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# Bertrand mate of null biharmonic curves in the Lorentzian Heisenberg group Heis$^3$

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**Abstract**

In this paper, we study null biharmonic curves and we characterize null biharmonic curves in terms of their curvature and torsion in the Lorentzian Heisenberg group Heis$^3$. Moreover, we construct parametric equations of Bertrand mate of null biharmonic curves and null biharmonic in the Lorentzian Heisenberg group Heis$^3$.

**key words.** Heisenberg group, biharmonic curve, null curve, helices.

**AMS subject classifications.** 58E20.

## 1 Introduction

In the theory of space curves in differential geometry, the associated curves, the curves for which at the corresponding points of them one of the Frenet vectors of a curve coincides with the one of the Frenet vectors of the other curve have an important role for the characterizations of space curves. The well-known examples of such curves are Bertrand curves. These special curves are very interesting and characterized as a kind of corresponding relation between two curves such that the curves have the common principal normal i.e., the Bertrand curve is a curve which shares the normal line with another curve. These curves have an important role in the theory of curves. Hereby, from the past to today, a lot of mathematicians have studied on Bertrand curves in different areas [22].

On the other hand, harmonic maps $f:(M,g)\longrightarrow(N,h)$ between Riemannian manifolds are the critical points of the energy

$$E(f)=\frac{1}{2}\int_M |df|^2\, v_g, \tag{1.1}$$

and they are therefore the solutions of the corresponding Euler–Lagrange equation. This equation is given by the vanishing of the tension field

$$\tau(f)=\mathrm{trace}\nabla df. \tag{1.2}$$

As suggested by Eells and Sampson in [4], we can define the bienergy of a map $f$ by

$$E_2(f) = \frac{1}{2}\int_M |\tau(f)|^2 v_g, \tag{1.3}$$

and say that is biharmonic if it is a critical point of the bienergy.

Jiang derived the first and the second variation formula for the bienergy in [7], showing that the Euler–Lagrange equation associated to $E_2$ is

$$\tau_2(f) = -\mathcal{J}^f(\tau(f)) = -\Delta\tau(f) - \mathrm{trace}R^N(df, \tau(f))\,df = 0, \tag{1.4}$$

where $\mathcal{J}^f$ is the Jacobi operator of $f$ . The equation $\tau_2(f) = 0$ is called the biharmonic equation. Since $\mathcal{J}^f$ is linear, any harmonic map is biharmonic. Therefore, we are interested in proper biharmonic maps, that is non-harmonic biharmonic maps.

Biharmonic maps have been extensively studied in the last decade.

In [1] the authors completely classified the biharmonic submanifolds of the three-dimensional sphere, while in [2] there were given new methods to construct biharmonic submanifolds of codimension greater than one in the n-dimensional sphere. The biharmonic submanifolds into a space of nonconstant sectional curvature were also investigated. The proper biharmonic curves on Riemannian surfaces were studied in [3]. Inoguchi classified the biharmonic Legendre curves and the Hopf cylinders in three-dimensional Sasakian space forms [6]. Then, Sasahara gave in [17] the explicit representation of the proper biharmonic Legendre surfaces in five-dimensional Sasakian space forms.

In this paper, we study null biharmonic curves and we characterize null biharmonic curves in terms of their curvature and torsion in the Lorentzian Heisenberg group Heis$^3$. Moreover, we construct parametric equations of Bertrand mate of null biharmonic curves and null biharmonic in the Lorentzian Heisenberg group Heis$^3$.

