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On acceleration pole points in special Frenet and Bishop motions

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Abstract

A special motion by the form $Y = AX + C$ with one parameter has been given Hacisalihoglu [2, 5] in Euclidean n-space. In this paper, we find a geometrical meaning for the determinant of the derivative matrices $\dot{A}, \ddot{A}$ and $\dddot{A}$ according to $\frac{\tau}{\kappa}$ or in Euclidean 3-space The ratio of torsion and curvature is taken as a constant in our study. Then we search, in this case, the geometry of the 1^{st} and 2^{nd} order acceleration pole points and acceleration axodes in generalized helix curves that yields a necessary condition for the Frenet and Bishop Motions, to accelerate pole points, and compare these points in two motions.

key words. Frenet motion- Bishop motion- Acceleration pole points- General helix

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1 Introduction

In Euclidean 3-space, Bottema and Bishop defined the Frenet and Bishop motions (see [3, 6]). Hacisalihoglu [5] also gives a necessary and enough condition for stationary direction of the Instantaneous Screw Axis (I.S.A) depending on $rank\dot{A}$ and $rank\ddot{A}$.

In this paper, we first find a geometrical meaning for $rank\dot{A}$ and $rank\ddot{A}$ to be 2 or 3, then use this theorem for discussion of existence of 1^{st} and 2^{nd} acceleration pole points. The 1^{st} order velocity of a fixed point X is $\dot{Y} = \dot{A}X + \dot{C}$ and for the 2^{nd} and 3^{rd} order velocity of this point, give us $\ddot{Y} = \ddot{A}X + \ddot{C}$ and $\dddot{Y} = \dddot{A}X + \dddot{C}$ respectively. $\dot{Y}$ is the sliding velocity and $\ddot{Y}$ and $\dddot{Y}$ are the 1^{st} and 2^{nd} sliding acceleration of the point X respectively. We will show that existence of the 1^{st} and 2^{nd} acceleration poles by the solution of the $\ddot{A}X + \ddot{C} = 0$, $\dddot{A}X + \dddot{C} = 0$ systems. The solution of these systems depend on $rank\ddot{A}$ and $rank\dddot{A}$.

2 Preliminaries

In one parameter motion of a body in Euclidean 3-space is generated by the transformation

$$(1)\qquad \begin{bmatrix} Y \\ 1 \end{bmatrix} = \begin{bmatrix} A & C \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ 1 \end{bmatrix} \quad \text{or} \quad Y = AX + C$$

Where $A \in SO(3)$ and X, Y, C are 3×1 real matrices and

$$SO(3) = \left\{ A \in R_3^3 \middle| A^t = A^{-1}, \ \det A = 1 \right\}.$$

A, C are C^∞ functions of a real parameter t, X and Y corresponding to the position vectors of the same point X, with respect to the orthonormal coordinate systems of the moving space S and the fixed space S_0, respectively. At the initial time $t = t_0$ we consider the coordinate system of S_0 and S are coincident. Denote by $\{T, N, B\}$ the moving Frenet frame and $\{T, N_1, N_2\}$ the moving Bishop frame along the regular curve $\alpha = \alpha(t)$ that are parameterized by arc-length parameter t, i.e,

$$\langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle = 1.$$

The Frenet trihedron consists of the tangent T, the principle normal N and the binormal B, and the Bishop trihedron consists of the tangent T, the 1^{st} principle normal N_1 and 2^{nd} principle normal N_2, which are three mutually orthogonal axes. Obviously, the geometry of this motions is completely defined by α. The Frenet formulas read

$$\dot{T} = \kappa N, \ \dot{N} = -\kappa T + \tau B, \ \dot{B} = -\tau N$$

and the Bishop formulas read

$$\dot{T} = \kappa_1 N_1 + \kappa_2 N_2, \dot{N}_1 = -\kappa_1 T, \dot{N}_2 = -\kappa_2 T.$$

In the Frenet formulas, $\kappa > 0$ being the curvature and τ the torsion of the curve α, so κ_1, κ_2 are the 1^{st} and 2^{nd} curvatures in the Bishop motion, respectively.

The Bishop frame is an alternative approach to defining a moving frame that is well defined even when the curve is vanishing the second derivative. We can parallel transport an orthonormal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $T(t)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(N_1(t), N_2(t))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $T(t)$ at each point. If the derivatives of $(N_1(t), N_2(t))$ depend only on $T(t)$ and not each other we can make $N_1(t)$ and $N_2(t)$ vary smoothly throughout the path regardless of the curvature.

