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A Generalization of Modular Sequence Spaces by Cesàro Mean of Order One

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Abstract

In this paper, we introduce the modular sequence spaces generated by Cesàro mean of order one and give several properties relevant to algebraic and topological structures of these spaces.

key words. Sequence space, Cesàro mean of order one, Orlicz function, completeness, AK-BK space. **AMS subject classifications.** 46A45, 40C05, 40A05, 46E30.

1 Introduction

An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \longrightarrow \infty$, as $x \longrightarrow \infty$.

If convexity of Orlicz function M is replaced by $M(x+y) \leq M(x)+M(y)$, then this function is called a modulus function introduced by Nakano [6].

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct sequence space

$$\ell_M = \{x = (x_k) \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, \text{ for some } \rho > 0\}.$$

The space ℓ_M becomes a Banach space, with the norm

$$\|x\| = \|(x_k)\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\}$$

which is called an Orlicz space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x) = x^p$ for $1 \leq p < \infty$. The study of Orlicz sequence spaces was initiated with a certain specific purpose in Banach space theory. Indeed, Lindberg got interested in Orlicz spaces in connection with finding Banach spaces with symmetric Schauder bases having complementary subspaces isomorphic to c_0 or ℓ_p $(1 \leq p < \infty)$. Subsequently Lindenstrauss and Tzafriri studied these Orlicz sequence spaces in more detail, and solved many important and interesting structural problems in Banach spaces. Later on, different classes of sequence spaces defined by Orlicz function were studied by many others.

Another generalization of Orlicz sequence spaces is due to Woo [10]. Let $\{M_k\}$ be a sequence of Orlicz functions. Define the vector space $\ell\{M_k\}$ by

$$\ell\{M_k\} = \{x = (x_k) \in w : \sum_{k=1}^{\infty} M_k\left(\frac{|x_k|}{\rho}\right) < \infty,\ for\ some\ \rho > 0\}$$

and equip this space with the norm

$$\|x\| = \|(x_k)\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M_k\left(\tfrac{|x_k|}{\rho}\right) \leq 1\}.$$

Then $\ell\{M_k\}$ becomes a Banach space and is called a modular sequence space. The space $\ell\{M_k\}$ also generalizes the concept of modulared sequence space introduced earlier by Nakano [7], who considered the space $\ell\{M_k\}$ when $M_k(x) = x^{\alpha_k}$, where $1 \leq \alpha_k < \infty$ for $k \geq 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values of u, if there exists a constant $K > 0$, such that $M(2u) \leq KM(u)$, $(u \geq 0)$. The Δ_2-condition is equivalent to the satisfaction of inequality $M(lu) \leq KluM(u)$ for all values of u and for $l > 1$(See[4]).

The above Δ_2-condition also implies $M(lu) \leq Kl^{\log_2 K}M(u)$, for all $u > 0$, $l > 1$.

A BK-space (introduced by Zeller [11]) $(X, \|.\|)$ is a Banach space of complex sequences $x = (x_k)$ in which the co-ordinate maps are continuous, that is $|x_k^n - x_k| \longrightarrow 0$, whenever $\|x^n - x\| \longrightarrow 0$ as $n \longrightarrow \infty$, where $x^n = (x_k^n)$, for all $n \in N$ and $x = (x_k)$.

Let A denotes the set of all complex sequences which have only a finite number of non-zero coordinates, λ denotes a BK-space of sequences $x = (x_k)$ which contains A. An element $x = (x_k)$ of λ will be called sectionally convergent if

$$x^{(n)} = \sum_{k=1}^{n} x_k e_k \longrightarrow x, \quad \text{as} \quad n \longrightarrow \infty$$

where $e_k = (\delta_{ki})$, where $\delta_{kk} = 1$, $\delta_{ki} = 0$ for $k \neq i$

λ will be called AK-space if and only if each of its elements is sectionallly convergent.

Let $M = (M_k)$ be a sequence of Orlicz function and $C = (c_{nk})_{n,k=0}^{\infty}$ be the Cesàro matrix of order one with $c_{nk} = \frac{1}{n+1}$ if $0 \leq k \leq n$ and $c_{nk} = 0$, otherwise. Then we define

$$\ell\{M_k, C\} = \left\{ x = (x_k) \in w : \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty, \text{ for some } \rho > 0 \right\}.$$

2 Main Results

In this section we give the theorems that characterize the structure of the class of sequences $\ell\{M_k, C\}$.