## 2 The Lorentzian Heisenberg Group Heis$^3$

The Lorentzian Heisenberg group Heis$^3$ can be seen as the space $\mathbb{R}^3$ endowed with the following multiplication:

$$(\overline{x}, \overline{y}, \overline{z})(x, y, z) = (\overline{x} + x, \overline{y} + y, \overline{z} + z - \overline{x}y + x\overline{y}).$$

Heis$^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Lorentz metric $g$ is given by

$$g = -dx^2 + dy^2 + (xdy + dz)^2,$$

where

$$\omega^1 = dz + xdy, \qquad \omega^2 = dy, \qquad \omega^3 = dx$$

is the left-invariant orthonormal coframe associated with the orthonormal left-invariant frame,

$$\mathbf{e}_1 = \frac{\partial}{\partial z}, \quad \mathbf{e}_2 = \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \quad \mathbf{e}_3 = \frac{\partial}{\partial x} \tag{2.1}$$

for which we have the Lie products

$$[\mathbf{e}_2, \mathbf{e}_3] = 2e_1, \quad [\mathbf{e}_3, \mathbf{e}_1] = 0, \quad [\mathbf{e}_2, \mathbf{e}_1] = 0,$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = 1, \quad g(\mathbf{e}_3, \mathbf{e}_3) = -1. \tag{2.2}$$

**Proposition 2.1.** *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g$, defined above the following is true:*

$$\nabla = \begin{pmatrix} 0 & \mathbf{e}_3 & \mathbf{e}_2 \\ \mathbf{e}_3 & 0 & \mathbf{e}_1 \\ \mathbf{e}_2 & -\mathbf{e}_1 & 0 \end{pmatrix}, \tag{2.3}$$

*where the $(i, j)$-element in the table above equals $\nabla_{e_i} e_j$ for our basis*

$$\{\mathbf{e}_k, k = 1, 2, 3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X, Y)Z = -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z.$$

The Riemannian curvature tensor is given by

$$R(X, Y, Z, W) = g(R(X, Y)Z, W).$$

Moreover we put

$$R_{abc} = R(\mathbf{e}_a, \mathbf{e}_b)\mathbf{e}_c, \quad R_{abcd} = R(\mathbf{e}_a, \mathbf{e}_b, \mathbf{e}_c, \mathbf{e}_d),$$

ewhere the indices $a, b, c$ and $d$ take the values $1, 2$ and $3$.

$$R_{121} = -\mathbf{e}_2, \quad R_{131} = -\mathbf{e}_3, \quad R_{232} = 3\mathbf{e}_3,$$

and

$$R_{1212} = -1, \quad R_{1313} = 1, \quad R_{2323} = -3. \tag{2.4}$$

# 3 Null Biharmonic Curves In The Lorentzian Heisenberg Group Heis$^3$

Let $\gamma : I \longrightarrow Heis^3$ be a null curve on the Lorentzian Heisenberg group Heis$^3$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group Heis$^3$ along $\gamma$ defined as follows:

$\mathbf{T}$ is the unit vector field $\gamma'$ tangent to $\gamma$, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to $\gamma$), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa_1\mathbf{N}, \\ \nabla_T N &= \kappa_2\mathbf{T} - \kappa_1\mathbf{B}, \\ \nabla_T B &= -\kappa_2\mathbf{N}, \end{aligned} \tag{3.1}$$

where

$$\begin{gathered} g(\mathbf{T},\mathbf{T}) = g(\mathbf{B},\mathbf{B}) = 0, g(\mathbf{N},\mathbf{N}) = 1, \\ g(\mathbf{T},\mathbf{N}) = g(\mathbf{N},\mathbf{B}) = 0, g(\mathbf{T},\mathbf{B}) = 1, \end{gathered} \tag{3.2}$$

and $\kappa_1$ is the curvature of $\gamma$ and $\kappa_2$ is its torsion.

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\begin{aligned} &\mathbf{T} = T_1\mathbf{e}_1 + T_2\mathbf{e}_2 + T_3\mathbf{e}_3, \\ &\mathbf{N} = N_1\mathbf{e}_1 + N_2\mathbf{e}_2 + N_3\mathbf{e}_3, \\ &\mathbf{B} = \mathbf{T} \times \mathbf{N} = B_1\mathbf{e}_1 + B_2\mathbf{e}_2 + B_3\mathbf{e}_3 \end{aligned} \tag{3.3}$$