Therefore, we have the alternative frame equations:

$$\begin{bmatrix} \dot{T} \\ \dot{N}_1 \\ \dot{N}_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

where

$$\kappa(t) = \sqrt{\kappa_1^2 + \kappa_2^2} \quad , \quad \theta(t) = \arctan\left(\frac{\kappa_2}{\kappa_1}\right) \quad , \quad \tau(t) = -\frac{d\theta(t)}{dt}, \tag{2}$$

[1,3,4] so that κ_1 and κ_2 effectively correspond to a Cartesian coordinate system for the polar coordinates κ, θ with $\theta = -\int \tau(t)dt$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0, which disappears from τ (and hence from the Frenet frame) due to the differentiation.

3 Acceleration Pole Points In Frenet Motion

Definition 3.1 *The first derivation of (1), with respect to t, we have*

$$\dot{Y} = \dot{A}X + \dot{C} + A\dot{X}$$

Where $\dot{Y}$ is the absolute velocity, $\dot{A}X + \dot{C}$ is the sliding velocity and $A\dot{X}$ is the relative velocity of the point X. The solution vector X of the system $\dot{A}X + \dot{C} = 0$ is the position vector of the point which may be considered as a fixed point of S_0 and S at the same time t. These points are called instantaneous pole points at the time t. The sliding velocity of a fixed point X in moving space S is

$$\dot{Y} = \dot{A}X + \dot{C} \tag{3}$$

and for the 2^{nd} order velocity (or the 1^{st} order sliding acceleration) of this point, (3) gives us

$$\ddot{Y} = \ddot{A}X + \ddot{C} \tag{4}$$

and for the 3^{rd} order velocity (or the 2^{nd} order sliding acceleration) of this point, (4) gives us

$$\dddot{Y} = \dddot{A}X + \dddot{C} \tag{5}$$

By using the Frenet formulas and

$$A = \begin{bmatrix} T & N & B \end{bmatrix}, \ \dot{A} = \begin{bmatrix} \dot{T} & \dot{N} & \dot{B} \end{bmatrix}, \ \ddot{A} = \begin{bmatrix} \ddot{T} & \ddot{N} & \ddot{B} \end{bmatrix}, \ \dddot{A} = \begin{bmatrix} \dddot{T} & \dddot{N} & \dddot{B} \end{bmatrix}$$

we can give,

$$\det \dot{A} = \begin{vmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{vmatrix} = 0.$$

Then the system $\dot{A}X + \dot{C} = 0$, has not unique solution. So, the Frenet motion hasnnot pole point. From $\det \dot{A} = 0 and \det \dot{A} = \begin{vmatrix} 0 & \kappa \\ -\kappa & 0 \end{vmatrix} = \kappa^2 \neq 0$, we have $rank A = 2$.

3.1 1^{st} acceleration pole points in Frenet motion

The discussion of existence of the 1^{st} acceleration poles and the 1^{st} acceleration axodes is the discussion of the solution of the system

$$\ddot{A}X + \ddot{C} = 0 \tag{6}$$

The solution of the system of (6) depend on $rank\ddot{A}$.
If $\{T, N, B\}$ is an adapted Frenet frame, then we have

$$\begin{aligned}
\ddot{T} &= -\kappa^2 T + \dot{\kappa} N + \kappa\tau B \\
\ddot{N} &= -\dot{\kappa} T - (\kappa^2 + \tau^2) N + \dot{\tau} B \\
\ddot{B} &= \kappa\tau T - \dot{\tau} N - \tau^2 B
\end{aligned}$$

So, we obtain

$$\begin{bmatrix} \ddot{T} \\ \ddot{N} \\ \ddot{B} \end{bmatrix} = \begin{bmatrix} -\kappa^2 & \dot{\kappa} & \kappa\tau \\ -\dot{\kappa} & -(\kappa^2 + \tau^2) & \dot{\tau} \\ \kappa\tau & -\dot{\tau} & -\tau^2 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

By using $A = [T, N, B] \in SO(3)$, we get

$$\det \ddot{A} = \begin{vmatrix} -\kappa^2 & \dot{\kappa} & \kappa\tau \\ -\dot{\kappa} & -(\kappa^2 + \tau^2) & \dot{\tau} \\ \kappa\tau & -\dot{\tau} & -\tau^2 \end{vmatrix} = -\left[\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot}\right]^2 \tag{7}$$