Theorem 1. *$\ell\{M_k, C\}$ is a linear space over the field C.*

Proof. Let $x, y \in \ell\{M_k, C\}$ and $\alpha, \beta \in C$. Then there exists some $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho_1(k+1)} \right) < \infty \quad \text{and} \quad \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} y_j \right|}{\rho_2(k+1)} \right) < \infty.$$

We consider $\rho_3 = max\,(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since each M_k is non-decreasing and convex, we have

$$\begin{aligned}
\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} (\alpha x_j + \beta y_j) \right|}{\rho_3(k+1)} \right) &\leq \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} (\alpha x_j) \right|}{\rho_3(k+1)} + \frac{\left| \sum_{j=0}^{k} (\beta y_j) \right|}{\rho_3(k+1)} \right) \\
&\leq \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{2\rho_1(k+1)} + \frac{\left| \sum_{j=0}^{k} y_j \right|}{2\rho_2(k+1)} \right) \\
&\leq \frac{1}{2} \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho_1(k+1)} \right) + \frac{1}{2} \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} y_j \right|}{\rho_2(k+1)} \right)
\end{aligned}$$

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} (\alpha x_j + \beta y_j) \right|}{\rho_3(k+1)} \right) < \infty, \text{ for some } \rho_3 > 0.$$

This completes the proof.

Theorem 2. *Let $\ell\{M_k, C\}$ is a normed linear space normed by*

$$\|x\| = \inf \left\{ \rho > 0 : \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) \leq 1 \right\}.$$

Proof. If $x = \theta$, then it is obvious that $\|x\| = 0$. Conversely assume $\|x\| = 0$. Then using the definition of norm, we have

$$\inf \left\{ \rho > 0 : \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) \leq 1 \right\} = 0$$

This implies that for a given $\varepsilon > 0$, there exists some ρ_ε $(0 < \rho_\varepsilon < \varepsilon)$ such that

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1$$

It follows that

$$M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1, \quad \forall \ \ k \in N$$

Thus $$M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\varepsilon(k+1)} \right) \leq M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho_\varepsilon(k+1)} \right) \leq 1, \quad \forall \ \ k \in N$$

Suppose $\dfrac{\sum_{j=0}^{n_i} x_j}{n_i + 1} \neq 0$, for some i. Let $\varepsilon \longrightarrow 0$ then $\dfrac{\sum_{j=0}^{n_i} x_j}{\varepsilon(n_i + 1)} \longrightarrow \infty$.

$$\text{It follows that} \quad M_k\left(\frac{\left|\sum\limits_{j=0}^{k} x_j\right|}{\varepsilon(k+1)}\right) \longrightarrow \infty \quad \text{as } \varepsilon \longrightarrow 0 \quad \text{for some } n_i \in N.$$

This is a contradiction. Therefore

$$\frac{\sum\limits_{j=0}^{k} x_j}{k+1} = 0, \qquad \forall \;\; k \in N.$$

It follows that $x_k = 0$ for all $k \geq 1$. Hence $x = \theta$. Now let $x, y \in \ell\{M_k, C\}$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} x_j\right|}{\rho_1(k+1)}\right) \leq 1 \quad \text{and} \quad \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} y_j\right|}{\rho_2(k+1)}\right) \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we have

$$\sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} (x_j + y_j)\right|}{\rho(k+1)}\right) \leq \left(\frac{\rho_1}{\rho_1 + \rho_2}\right) \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} x_j\right|}{\rho_1(k+1)}\right)$$

$$+ \left(\frac{\rho_2}{\rho_1 + \rho_2}\right) \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} y_j\right|}{\rho_2(k+1)}\right) \leq 1.$$

Hence $\|x + y\| \leq \|x\| + \|y\|$. Finally let λ be a given non-zero scalar, then we have

$$\|\lambda x\| = \inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} (\lambda x_j)\right|}{\rho(k+1)}\right) \leq 1\right\}$$

$$= \inf\left\{(|\lambda| s) > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} x_j\right|}{s(k+1)}\right) \leq 1\right\}, \quad \text{where} \quad s = \frac{\rho}{|\lambda|}.$$

$\|\lambda x\| = |\lambda| \; \|x\|$. This completes the proof.