**Theorem 3.1.** *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic null curve parametrized by arc length. $\gamma$ is a non-geodesic null biharmonic curve if and only if*

$$\begin{aligned} \kappa_1\kappa_1' &= 0, \\ \kappa_1'' + 2\kappa_1^2\kappa_2 &= \kappa_1 R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}), \\ 2\kappa_1'\kappa_2 + \kappa_2'\kappa_1 &= \kappa_1 R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}). \end{aligned} \tag{3.4}$$

**Proof.** Using Eq. (1.4) and Eq. (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \nabla_{\mathbf{T}}^3\mathbf{T} - \kappa_1 R(\mathbf{T},\mathbf{N})\mathbf{T} \\ &= (2\kappa_2'\kappa_1 + \kappa_1'\kappa_2)\mathbf{T} + (\kappa_1'' + 2\kappa_1^2\kappa_2)\mathbf{N} + (3\kappa_1\kappa_1')\mathbf{B} - \kappa_1 R(\mathbf{T},\mathbf{N})\mathbf{T} \\ &= 0. \end{aligned}$$

By Eq. (3.2), we see that $\gamma$ is a biharmonic curve if and only if

$$\begin{aligned} \kappa_1\kappa_1' &= 0, \\ \kappa_1'' + 2\kappa_1^2\kappa_2 &= \kappa_1 R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}), \\ 2\kappa_2\kappa_1' + \kappa_1\kappa_2' &= \kappa_1 R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}). \end{aligned} \tag{3.5}$$

The proof is completed.

**Theorem 3.2.** *Let $\gamma : I \longrightarrow Heis^3$ be a non-geodesic null curve parametrized by arc length. $\gamma$ is a non-geodesic null biharmonic curve if and only if*

$$\begin{aligned} \kappa_1 &= \textit{constant} \neq 0, \\ \kappa_1\kappa_2 &= -2B_1^2, \\ \kappa_2' &= 4N_1B_1. \end{aligned} \tag{3.6}$$

**Proof.** Using Eq. (3.4), we have

$$\begin{aligned} \kappa_1 &= \textit{constant} \neq 0, \\ 2\kappa_1\kappa_2 &= -R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}), \\ \kappa_2' &= -R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}). \end{aligned} \tag{3.7}$$

A direct computation using Eq. (2.4), yields

$$\begin{gathered} R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{N}) = -4B_1^2, \\ R(\mathbf{T},\mathbf{N},\mathbf{T},\mathbf{B}) = 4N_1B_1. \end{gathered} \tag{3.8}$$

These, together with Eq. (3.7), complete the proof of the theorem.

**Theorem 3.3.** *(see [21]) Let $\gamma : I \longrightarrow Heis^3$ be a null biharmonic curve. Then $\gamma$ is a helix.*

**Corollary 3.4.** *$\gamma : I \longrightarrow Heis^3$ is null biharmonic if and only if*

$$\begin{aligned} \kappa_1 &= \textit{constant} \neq 0, \\ \kappa_2 &= \textit{constant}, \\ N_1B_1 &= 0, \\ \kappa_1\kappa_2 &= -2B_1^2. \end{aligned} \tag{3.9}$$

**Corollary 3.5.** *If $\gamma$ is null biharmonic, then $N_1 = 0$.*

**Corollary 3.6.** *(see [21]) If $N_1 = 0$, then*

$$\mathbf{T}(s) = \sinh\Psi_0\mathbf{e}_1 + \sinh\Psi_0\sinh\Phi(s)\mathbf{e}_2 + \sinh\Psi_0\cosh\Phi(s)\mathbf{e}_3, \tag{3.10}$$

*where $\Psi_0 \in \mathbb{R}$.*

# 4 Bertrand Mate Of Null Biharmonic Curves In The Lorentzian Heisenberg Group Heis$^3$

**Definition 4.1.** *A curve* $\gamma : I \longrightarrow M$ *with* $\kappa_1 \neq 0$ *is called a Bertrand curve if there exist a curve* $\widetilde{\gamma} : I \longrightarrow M$ *such that the principal normal lines of* $\gamma$ *and* $\widetilde{\gamma}$ *at* $s \in I$ *are equal. In this case* $\widetilde{\gamma}$ *is called a Bertrand mate of* $\gamma$ *on Lorentzian manifold M [9].*

