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Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

# Biharmonic curves in the special three-dimensional Kentmotsu manifold $\mathbb{K}$ with $\eta$-parallel Ricci tenso

Talat KÖRPINAR and Essin TURHAN

**Abstract**

In this paper, we study biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with $\eta$-parallel Ricci tensor. We characterize the biharmonic curves in terms of their curvature and torsion.

**key words.** Biharmonic curve, Kenmotsu manifold, curvature, torsion. **AMS subject classifications.** 53C41, 53A10.

## 1 Introduction

Let $\varphi : (M, g) \longrightarrow (N, h)$ be a smooth map between Riemannian manifolds with $M$ compact. Then $\varphi$ is called biharmonic if it is an extremal of the functional

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 v^M,$$

where $\tau(\varphi)$ denotes the tension field of the map $\varphi$, and $v^M$ is the volume form on $M$ [7, 10, 11]. Clearly every harmonic map is biharmonic (see [8] for a background on harmonic maps). If we set

$$E_1(\varphi) = \frac{1}{2}\int_M |d\varphi|^2 v^M,$$

to be the energy of $\varphi$, then we recall the first variation formula

$$\frac{\partial}{\partial s}E_1(\varphi_s)|_{s=0} = -\int_M \langle \tau(\varphi), v\rangle v^M,$$

where

$$v = \frac{\partial \varphi}{\partial s}|_{s=0} \in \Gamma\left(\varphi^{-1}TN\right)$$

is an arbitrary variation of $\varphi = \varphi_0$. If $v$ is taken to be in the direction of $\tau(\varphi)$, then

$$\frac{\partial}{\partial s}E_1(\varphi_s)\,|_{s=0} = -\int_M |\tau(\varphi)|^2\, v^M = -E_2(\varphi).$$

Now take an arbitrary variation of $E_2(\varphi)$ in the direction $w = \frac{\partial \varphi}{\partial t}|_{t=0}$, we have

$$\frac{\partial}{\partial t}E_2(\varphi_t)\,|_{t=0} = -\frac{\partial^2}{\partial s \partial t}E_1(\varphi_{s,t})\,|_{s,t=0} = -\int_M \langle J_\varphi(\tau(\varphi)), v\rangle\, v^M,$$

where $J_\varphi$ is the Jacobi operator corresponding to the second variation of $E_1(\varphi)$. The Euler–Lagrange equations for a biharmonic map are therefore given by the negative of the Jacobi operator acting on the tension field:

$$\tau_2(\varphi) \equiv -\mathtt{Tr}_g(\nabla^\varphi)^2\,\tau(\varphi) - \mathtt{Tr}_g R^N(\tau(\varphi), d\varphi)\,d\varphi = 0. \tag{1.1}$$

Here, our convention for the curvature is

$$R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z$$

and

$$(\nabla^\varphi)^2_{X,Y}\, v = \nabla^\varphi_X\left(\nabla^\varphi_Y v\right) - \nabla^\varphi_{\nabla^M_X Y} v$$

with $\nabla^\varphi$ representing the connection in the pull-back bundle $\varphi^{-1}(TN)$ and $\nabla^M$ the Levi-Civita connection on $M$. More generally, in the case when $M$ is no longer compact, we call a smooth map $\varphi$ biharmonic if it satisfies (1.1).

In general, the fourth-order equation (1.1) is difficult to solve. Natural candidates for solutions are submanifolds of parallel mean curvature, see [10, 11]; or submanifolds with harmonic mean curvature, see [5, 6, 9]. In [1, 2], examples of biharmonic nonminimal submanifolds of spheres are given, as well as a complete classification of biharmonic curves in a sphere. Biharmonic curves on a surface are studied in [3]. We adopt a different approach here to construct biharmonic, nonharmonic maps.