Obviously as a consequence of equation (7) we have the following:

$$\det \ddot{A} = 0 \Leftrightarrow \frac{\tau}{\kappa} = \text{ constant}$$

From this case we obtain that at any moment t, if the curve $\alpha(t)$ is a generalized helix then the solution systems of (6) are not unique in fixed space S_0. The Frenet motion $Y = AX + C$ has not the 1^{st} acceleration pole point. If $\det \ddot{A} \neq 0$ then $\alpha(t)$ is not general helix.
Thus we can give the following theorem:

Theorem 3.2 *The curve $\alpha(t)$ is not general helix $\Leftrightarrow$ in the moving space S, the Frenet motion has a 1^{st} acceleration pole point; $X = -(\ddot{A})^{-1}\ddot{C}$.*

On the other hand, by means of $\det \ddot{A} = 0$ we have that $rank\ddot{A} < 3$ and since

$$\begin{vmatrix} -\kappa^2 & \dot{\kappa} \\ -\dot{\kappa} & -(\kappa^2 + \tau^2) \end{vmatrix} = \kappa^4 + \kappa^2\tau^2 + \dot{\kappa}^2 \neq 0$$

Such being the case, $rank\ddot{A} = 2$ then, the solution of (6) is a line, at every instant t.
Therefore as a consequence of theorem 3.4 [5], for $n = 3$, we can write, the direction of the Instantaneous Screw Axis (ISA) is stationary.

Theorem 3.3 [5] *If $A \in SO(n)$ and $rank\dot{A} = n-1$, then the direction of the I.S.A is stationary $\Leftrightarrow$ $rank\ \ddot{A} = n-1$.*

Definition 3.4 *As the instantaneous screw always intersect the principal normal at a right angle, its locus - the moving axode - is a special type of ruled surface called a Coned. The axode in the fixed space follows by the means of $Y = AX + C$.*

To sum up, we have proved the following theorem:

Theorem 3.5 *Structure of a Coned has a surface cylindrical $\Leftrightarrow$ the Frenet motion $Y = AX+C$, has not the 1^{st} acceleration pole points.*

3.2 2^{nd} acceleration pole points in Frenet motion

The discussion of existence of the 2^{nd} acceleration pole points and the 2^{nd} acceleration axodes is the discussion of the solution of the system

$$\dddot{A}\, X + \dddot{C} = 0 \tag{8}$$

If T, N and B is an adapted Frenet frame, then we have;

$$\begin{aligned}
\dddot{T} &= (-3\kappa\dot{\kappa})T - (\kappa^3 + \kappa\tau^2 - \ddot{\kappa})N + (\kappa\dot{\tau} + 2\dot{\kappa}\tau)B \\
\dddot{N} &= (\kappa^3 + \kappa\tau^2 - \ddot{\kappa})T - 3(\kappa\dot{\kappa} + \tau\dot{\tau})N - (\tau^3 + \kappa^2\tau - \ddot{\tau})B \\
\dddot{B} &= (2\kappa\dot{\tau} + \dot{\kappa}\tau)T + (\tau^3 + \kappa^2\tau - \ddot{\tau})N - (3\tau\dot{\tau})B
\end{aligned}$$

So, we obtain

$$\begin{bmatrix} \dddot{T} \\ \dddot{N} \\ \dddot{B} \end{bmatrix} = \begin{bmatrix} (-3\kappa\dot{\kappa}) & -(\kappa^3 + \kappa\tau^2 - \ddot{\kappa}) & (\kappa\dot{\tau} + 2\dot{\kappa}\tau) \\ (\kappa^3 + \kappa\tau^2 - \ddot{\kappa}) & -3(\kappa\dot{\kappa} + \tau\dot{\tau}) & -(\tau^3 + \kappa^2\tau - \ddot{\tau}) \\ (2\kappa\dot{\tau} + \dot{\kappa}\tau) & (\tau^3 + \kappa^2\tau - \ddot{\tau}) & -(3\tau\dot{\tau}) \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

By using $\det A = 1$, we get

$$\begin{aligned}
\det \dddot{A} &= 3\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot} \left[2(\kappa\dot{\kappa} + \tau\dot{\tau})(\kappa\dot{\tau} - \dot{\kappa}\tau) - (\kappa^2 + \tau^2)(\kappa\ddot{\tau} - \ddot{\kappa}\tau)\right] \\
&+ 3\left[\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot}\right]^{\cdot} (\dot{\kappa}\ddot{\tau} - \ddot{\kappa}\dot{\tau})
\end{aligned} \tag{9}$$

As a consequence of equation of (8) we have the following:
($\frac{\tau}{\kappa}$ = constant) $\Rightarrow \left(\frac{\tau}{\kappa}\right)^{\cdot} = 0, \left(\frac{\dot{\tau}}{\dot{\kappa}}\right)^{\cdot} = 0 \Rightarrow \det \dddot{A} = 0$ From this case we obtain, at any time t, the curve $\alpha(t)$ is a generalized helix, and the solution of system (8) are not unique and in fixed space S_0, the Frenet motion $Y = AX + C$ has not the 2^{nd} acceleration pole point.
Now we can give the following theorem:

Theorem 3.6 *$\forall t$ the curve $\alpha(t)$ is a generalized helix $\Rightarrow$ in fixed space S_0 the Frenet motion has not a 2^{nd} acceleration pole point.*

Let now see example of non-planar curves.

