Revista Notas de Matemática
Vol.6(2), No. 298, 2010, pp. 63-71
http://www.saber.ula.ve/notasdematematica
Pre-prints
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On characterization dual spacelike biharmonic curves with spacelike principal normal according to dual Bishop frames in the dual Lorentzian space $\mathbb{D}_1^3$

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Abstract

In this paper, we study dual spacelike biharmonic curves with spacelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with spacelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

key words. Dual space curve, dual Bishop frame, biharmonic curve.

AMS subject classifications. 58E20.

1 Introduction

Dual numbers were introduced by W. K. Clifford (1849-1879) as a tool for his geometrical investigations [2]. After him E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [5]. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the points of dual unit sphere $\mathbb{S}^2$ and the directed lines in $\mathbb{R}^3$.

E. Study devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere $\mathbb{S}^2$ and the directed lines of space of lines $\mathbb{R}^3$ [5]. The existence of the dual numbers has been noticed in some papers concerning supermathematics e.g. [9]. The most interesting use of dual numbers in field theory can be found in a series of papers by Wald [10].

Harmonic maps $f:(M,g)\longrightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2 v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\operatorname{trace}\nabla df. \tag{1.2}$$

The bienergy of a map f by

$$E_2(f)=\frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [3], showing that the Euler–Lagrange equation associated to E_2 is

$$\tau_2(f)=-\mathcal{J}^f(\tau(f))=-\Delta\tau(f)-\operatorname{trace}R^N(df,\tau(f))\,df=0 \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of f . The equation $\tau_2(f)=0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

In this paper, we study dual spacelike biharmonic curves with spacelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with spacelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

2 Preliminaries

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$. If φ and φ^* are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a dual number. The symbol ε designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which φ and φ^* are, respectively, the projected angle and the shortest distance between the two lines.

By a dual number $\hat{x}$, we mean an ordered pair of the form (x, x^*) for all $x, x^* \in \mathbb{R}$. Let the set $\mathbb{R} \times \mathbb{R}$ be denoted as $\mathbb{D}$. Two inner operations and an equality on $\mathbb{D} = \{(x, x^*) | x, x^* \in \mathbb{R}\}$ are defined as follows:

(i) $\oplus : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*)$ defined as

$$\hat{x} \oplus \hat{y} = (x, x^*) \oplus (y, y^*) = (x + y, x^* + y^*)$$

is called the addition in $\mathbb{D}$.

(ii) $\otimes : \mathbb{D} \times \mathbb{D} \longrightarrow \mathbb{D}$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*)$ defined as

$$\hat{x} \otimes \hat{y} = (x, x^*) \otimes (y, y^*) = (xy, xy^* + x^*y)$$

is called the multiplication in $\mathbb{D}$.

The set $\mathbb{D}$ of dual numbers is a commutative ring.

(iii) If $x = y$, $x^* = y^*$ for $\hat{x} = (x, x^*)$, $\hat{y} = (y, y^*) \in \mathbb{D}$, $\hat{x}$ and $\hat{y}$ are equal, and it is indicated as $\hat{x} = \hat{y}$.

If the operations of addition, multiplication and equality on $\mathbb{D} = \mathbb{R} \times \mathbb{R}$ with set of real numbers $\mathbb{R}$ are defined as above, the set $\mathbb{D}$ is called the dual numbers system and the element (x, x^*) of $\mathbb{D}$ is called a dual number. In a dual number $\hat{x} = (x, x^*) \in \mathbb{D}$, the real number x is called the real part of $\hat{x}$ and the real number x^* is called the dual part of $\hat{x}$. The dual number $(1, 0) = 1$ is called unit element of multiplication operation in $\mathbb{D}$ or real unit in $\mathbb{D}$. The dual number $(0, 1)$ is to be denoted with " in short, and the $(0, 1) = \varepsilon$ is to be called dual unit. In accordance with the definition of the operation of multiplication, it can easily be seen that $\varepsilon^2 = 0$. Also, the dual number $\hat{x} = (x, x^*) \in \mathbb{D}$ can be written as $\hat{x} = x + \varepsilon x^*$.

The set

$$\mathbb{D}^3 = \left\{ \hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon\varphi^*, \ \varphi, \varphi^* \in \mathbb{E}^3 \right\}$$

is a module over the ring $\mathbb{D}$.

For any $\overrightarrow{\hat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*}$, $\overrightarrow{\hat{y}} = \overrightarrow{y} + \varepsilon \overrightarrow{y^*} \in \mathbb{D}^3$, the scalar or inner product $\overrightarrow{\hat{x}}$ and $\overrightarrow{\hat{y}}$ is defined by

$$\left\langle \overrightarrow{\hat{x}}, \overrightarrow{\hat{y}} \right\rangle = \langle \overrightarrow{x}, \overrightarrow{y} \rangle + \varepsilon \left(\left\langle \overrightarrow{x}, \overrightarrow{y^*} \right\rangle + \left\langle \overrightarrow{x^*}, \overrightarrow{y} \right\rangle \right),$$