Recently, there has been a growing interest in the theory of biharmonic maps which can be divided in two main research directions. On the one side, constructing the examples and classification results have become important from the differential geometric aspect. The other side is the analytic aspect from the point of view of partial differential equations [2, 12, 15, 20, 21], because biharmonic maps are solutions of a fourth order strongly elliptic semilinear PDE. In differential geometry, harmonic maps, candidate minimisers of the Dirichlet energy, can be described as constraining a rubber sheet to fit on a marble manifold in a position of elastica equilibrium, i.e. without tension [7]. However, when this scheme falls through, and it can, as corroborated by the case of the two-torus and the two-sphere [8], a best map will minimise this failure, measured by the total tension, called bienergy. In the more geometrically meaningful context of immersions, the fact that the tension field is normal to the image submanifold, suggests that the most effective deformations must be sought in the normal direction.

In this paper, we study biharmonic curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with $\eta$-parallel ricci tensor. We characterize the biharmonic curves in terms of their curvature and torsion.

## 2 Preliminaries

Let $M^{2n+1}(\phi,\xi,\eta,g)$ be an almost contact Riemannian manifold with 1-form $\eta$, the associated vector field $\xi$, $(1,1)$-tensor field $\phi$ and the associated Riemannian metric $g$. It is well known that [1]

$$\phi\xi=0,\quad \eta(\xi)=1,\quad \eta(\phi X)=0, \tag{2.1}$$

$$\phi^2(X)=-X+\eta(X)\xi, \tag{2.2}$$

$$g(X,\xi)=\eta(X), \tag{2.3}$$

$$g(\phi X,\phi Y)=g(X,Y)-\eta(X)\eta(Y), \tag{2.4}$$

for any vector fields $X$, $Y$ on $M$. Moreover,

$$(\nabla_X\phi)Y=-\eta(Y)\phi(X)-g(X,\phi Y)\xi,\qquad X,Y\in\chi(M), \tag{2.5}$$

$$\nabla_X\xi=X-\eta(X)\xi, \tag{2.6}$$

where $\nabla$ denotes the Riemannian connection of $g$, then $(M,\phi,\xi,\eta,g)$ is called an almost Kenmotsu manifold [1].

In Kenmotsu manifolds the following relations hold [1]:

$$(\nabla_X\eta)Y=g(\phi X,\phi Y) \tag{2.7}$$

$$\eta\left(R\left(X,Y\right)Z\right)=\eta\left(Y\right)g\left(X,Z\right)-\eta\left(X\right)g\left(Y,Z\right) \tag{2.8}$$

$$R\left(X,Y\right)\xi=\eta\left(X\right)Y-\eta\left(Y\right)X \tag{2.9}$$

$$R\left(\xi,X\right)Y=\eta\left(Y\right)X-g\left(X,Y\right)\xi \tag{2.10}$$

$$R\left(\xi,X\right)\xi=X-\eta\left(X\right)\xi \tag{2.11}$$

$$S\left(\phi X,\phi Y\right)=S\left(X,Y\right)+2n\eta\left(X\right)\eta\left(Y\right) \tag{2.12}$$

$$S\left(X,\xi\right)=-2n\eta\left(X\right) \tag{2.13}$$

$$\left(\nabla_X R\right)\left(X,Y\right)\xi=g\left(Z,X\right)Y-g\left(Z,Y\right)X-R\left(X,Y\right)Z \tag{2.14}$$

where $R$ is the Riemannian curvature tensor and $S$ is the Ricci tensor. In a Riemannian manifold we also have

$$g\left(R\left(W,X\right)Y,Z\right)+g\left(R\left(W,X\right)Z,Y\right)=0, \tag{2.15}$$

for every vector fields $X,Y,Z$.