**Theorem 4.2.** *Let* $\gamma : I \longrightarrow M$ *be a Bertrand curve parametrized by arc length . A Bertrand mate of* $\gamma$ *is as follows:*

$$\widetilde{\gamma}(s) = \gamma(s) + \lambda N(s), \quad \forall s \in I, \tag{4.1}$$

*where* $\lambda$ *is constant [9].*

**Theorem 4.3.** *Let* $\gamma : I \longrightarrow Heis^3$ *be a null biharmonic curve parametrized by arc length. If* $\widetilde{\gamma}$ *is a Bertrand mate of* $\gamma$*, then the parametric equations of* $\widetilde{\gamma}$ *are*

$$\begin{aligned}
\widetilde{x}(s) &= \Omega_1 \sinh\Psi_0 \sinh((a - 2\sinh\Psi_0)\, s + \sigma) + c_1, \\
\widetilde{y}(s) &= \Omega_1 \sinh\Psi_0 \cosh((a - 2\sinh\Psi_0)\, s + \sigma) + c_2, \\
\widetilde{z}(s) &= \left[\sinh\Psi_0 - \frac{[\sinh\Psi_0]^2}{\delta} + \frac{\lambda \overline{c}_1}{\kappa_1}\right] s \\
&\quad -\Omega_2\,(a + 2\sinh\Psi_0)\sinh^2\Psi_0 \sinh 2((a - 2\sinh\Psi_0)\, s + \sigma) \\
&\quad + \left[-\frac{c_1}{\delta} + \overline{c}_2\,(a + 2\sinh\Psi_0)\right] \sinh\Psi_0 \cosh((a - 2\sinh\Psi_0)\, s + \sigma) + c_3,
\end{aligned} \tag{4.2}$$

*where* $\overline{c}_1$, $\overline{c}_2$, $c_1$, $c_2$, $c_3$ *are constants of integration and*

$$\begin{aligned}
\Omega_1 &= \left[\frac{1}{a - 2\sinh\Psi_0} + \frac{\lambda}{\kappa_1}(a + 2\sinh\Psi_0)\right], \\
\Omega_1 &= \left[\frac{1}{2\,(a - 2\sinh\Psi_0)^2} - \frac{\lambda}{2\kappa_1^2\,(a - 2\sinh\Psi_0)^2}\right].
\end{aligned}$$

**Proof.** The covariant derivative of the vector field $\mathbf{T}$ is:

$$\nabla_{\mathbf{T}}\mathbf{T} = T_1'\mathbf{e}_1 + (T_2' + 2T_1T_3)\mathbf{e}_2 + (T_3' + 2T_1T_2)\mathbf{e}_3. \tag{4.3}$$

From Eq. (3.10) and Eq. (4.3), we have

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= (\Phi'(s)\sinh\Psi_0\cosh\Phi(s) + 2\sinh^2\Psi_0\cosh\Phi(s))\mathbf{e}_2 \\
&\quad +(\Phi'(s)\sinh\Psi_0\sinh\Phi(s) + 2\sinh^2\Psi_0\sinh\Phi(s))\mathbf{e}_3.
\end{aligned}$$

Since $|\nabla_{\mathbf{T}}\mathbf{T}| = \kappa_1$, we obtain

$$\Phi(s) = ((a - 2\sinh\Psi_0)\,s + \sigma)s + \sigma, \tag{4.4}$$

where $a = \pm\sqrt{4\sinh\Psi_0 + \frac{\kappa_1}{\sinh^3\Psi_0}}$ and $\sigma \in \mathbb{R}$.