$$\overrightarrow{\hat{x}} \wedge \overrightarrow{\hat{y}} = (\hat{x}_2\hat{y}_3 - \hat{x}_3\hat{y}_2, \hat{x}_3\hat{y}_1 - \hat{x}_1\hat{y}_3, \hat{x}_1\hat{y}_2 - \hat{x}_2\hat{y}_1),$$

where $\hat{x}_i = x_i + \varepsilon x_i^*$, $\hat{y}_i = y_i + \varepsilon y_i^* \in \mathbb{D}$, $1 \leq i \leq 3$. If $x \neq 0$,the norm$\left\| \overrightarrow{\hat{x}} \right\|$ of $\overrightarrow{\hat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*}$ is defined by

$$\left\| \overrightarrow{\hat{x}} \right\| = \sqrt{\left\langle \overrightarrow{\hat{x}}, \overrightarrow{\hat{x}} \right\rangle} = \| \overrightarrow{x} \| + \varepsilon \frac{\left\langle \overrightarrow{x}, \overrightarrow{x^*} \right\rangle}{\| \overrightarrow{x} \|}.$$

A dual vector $\overrightarrow{\hat{x}}$ with norm 1 is called a dual unit vector.

Let $\overrightarrow{\hat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*} \in \mathbb{D}^3$. The set

$$\mathbb{S}^2 = \left\{ \overrightarrow{\hat{x}} = \overrightarrow{x} + \varepsilon \overrightarrow{x^*} \;\mid \left\| \overrightarrow{\hat{x}} \right\| = (1,0)\,; \overrightarrow{x}, \overrightarrow{x^*} \epsilon \mathbb{R}^3 \right\}$$

is called the dual unit sphere with the center $\hat{O}$ in $\mathbb{D}^3$.

If every $x_i(s)$ and $x_i^*(s)$, $1 \leq i \leq 3$, real valued functions,are differentiable, the dual space curve

$$\begin{aligned} \hat{x} \quad &: \quad I \subset \mathbb{R} \to \mathbb{D}^3 \\ s \quad &\to \quad \hat{x}(s) = \left(x_1(s) + \varepsilon x_1^*(s), x_2(s) + \varepsilon x_2^*(s), x_3(s) + \varepsilon x_3^*(s) \right), \end{aligned}$$

in $\mathbb{D}^3$ is differentiable.

3 Spacelike Dual Biharmonic Curves with Spacelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon \gamma^* : I \subset R \to \mathbb{D}_1^3$ be a C^4 dual spacelike curve with spacelike principal normal by the arc length parameter s. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \frac{1}{\hat{\kappa}} \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\| \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}} \right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Hence, the triple $\left\{\hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}}\right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} &= \hat{\kappa}\hat{\mathbf{n}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{n}} &= -\hat{\kappa}\hat{\mathbf{t}} + \hat{\tau}\hat{\mathbf{b}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{b}} &= \hat{\tau}\hat{\mathbf{n}}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon\tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. In addition,

$$\begin{aligned} &g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) = 1,\ g\left(\hat{\mathbf{n}}, \hat{\mathbf{n}}\right) = 1,\ g\left(\hat{\mathbf{b}}, \hat{\mathbf{b}}\right) = -1, \\ &g\left(\hat{\mathbf{t}}, \hat{\mathbf{n}}\right) = g\left(\hat{\mathbf{t}}, \hat{\mathbf{b}}\right) = g\left(\hat{\mathbf{n}}, \hat{\mathbf{b}}\right) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\hat{\kappa} \neq 0$ and $\hat{\tau} \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} &= \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_1 &= -\hat{k}_1\hat{\mathbf{t}}, \\ \nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_2 &= -\hat{k}_2\hat{\mathbf{t}}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) = 1,\ g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\right) = 1,\ g\left(\hat{\mathbf{m}}_2, \hat{\mathbf{m}}_2\right) = -1, \\ &g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_1\right) = g\left(\hat{\mathbf{t}}, \hat{\mathbf{m}}_2\right) = g\left(\hat{\mathbf{m}}_1, \hat{\mathbf{m}}_2\right) = 0. \end{aligned} \tag{3.4}$$

Here, we shall call the set $\{\hat{\mathbf{t}}, \hat{\mathbf{m}}_1, \hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures. where $\hat{\theta}\left(s\right) = \arctan\frac{\hat{k}_2}{\hat{k}_1}$, $\tau(s) = \hat{\theta}'\left(s\right)$ and $\hat{\kappa}(s) = \sqrt{\left|\hat{k}_1^2 - \hat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by

$$\begin{aligned} \hat{k}_1 &= \hat{\kappa}(s)\cosh\hat{\theta}\left(s\right), \\ \hat{k}_2 &= \hat{\kappa}(s)\sinh\hat{\theta}\left(s\right). \end{aligned} \tag{3.5}$$

Theorem 3.1. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} &\hat{k}_2^2 - \hat{k}_1^2 = \hat{\Omega}, \\ &\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 = 0, \\ &-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 = 0, \end{aligned} \tag{3.6}$$

where $\hat{\Omega}$ is dual constant of integration.