# 3 Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with $\eta$-Parallel Ricci Tensor

**Definition 3.1.** The Ricci tensor $S$ of a Kenmotsu manifold is called $\eta$-parallel if it satisfies

$$\left(\nabla_X S\right)\left(\phi Y,\phi Z\right)=0.$$

We consider the three-dimensional manifold

$$\mathbb{K} = \left\{ \left(x^1, x^2, x^3\right) \in \mathbb{R}^3 : \left(x^1, x^2, x^3\right) \neq (0,0,0) \right\},$$

where $\left(x^1, x^2, x^3\right)$ are the standard coordinates in $\mathbb{R}^3$. The vector fields

$$\mathbf{e}_1 = x^3 \frac{\partial}{\partial x^1}, \quad \mathbf{e}_2 = x^3 \frac{\partial}{\partial x^2}, \quad \mathbf{e}_3 = -x^3 \frac{\partial}{\partial x^3} \tag{3.1}$$

are linearly independent at each point of $\mathbb{K}$. Let $g$ be the Riemannian metric defined by

$$\begin{aligned} g\left(\mathbf{e}_1, \mathbf{e}_1\right) &= g\left(\mathbf{e}_2, \mathbf{e}_2\right) = g\left(\mathbf{e}_3, \mathbf{e}_3\right) = 1, \\ g\left(\mathbf{e}_1, \mathbf{e}_2\right) &= g\left(\mathbf{e}_2, \mathbf{e}_3\right) = g\left(\mathbf{e}_1, \mathbf{e}_3\right) = 0. \end{aligned} \tag{3.2}$$

The characterising properties of $\chi(\mathbb{K})$ are the following commutation relations:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_2. \tag{3.3}$$

Let $\eta$ be the 1-form defined by

$$\eta(Z) = g(Z, \mathbf{e}_3) \text{ for any } Z \in \chi(M)$$

Let be the (1,1) tensor field defined by

$$\phi(\mathbf{e}_1) = -\mathbf{e}_2, \ \phi(\mathbf{e}_2) = \mathbf{e}_1, \ \phi(\mathbf{e}_3) = 0.$$

Then using the linearity of and $g$ we have

$$\eta(\mathbf{e}_3) = 1, \tag{3.4}$$

$$\phi^2(Z) = -Z + \ \eta(Z)\mathbf{e}_3, \tag{3.5}$$

$$g\left(\phi Z, \phi W\right) = g\left(Z, W\right) - \eta(Z)\eta(W), \tag{3.6}$$

for any $Z, W \in \chi(M)$. Thus for $\mathbf{e}_3 = \xi$, $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $\mathbb{M}$.

The Riemannian connection $\nabla$ of the metric $g$ is given by

$$\begin{aligned} 2g\left(\nabla_X Y, Z\right) &= Xg\left(Y, Z\right) + Yg\left(Z, X\right) - Zg\left(X, Y\right) \\ &\quad - g\left(X, [Y, Z]\right) - g\left(Y, [X, Z]\right) + g\left(Z, [X, Y]\right), \end{aligned}$$

which is known as Koszul's formula.

Koszul's formula yields

$$\begin{aligned}
&\nabla_{\mathbf{e}_1}\mathbf{e}_1 = 0, \;\; \nabla_{\mathbf{e}_1}\mathbf{e}_2 = 0, \;\; \nabla_{\mathbf{e}_1}\mathbf{e}_3 = \mathbf{e}_1, \\
&\nabla_{\mathbf{e}_2}\mathbf{e}_1 = 0, \;\; \nabla_{\mathbf{e}_2}\mathbf{e}_2 = 0, \;\; \nabla_{\mathbf{e}_2}\mathbf{e}_3 = \mathbf{e}_2, \\
&\nabla_{\mathbf{e}_3}\mathbf{e}_1 = 0, \;\; \nabla_{\mathbf{e}_3}\mathbf{e}_2 = 0, \; \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0.
\end{aligned} \tag{3.7}$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \quad R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices $i, j, k$ and $l$ take the values $1, 2$ and $3$.

$$R_{121} = 0, \quad R_{131} = R_{232} = \mathbf{e}_3$$

and

$$R_{1212} = 0, \quad R_{1313} = R_{2323} = 1. \tag{3.8}$$

## 4 Biharmonic Curves in the Special Three-Dimensional Kenmotsu Manifold $\mathbb{K}$ with $\eta$-Parallel Ricci Tensor

Biharmonic equation for the curve $\gamma$ reduces to

$$\nabla^3_{\mathbf{T}}\mathbf{T} - R\left(\mathbf{T}, \nabla_{\mathbf{T}}\mathbf{T}\right)\mathbf{T} = 0, \tag{4.1}$$

that is, $\gamma$ is called a biharmonic curve if it is a solution of the equation (4.1).