Example: The helix $\alpha(t) = (t, \cosh t, \sinh t)$, this curve is a Euclidean helix.

$$\dot{\alpha} = (1, \sinh t, \cosh t)$$
$$\ddot{\alpha} = (0, \cosh t, \sinh t)$$
$$\dddot{\alpha}\,(t) = (0, \sinh t, \cosh t)$$
$$\langle \dot{\alpha}(t), \dot{\alpha}(t) \rangle = 1 + \cosh 2t = 2cos^2 ht \neq 1$$
$$\|\dot{\alpha}\| = \sqrt{1 + \cosh 2t} = \sqrt{2} \cosh t$$
$$\dot{\alpha} \wedge \ddot{\alpha} = \begin{vmatrix} i & j & k \\ 1 & \sinh t & \cosh t \\ 0 & \cosh t & \sinh t \end{vmatrix} = (-1, -\sinh t, \cosh t)$$
$$\|\dot{\alpha} \wedge \ddot{\alpha}\| = \sqrt{2} \cosh t$$
$$\det(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}\,) = \begin{vmatrix} 1 & \sinh t & \cosh t \\ 0 & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{vmatrix} = 1$$
$$T = \frac{\dot{\alpha}}{\|\dot{\alpha}\|} = \left(\frac{1}{\sqrt{2}\cosh t}, \frac{\sinh t}{\sqrt{2}\cosh t}, \frac{1}{\sqrt{2}} \right)$$
$$B = \frac{\dot{\alpha} \wedge \ddot{\alpha}}{\|\dot{\alpha} \wedge \ddot{\alpha}\|} = \left(\frac{-1}{\sqrt{2}\cosh t}, \frac{-\sinh t}{\sqrt{2}\cosh t}, \frac{1}{\sqrt{2}} \right)$$
$$N = B \wedge T = \left(\frac{-\sinh t}{\cosh t}, \frac{1}{\cosh t}, 0 \right)$$
$$\kappa = \frac{\|\dot{\alpha} \wedge \ddot{\alpha}\|}{\|\dot{\alpha}\|^3} = \frac{1}{2\cos^2 ht}$$
$$\tau = \frac{\det(\dot{\alpha}, \ddot{\alpha}, \dddot{\alpha}\,)}{\|\dot{\alpha} \wedge \ddot{\alpha}\|^2} = \frac{1}{2\cos^2 ht}$$
$$\frac{\tau}{\kappa} = 1 = \text{constrant} \Rightarrow \left(\frac{\tau}{\kappa}\right)^{\cdot} = 0$$
$$A = \begin{bmatrix} T & N & B \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}\cosh t} & \frac{-\sinh t}{\cosh t} & \frac{-1}{\sqrt{2}\cosh t} \\ \frac{\sinh t}{\sqrt{2}\cosh t} & \frac{1}{\cosh t} & \frac{-\sinh t}{\sqrt{2}\cosh t} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$
$$\det A = 1$$
$$\det \dot{A} = \det(\dot{T}, \dot{N}, \dot{B}) = 0$$
$$\det \ddot{A} = \det(\ddot{T}, \ddot{N}, \ddot{B}) = \left(\kappa^2 \left(\frac{\tau}{\kappa}\right)^{\cdot} \right)^2 = 0$$
$$9 \Rightarrow \det \dddot{A} \; = 0$$
$$\overrightarrow{\omega} = (\tau, 0, \kappa) = \tau T + \kappa B$$
$$= \frac{1}{2\cos^2 ht} \left(\frac{1}{\sqrt{2}\cosh t}, \frac{\sinh t}{\sqrt{2}\cosh t}, \frac{1}{\sqrt{2}} \right) + \frac{1}{2\cos^2 ht} \left(\frac{-1}{\sqrt{2}\cosh t}, \frac{-\sinh t}{\sqrt{2}\cosh t}, \frac{1}{\sqrt{2}} \right)$$
$$\overrightarrow{\omega} = \sqrt{2}\kappa(0, 0, 1)$$

$rank\dot{A} = rank\ddot{A} = 2 \Rightarrow$ The direction of the I.S.A is stationary. The conclusion is (Fig.3.1): the instantaneous screw axis intersects ω, it is parallel to the plane O_{xz}; the components of ω are $(\tau, 0, \kappa)$ and that of the translation vector $(0, 0, 1)$.