Using Eq. (2.1) in Eq. (3.10), we easily have:

$$\begin{aligned}
\frac{dx}{ds} &= \sinh\Psi_0\cosh((a - 2\sinh\Psi_0)\,s + \sigma),\\
\frac{dy}{ds} &= \sinh\Psi_0\sinh((a - 2\sinh\Psi_0)\,s + \sigma),\\
\frac{dz}{ds} &= \sinh\Psi_0\cosh((a - 2\sinh\Psi_0)\,s + \sigma)\\
&\quad -x(s)\sinh\Psi_0\sinh((a - 2\sinh\Psi_0)\,s + \sigma).
\end{aligned} \tag{4.5}$$

On the other hand, using Eq. (4.3) and Eq. (4.4), we have

$$\begin{aligned}
\nabla_{\mathbf{T}}\mathbf{T} &= (a + 2\sinh\Psi_0)\sinh\Psi_0\cosh((a - 2\sinh\Psi_0)\,s + \sigma)\mathbf{e}_2\\
&\quad + (a + 2\sinh\Psi_0)\sinh\Psi_0\sinh((a - 2\sinh\Psi_0)\,s + \sigma)\mathbf{e}_3.
\end{aligned} \tag{4.6}$$

By the use of Frenet formulas (3.1), we get

$$\begin{aligned}
\mathbf{N} &= \frac{1}{\kappa_1}\nabla_{\mathbf{T}}\mathbf{T}\\
&= \frac{1}{\kappa_1}(a + 2\sinh\Psi_0)\sinh\Psi_0[\cosh((a - 2\sinh\Psi_0)\,s + \sigma)\mathbf{e}_2\\
&\quad + \sinh((a - 2\sinh\Psi_0)\,s + \sigma)\mathbf{e}_3]
\end{aligned} \tag{4.7}$$

From Eq. (2.1), we have

$$\frac{\partial}{\partial x} = \mathbf{e}_3,\quad \frac{\partial}{\partial y} = \mathbf{e}_2 + x\mathbf{e}_3,\quad \frac{\partial}{\partial z} = \mathbf{e}_1. \tag{4.8}$$

Substituting Eq. (4.8) in Eq. (4.7), we have

$$\begin{aligned}
\mathbf{N} = \frac{1}{\kappa_1}&(a + 2\sinh\Psi_0)\sinh\Psi_0(\sinh((a - 2\sinh\Psi_0)\,s + \sigma),\\
&\cosh((a - 2\sinh\Psi_0)\,s + \sigma), -x(s)\cosh((a - 2\sinh\Psi_0)\,s + \sigma)).
\end{aligned} \tag{4.9}$$

Finally, we substitute Eq. (4.9) and Eq. (4.5) into Eq. (4.1), we get Eq. (4.2). The proof is completed.

**Corollary 4.4.** *The parametric equations of* $\gamma$ *null biharmonic curve are*

$$\begin{aligned}
x(s) &= \frac{1}{(a-2\sinh\Psi_0)}\sinh\Psi_0\sinh((a-2\sinh\Psi_0)\,s+\sigma)+c_1,\\
y(s) &= \frac{1}{(a-2\sinh\Psi_0)}\sinh\Psi_0\cosh((a-2\sinh\Psi_0)\,s+\sigma)+c_2,\\
z(s) &= \left[\sinh\Psi_0-\frac{[\sinh\Psi_0]^2}{a-2\sinh\Psi_0}\right]s\\
&\quad -\frac{1}{2\,(a-2\sinh\Psi_0)^2}\,[\sinh\Psi_0]^2\sinh 2((a-2\sinh\Psi_0)\,s+\sigma)\\
&\quad -\frac{c_1}{a-2\sinh\Psi_0}\sinh\Psi_0\cosh((a-2\sinh\Psi_0)\,s+\sigma)+c_3,
\end{aligned}\tag{4.10}$$

*where* $a=\pm\sqrt{4\sinh\Psi_0+\frac{\kappa_1}{\sinh^3\Psi_0}}$ *and* $\Psi_0, c_1, c_2, c_3, \sigma\in\mathbb{R}$.

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