Let us consider biharmonicity of curves in the special three-dimensional Kenmotsu manifold $\mathbb{K}$ with $\eta$-parallel ricci tensor. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet–Serret equations:

$$\begin{aligned}
&\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N}, \\
&\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \\
&\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N},
\end{aligned} \tag{4.2}$$

where $\kappa = |\mathcal{T}(\gamma)| = |\nabla_{\mathbf{T}}\mathbf{T}|$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$\begin{aligned}
g\left(\mathbf{T}, \mathbf{T}\right) &= 1, \; g\left(\mathbf{N}, \mathbf{N}\right) = 1, \; g\left(\mathbf{B}, \mathbf{B}\right) = 1, \\
g\left(\mathbf{T}, \mathbf{N}\right) &= g\left(\mathbf{T}, \mathbf{B}\right) = g\left(\mathbf{N}, \mathbf{B}\right) = 0.
\end{aligned}$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\begin{aligned} \mathbf{T} &= T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ \mathbf{N} &= N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ \mathbf{B} &= \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3. \end{aligned} \tag{4.3}$$

**Theorem 4.1.** $\gamma : I \longrightarrow \mathbb{K}$ *is a biharmonic curve if and only if*

$$\begin{aligned} &\kappa = \text{constant} \neq 0, \\ &\kappa^2 + \tau^2 = 1 - B_3^2, \\ &\tau' = N_3 B_3. \end{aligned} \tag{4.4}$$

**Proof.** Using (4.1) and Frenet formulas (4.2), we have (4.4).

**Theorem 4.2.** *Let* $\gamma : I \longrightarrow \mathbb{K}$ *be a non-geodesic curve on the special three-dimensional Kenmotsu manifold* $\mathbb{K}$ *with* $\eta$*-parallel ricci tensor parametrized by arc length. If* $\kappa$ *is constant and* $N_3B_3 \neq 0$*, then* $\gamma$ *is not biharmonic.*

**Proof.** Using Frenet formulas (4.2) and $\nabla_{\mathbf{T}}\mathbf{B}$, we have

$$B_3' = -\tau N_3. \tag{4.5}$$

Assume now that $\gamma$ is biharmonic. Then, using $\tau' = N_3B_3 \neq 0$ and from (4.4), we obtain

$$\tau\tau' = -B_3 B_3',$$

and

$$\tau N_3 B_3 = B_3 B_3' \ . \tag{4.6}$$

Substituting $B_3'$ in equation (4.5), we find

$$\tau = 0. \tag{4.7}$$

Therefore, $\tau$ is constant and we have a contradiction.

**Theorem 4.3.** *Let* $\gamma : I \longrightarrow \mathbb{K}$ *be a unit speed non-geodesic curve with constant curvature. Then, the parametric equations of* $\gamma$ *are*

$$
\begin{aligned}
x^1(s) &= C_2 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s} \Big( \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} \cos\left[ \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C \right] \\
&\quad - \cos\varphi \sin\left[ \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C \right] \Big), \\
x^2(s) &= C_3 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s} \Big( -\cos\varphi \cos\left[ \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C \right] \\
&\quad + \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} \sin\left[ \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}} s + C \right] \Big), \\
x^3(s) &= C_1 e^{-\cos\varphi s},
\end{aligned}
\tag{4.8}
$$

*where $C$, $C_1$, $C_2$, $C_3$ are constants of integration.*

**Proof.** Since $\gamma$ is biharmonic, $\gamma$ is a helix. So, without loss of generality, we take the axis of $\gamma$ is parallel to the vector $\mathbf{e}_3$. Then,

$$g(\mathbf{T}, \mathbf{e}_3) = T_3 = \cos\varphi, \tag{4.9}$$

where $\varphi$ is constant angle.