Proposition 3. *Let $M = (M_k)$ and $T = (T_k)$ be sequences of Orlicz functions. Then we have $\ell\{M_k, C\} \cap \ell\{T_k, C\} \subset \ell\{M_k + T_k, C\}$.*

Proof. The proof is easy, so omitted.

Theorem 4. *Let $M = (M_k)$ and $T = (T_k)$ be sequences of Orlicz functions which satisfy Δ_2-condition, then $\ell\{M_k, C\} \subseteq \ell\{T_k \circ M_k, C\}$.*

Proof. Let $x \in \ell\{M_k, C\}$ and $\varepsilon > 0$. We choose $0 < \delta < 1$ such that for each k, $T_k(u) < k^{-s}(s > 1) < \varepsilon$ for $0 \leq u \leq \delta$. We write

$$y_k = M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right)$$

and consider $\sum_{k=0}^{\infty} T_k(y_k) = \sum_1 T_k(y_k) + \sum_2 T_k(y_k)$ where the first summation is over $y_k \leq \delta$ and the second summation over $y_k > \delta$. Now we have $\sum_1 T_k(y_k) < \infty$. For $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \left(\frac{y_k}{\delta} \right).$$

Since for each T_k is non-decreasing and convex, it follows that

$$T_k(y_k) < T_k\left(1 + \frac{y_k}{\delta}\right) < \frac{1}{2}T_k(2) + \frac{1}{2}T_k\left(2\frac{y_k}{\delta}\right), \text{ for each } k \in N.$$

Since each T_k satisfy Δ_2-condition, we have

$$T_k(y_k) < \frac{1}{2}K(\frac{y_k}{\delta})T_k(2) + \frac{1}{2}K(\frac{y_k}{\delta})T_k(2) = K y_k \delta^{-1} T_k(2)$$

Hence $$\sum_2 T_k(y_k) \leq \max\left(1, \left(K\delta^{-1}M(2)\right)\right) \sum_{k=0}^{\infty} y_k < \infty$$

Thus $$\sum_{k=0}^{\infty} T_k(y_k) = \sum_1 T_k(y_k) + \sum_2 T_k(y_k) < \infty.$$

Hence $x \in \ell\{T_k \circ M_k, C\}$. This completes the proof.

Taking $M_k(x) = x$, for all $x \in [0, \infty)$ and k in N, in Theorem 4, we get the next Corollary.

Corollary 5. *Let $M = (M_k)$ be any sequence of Orlicz functions which satisfy Δ_2-condition and $s > 1$, then $\ell\{C\} \subseteq \ell\{M_k, C\}$.*

We will write $f \approx g$ for non-negative functions f and g whenever $C_1 f \leq g \leq C_2 f$ for some $C_j > 0$, $j = 1, 2$.

Proposition 6. *Let $M = (M_k)$ and $T = (T_k)$ be sequences of Orlicz functions. If $M_k \approx T_k$ for each $k \in N$, then $\ell\{M_k, C\} = \ell\{T_k, C\}$.*

Proof. Proof is obvious.

Proposition 7. *Let $M = (M_k)$ be a sequence of Orlicz functions. If $\lim\limits_{t\to 0} \frac{M_k(t)}{t} > 0$ and $\lim\limits_{t\to 0} \frac{M_k(t)}{t} < \infty$, for each $k \in N$, then $\ell\{M_k, C\} = \ell\{C\}$.*

Proof. If the given conditions are satisfied, we have $M_k(t) \approx t$ for each k and the proof follows from Proposition 6.

Theorem 8. *$\ell\{M_k, C\}$ is a Banach space normed by*

$$\|x\| = \inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} x_j\right|}{\rho(k+1)}\right) \leq 1\right\}.$$

Proof. Let (x^i) be a Cauchy sequence in $\ell\{M_k, C\}$. Let $\delta > 0$ be fixed and $r > 0$ be such that for a given $0 < \varepsilon < 1$, $\frac{\varepsilon}{r\delta} > 0$, and $r\delta \geq 1$. Then there exists a positive integer n_0 such that $\|x^s - x^t\| < \frac{\varepsilon}{r\delta}$, for all $s, t \geq n_0$

$$\implies \quad \inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} (x_j^s - x_j^t)\right|}{\rho(k+1)}\right) \leq 1\right\} < \frac{\varepsilon}{r\delta}, \quad \text{for all} \;\; s, t \geq n_0.$$