Proof. From (1.4), we get the biharmonic equation of $\hat{\gamma}$

$$\tau_2(\hat{\gamma}) = \nabla^3_{\hat{\mathbf{t}}}\hat{\mathbf{t}} - R\left(\hat{\mathbf{t}},\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}\right)\hat{\mathbf{t}} = 0. \tag{3.7}$$

Next, using the Bishop equations (3.3) we obtain

$$\nabla^3_{\hat{\mathbf{t}}}\hat{\mathbf{t}} = \left(-3\hat{k}_1'\hat{k}_1 + 3\hat{k}_2'\hat{k}_2\right)\hat{\mathbf{t}} + \left(\hat{k}_1'' - \hat{k}_1^3 + \hat{k}_2^2\hat{k}_1\right)\hat{\mathbf{m}}_1 + \left(-\hat{k}_2'' - \hat{k}_2^3 + \hat{k}_1^2\hat{k}_2\right)\hat{\mathbf{m}}_2. \tag{3.8}$$

Thus, (3.7) and (3.8) imply

$$\left(-3\hat{k}_1'\hat{k}_1 + 3\hat{k}_2'\hat{k}_2\right)\hat{\mathbf{t}} + \left(\hat{k}_1'' - \hat{k}_1^3 + \hat{k}_2^2\hat{k}_1\right)\hat{\mathbf{m}}_1 + \left(-\hat{k}_2'' - \hat{k}_2^3 + \hat{k}_1^2\hat{k}_2\right)\hat{\mathbf{m}}_2 - R\left(\hat{\mathbf{t}},\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}}\right)\hat{\mathbf{m}}_2 = 0. \tag{3.9}$$

In $\mathbb{D}^3$, the Riemannian curvature is zero, we have

$$\left(-3\hat{k}_1'\hat{k}_1 + 3\hat{k}_2'\hat{k}_2\right)\hat{\mathbf{t}} + \left(\hat{k}_1'' - \hat{k}_1^3 + \hat{k}_2^2\hat{k}_1\right)\hat{\mathbf{m}}_1 + \left(-\hat{k}_2'' - \hat{k}_2^3 + \hat{k}_1^2\hat{k}_2\right)\hat{\mathbf{m}}_2 = 0. \tag{3.10}$$

From Bishop frame, we have

$$-3\hat{k}_1'\hat{k}_1 + 3\hat{k}_2'\hat{k}_2 = 0. \tag{3.11}$$

Also, from (3.11) we get

$$-\hat{k}_1^2 + \hat{k}_2^2 = \hat{\Omega}, \tag{3.12}$$

where $\hat{\Omega}$ is dual constant of integration. Using (3.10), we get

$$\hat{k}_1'' - \hat{k}_1^3 + \hat{k}_2^2\hat{k}_1 = 0, \tag{3.13}$$

$$-\hat{k}_2'' - \hat{k}_2^3 + \hat{k}_1^2\hat{k}_2 = 0. \tag{3.14}$$

The proof is completed.

Lemma 3.2. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\begin{aligned} -\hat{k}_1^2 + \hat{k}_2^2 &= \hat{\Omega}, \\ \hat{k}_1'' + \hat{k}_1 \hat{\Omega} &= 0, \\ \hat{k}_2'' - \hat{k}_2 \hat{\Omega} &= 0, \end{aligned} \tag{3.15}$$

where $\hat{\Omega} = \Omega + \varepsilon \Omega^*$ is constant of integration.

Lemma 3.3. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$k_1^2 - k_2^2 = -\Omega, \tag{3.16}$$

$$k_1 k_1^* - k_2 k_2^* = -\Omega^*. \tag{3.17}$$

Proof. Using (3.12), we have (3.16) and (3.17).

Corollary 3.4. *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. If $\hat{\Omega} = 0$, then*

$$\begin{aligned} & k_1^2 = k_2^2, \\ & k_1 k_1^* = k_2 k_2^*, \\ & \hat{k}_1 = \hat{C}_1 \hat{s} + \hat{C}_2, \\ & \hat{k}_2 = \hat{C}_3 \hat{s} + \hat{C}_4, \end{aligned} \tag{3.18}$$

where $\hat{C}_1, \hat{C}_2, \hat{C}_3, \hat{C}_4$ are dual constants of integration.

Proof. Suppose that $\hat{\Omega} = 0$. Substituting $\Omega = \Omega^* = 0$ in Lemma 3.3, we have

$$\begin{aligned} k_1^2 &= k_2^2, \\ k_1 k_1^* &= k_2 k_2^*. \end{aligned}$$

On the other hand, using second and third equation of (3.2), we get

$$\hat{k}_1'' = 0 \text{ and } \hat{k}_2'' = 0.$$

If we integrate above equation, we have

$$\begin{aligned}\hat{k}_1 &= \hat{C}_1\hat{s}+\hat{C}_2,\\ \hat{k}_2 &= \hat{C}_3\hat{s}+\hat{C}_4,\end{aligned}$$

where $\hat{C}_1,\hat{C}_2,\hat{C}_3,\hat{C}_4$ are dual constants of integration. The proof is completed.

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