The tangent vector can be written in the following form

$$\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3. \tag{4.10}$$

On the other hand the tangent vector $\mathbf{T}$ is a unit vector, so the following condition is satisfied

$$T_1^2 + T_2^2 = 1 - \cos^2\varphi. \tag{4.11}$$

Noting that $\cos^2\varphi + \sin^2\varphi = 1$, we have

$$T_1^2 + T_2^2 = \sin^2\varphi. \tag{4.12}$$

The general solution of (4.12) can be written in the following form

$$
\begin{aligned}
T_1 &= \sin\varphi \sin\mu, \\
T_2 &= \sin\varphi \cos\mu,
\end{aligned}
\tag{4.13}
$$

where $\mu$ is an arbitrary function of $s$.

So, substituting the components $T_1$, $T_2$ and $T_3$ in the equation (4.10), we have the following equation

$$\mathbf{T} = \sin\varphi\sin\mu\mathbf{e}_1 + \sin\varphi\cos\mu\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \tag{4.14}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa$, we obtain

$$\mu = \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C, \tag{4.15}$$

where $C \in \mathbb{R}$.

Thus (4.14) and (4.15), imply

$$\begin{aligned}\mathbf{T} &= \sin\varphi\sin\left[\sqrt{-\cos^2\varphi + \tfrac{\kappa^2}{\sin^2\varphi}}s + C\right]\mathbf{e}_1 \\ &\quad + \sin\varphi\cos\left[\sqrt{-\cos^2\varphi + \tfrac{\kappa^2}{\sin^2\varphi}}s + C\right]\mathbf{e}_2 + \cos\varphi\mathbf{e}_3.\end{aligned} \tag{4.16}$$

Using (3.1) in (4.16), we obtain

$$\begin{aligned}\mathbf{T} = (&x^3\sin\varphi\sin\left[\sqrt{-\cos^2\varphi + \tfrac{\kappa^2}{\sin^2\varphi}}s + C\right], x^3\sin\varphi\cos\left[\sqrt{-\cos^2\varphi + \tfrac{\kappa^2}{\sin^2\varphi}}s + C\right], \\ &-x^3\cos\varphi).\end{aligned} \tag{4.17}$$

From third component of $\mathbf{T}$, we have

$$\frac{dx^3}{ds} = -x^3\cos\varphi.$$

$$x^3(s) = C_1 e^{-\cos\varphi s}$$

By direct calculations we have

$$\frac{dx^1}{ds} = C_1\sin\varphi e^{-\cos\varphi s}\sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right], \tag{4.18}$$

$$\frac{dx^2}{ds} = C_1\sin\varphi e^{-\cos\varphi s}\cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right]. \tag{4.19}$$

Moreover, (4.18) and (4.19) imply

$$\begin{aligned}x^1(s) = C_2 - \frac{C_1\sin^3\varphi}{\kappa^2}e^{-\cos\varphi s}(&\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}\cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right] \\ &-\cos\varphi\sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right]),\end{aligned} \tag{4.20}$$

and

$$x^2(s) = C_3 - \frac{C_1 \sin^3\varphi}{\kappa^2} e^{-\cos\varphi s}(-\cos\varphi\cos\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right] + \sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}\sin\left[\sqrt{-\cos^2\varphi + \frac{\kappa^2}{\sin^2\varphi}}s + C\right]), \tag{4.21}$$

which proves our assertion.

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**Talat KÖRPINAR and Essin TURHAN**
Fırat University, Department of Mathematics,
23119, Elazığ, TURKEY
e-mail: essin.turhan@gmail.com