Hence we have

$$\sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum\limits_{j=0}^{k} (x_j^s - x_j^t)\right|}{\|x^s - x^t\|(k+1)}\right) \leq 1, \quad \text{for all} \;\; s, t \geq n_0.$$

It follows that

$$M_k\left(\frac{\left|\sum\limits_{j=0}^{k} (x_j^s - x_j^t)\right|}{\|x^s - x^t\|(k+1)}\right) \leq 1, \quad \text{for all} \;\; s, t \geq n_0 \quad \text{and } k \in N.$$

For $r > 0$ with $M_k\left(\frac{r\delta}{2}\right) \geq 1$, we have

$$M_k\left(\frac{\left|\sum_{j=0}^{k}(x_j^s - x_j^t)\right|}{\|x^s - x^t\|(k+1)}\right) \leq M_k\left(\frac{r\delta}{2}\right), \quad \text{for all } \; s,t \geq n_0 \quad \text{and } k \in N.$$

Since M_k is non-decreasing for each $k \in N$, we have

$$\frac{\left|\sum_{j=0}^{k}(x_j^s - x_j^t)\right|}{(k+1)} \leq \frac{r\delta}{2}.\frac{\varepsilon}{r\delta} = \frac{\varepsilon}{2}$$

Hence it follows that (x_k^s) is a Cauchy sequence in C for each $k \in N$. But C is complete and so (x_k^s) is converges in C for each $k \in N$. Let $\lim\limits_{s\to\infty} x_k^s = x_k$ exists for each $k \in N$. Now we have for all $s,t \geq n_0$.

$$\inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum_{j=0}^{k}(x_j^s - x_j^t)\right|}{\rho(k+1)}\right) \leq 1\right\} < \varepsilon$$

Then we have

$$\lim_{t\to\infty}\left\{\inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum_{j=0}^{k}(x_j^s - x_j^t)\right|}{\rho(k+1)}\right) \leq 1\right\}\right\} < \varepsilon, \quad \text{for all } \; s \geq n_0.$$

Using the continuity of Orlicz functions, we have

$$\inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum_{j=0}^{k}(x_j^s - \lim\limits_{t\to\infty} x_j^t)\right|}{\rho(k+1)}\right) \leq 1\right\} < \varepsilon, \quad \text{for all } \; s \geq n_0.$$

This implies that

$$\inf\left\{\rho > 0 : \sum_{k=0}^{\infty} M_k\left(\frac{\left|\sum_{j=0}^{k}(x_j^s - x_j)\right|}{\rho}\right) \leq 1\right\} < \varepsilon, \quad \text{for all } \; s \geq n_0.$$

It follows that $(x^s - x) \in \ell\{M_k, C\}$. Since $(x^s) \in \ell\{M_k, C\}$ and $\ell\{M_k, C\}$ is a linear space, so we have $x = x^s - (x^s - x) \in \ell\{M_k, C\}$. This completes the proof.

It is easy to see that $\|x^i\| \longrightarrow 0$ implies that $x_k^i \longrightarrow 0$ for each $i \geq 1$. Hence we have the following Proposition.

Proposition 9. *The space $\ell\{M_k, C\}$ is a BK-space.*

Now we study the AK-characteristic of the space $\ell\{M_k, C\}$. Before that we give a new definition and prove some results those will be required.

Definition 10. *For any sequence of Orlicz functions $M = (M_k)$, we define*

$$h\{M_k, C\} = \left\{ x = (x_k) \in w : \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty, \quad \text{for every } \rho > 0 \right\}.$$

Clearly $h\{M_k, C\}$ is a subspace of $\ell\{M_k, C\}$. The topology of $h\{M_k, C\}$ is the one it inherits from $\|.\|$.

