(fk_1) + \frac{\partial^2 g}{\partial s^2} \right)\mathbf{M}_1 + \left(fk_1 + \frac{\partial g}{\partial s} \right)(-k_1\mathbf{T}) \\
&\quad + \left(\frac{\partial}{\partial s}(fk_2) + \frac{\partial^2 h}{\partial s^2} \right)\mathbf{M}_2 + \left(fk_2 + \frac{\partial h}{\partial s} \right)(-k_2\mathbf{T}).
\end{aligned}$$

On the other hand, from Bishop frame we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{T}}{\partial t} &= \frac{\partial}{\partial t}(k_1\mathbf{M}_1 + k_2\mathbf{M}_2) \\
&= \frac{\partial k_1}{\partial t}\mathbf{M}_1 + \left(-\left(fk_1 + \frac{\partial g}{\partial s} \right)\mathbf{T} + \psi\mathbf{M}_2 \right) + \frac{\partial k_1}{\partial t}\mathbf{M}_2 \\
&\quad + k_2\left(-\left(fk_2 + \frac{\partial h}{\partial s} \right)\mathbf{T} - \psi\mathbf{M}_1 \right).
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
\frac{\partial}{\partial s}\frac{\partial \mathbf{M}_2}{\partial t} &= \frac{\partial}{\partial s}\left[-\left(fk_2 + \frac{\partial h}{\partial s} \right)\mathbf{T} - \psi\mathbf{M}_1 \right] \\
&= \left(-\frac{\partial}{\partial s}(fk_2) - \frac{\partial^2 h}{\partial s^2} - \psi k_1 \right)\mathbf{T} - k_1\left(fk_2 + \frac{\partial h}{\partial s} - \frac{\partial \psi}{\partial s} \right)\mathbf{M}_1 + k_2\left(fk_2 + \frac{\partial h}{\partial s} \right)\mathbf{M}_2,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial t}\frac{\partial M_2}{\partial s} &= \frac{\partial}{\partial t}(-k_2\mathbf{T}) \\
&= -\frac{\partial k_2}{\partial t}\mathbf{T} - k_2\left[\left(fk_1 + \frac{\partial g}{\partial s} \right)\mathbf{M}_1 + \left(fk_2 + \frac{\partial h}{\partial s} \right)\mathbf{M}_2 \right].
\end{aligned}$$

Thus, we have (3.13).

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