Proposition 11. *Let $M = (M_k)$ be a sequence of Orlicz functions which satisfy Δ_2-condition then $\ell\{M_k, C\} = h\{M_k, C\}$.*

Proof. It is enough to prove that $\ell\{M_k, C\} \subseteq h\{M_k, C\}$. Let $x \in \ell\{M_k, C\}$, then for some $\rho > 0$,

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty$$

Choose an arbitrary $\eta > 0$. If $\rho \leq \eta$ then

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\eta(k+1)} \right) < \sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty$$

Let now $\eta < \rho$ and put $l = \frac{\rho}{\eta} > 1$. Since each M_k satisfies the Δ_2-condition, there exists constants K_k such that

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\eta(k+1)} \right) \leq \sum_{k=0}^{\infty} K_k \left(\frac{\rho}{\eta} \right)^{log_2 K_k} M_k \left(\frac{\left| \sum\limits_{j=0}^{k} x_j \right|}{\rho(k+1)} \right)$$

Let $S = \sup_k \left(\frac{\rho}{\eta}\right)^{log_2 K_k}$. Then for every $\eta > 0$

$$\sum_{k=0}^{\infty} M_k \left(\frac{\left|\sum_{j=0}^{k} x_j\right|}{\eta(k+1)} \right) \leq S \sum_{k=0}^{\infty} M_k \left(\frac{\left|\sum_{j=0}^{k} x_j\right|}{\rho(k+1)} \right) < \infty$$

This completes the proof.

Proposition 12. *$h\{M_k, C\}$ is an AK-space.*

Proof. Let $x \in h\{M_k, C\}$. Then for each ε, $0 < \varepsilon < 1$, we can find an s_0 such that

$$\sum_{k \geq s_0} M_k \left(\frac{\left|\sum_{j=0}^{k} x_j\right|}{\varepsilon(k+1)} \right) \leq 1$$

Hence for $s \geq s_0$,

$$\|x - x^{(s)}\| = \inf \left\{ \rho > 0 : \sum_{k \geq s+1} M_k \left(\frac{\left|\sum_{j=0}^{k} x_j\right|}{\rho(k+1)} \right) \leq 1 \right\}$$

$$\leq \inf \left\{ \rho > 0 : \sum_{k \geq s} M_k \left(\frac{\left|\sum_{j=0}^{k} x_j\right|}{\rho(k+1)} \right) \leq 1 \right\} < \varepsilon.$$

Thus we can conclude that $h\{M_k, C\}$ is an AK-space.

Combining Proposition 9 and Proposition 11, we have the following Theorem.

Theorem 13. *Let $M = (M_k)$ be a sequence of Orlicz functions which satisfy Δ_2-condition, then $x \in \ell\{M_k, C\}$ is an AK-space.*

Proposition 14. *The space $h\{M_k, C\}$ is a closed subspace of $\ell\{M_k, C\}$.*

Proof. Let $\{x^s\}$ be a sequence in $h\{M_k, C\}$ such that $\|x^s - x\| \longrightarrow 0$, where $x \in \ell\{M_k, C\}$. To complete the proof we need to show that $x \in h\{M_k, C\}$, i.e.,

$$\sum_{k\geq 0} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty, \quad \text{for every} \quad \rho > 0.$$

To $\rho > 0$ there corresponds an l such that $\|x^l - x\| \leq \frac{\rho}{2}$. Then using convexity of each M_k,

$$\sum_{k\geq 0} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) = \sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} x_j^l \right| - 2\left(\left| \sum_{j=0}^{k} x_j^l \right| - \left| \sum_{j=0}^{k} x_j \right| \right)}{2\rho(k+1)} \right)$$

$$\leq \frac{1}{2} \sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} x_j^l \right|}{\rho(k+1)} \right) + \frac{1}{2} \sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} (x_j^l - x_j) \right|}{\rho(k+1)} \right)$$

$$\leq \frac{1}{2} \sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} x_j^l \right|}{\rho(k+1)} \right) + \frac{1}{2} \sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} (x_j^l - x_j) \right|}{\|x^l - x\|(k+1)} \right)$$

Now from Theorem 9, using definition of norm $\|.\|$, we have

$$\sum_{k\geq 0} M_k \left(\frac{2\left| \sum_{j=0}^{k} (x_j^l - x_j) \right|}{\|x^l - x\|(k+1)} \right) \leq 1$$

It follows that

$$\sum_{k\geq 0} M_k \left(\frac{\left| \sum_{j=0}^{k} x_j \right|}{\rho(k+1)} \right) < \infty, \quad \text{for every} \quad \rho > 0.$$

Thus $x \in h\{M_k, C\}$.

Hence we have the following Corollary.

Corollary 15. *The space $h\{M_k, C\}$ is a BK-space.*

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