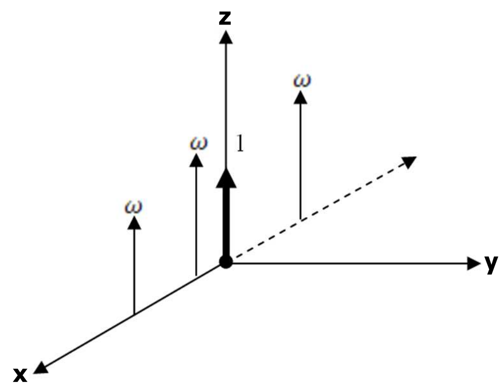

Fig 3.1

4 Acceleration Pole Points In Bishop Motion

By using the Bishop formulas and

$$A = \begin{bmatrix} T & N_1 & N_2 \end{bmatrix}, \quad \dot{A} = \begin{bmatrix} \dot{T} & \dot{N}_1 & \dot{N}_2 \end{bmatrix},$$
$$\ddot{A} = \begin{bmatrix} \ddot{T} & \ddot{N}_1 & \ddot{N}_2 \end{bmatrix}, \quad \dddot{A} = \begin{bmatrix} \dddot{T} & \dddot{N}_1 & \dddot{N}_2 \end{bmatrix}$$

we can give,

$$\det \dot{A} = \begin{bmatrix} 0 & \kappa_1 & \kappa_2 \\ -\kappa_1 & 0 & 0 \\ -\kappa_2 & 0 & 0 \end{bmatrix} = 0$$

Then the system $\dot{A}X + \dot{C} = 0$ has not unique solution. So, the Bishop motion has not pole point. From $\det \dot{A} = 0$ and $\begin{vmatrix} 0 & \kappa_1 \\ -\kappa_1 & 0 \end{vmatrix} = \kappa_1^2 \neq 0$, we have $rankA = 2$.

4.1 1^{st} acceleration pole points in Bishop motion

If $\{T, N_1, N_2\}$ is an adapted Bishop frame, then we have

$$\ddot{T} = -(\kappa_1^2 + \kappa_2^2)T + \dot{\kappa}_1 N_1 + \dot{\kappa}_2 N_2$$
$$\ddot{N}_1 = -\dot{\kappa}_1 T - \kappa_1^2 N_1 - \kappa_1\kappa_2 N_2$$
$$\ddot{N}_2 = -\dot{\kappa}_2 T - \kappa_1\kappa_2 N_1 - \kappa_2^2 N_2$$

So, we obtain

$$\begin{bmatrix} \ddot{T} \\ \ddot{N}_1 \\ \ddot{N}_2 \end{bmatrix} = \begin{bmatrix} -(\kappa_1^2+\kappa_2^2) & \dot{\kappa}_1 & \dot{\kappa}_2 \\ -\dot{\kappa}_1 & -\kappa_1^2 & -\kappa_1\kappa_2 \\ -\dot{\kappa}_2 & -\kappa_1\kappa_2 & -\kappa_2^2 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

By using $A = \begin{bmatrix} T & N_1 & N_2 \end{bmatrix}$, we get

$$\det \ddot{A} = \begin{vmatrix} -(\kappa_1^2+\kappa_2^2) & \dot{\kappa}_1 & \dot{\kappa}_2 \\ -\dot{\kappa}_1 & -\kappa_1^2 & -\kappa_1\kappa_2 \\ -\dot{\kappa}_2 & -\kappa_1\kappa_2 & -\kappa_2^2 \end{vmatrix} = -\left[\kappa_1^2\left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot}\right]^2 \tag{10}$$

By using equations 2 we have,

$$\begin{aligned} \kappa^2 &= \kappa_1^2+\kappa_2^2 \\ \frac{\kappa_2}{\kappa_1} &= \tan(\theta) \quad \Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = (1+\tan^2(\theta))\frac{d\theta}{dt} \\ &\Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = \left(1+\frac{\kappa_2^2}{\kappa_1^2}\right)\frac{d\theta}{dt} = -\left(\frac{\kappa_1^2+\kappa_2^2}{\kappa_1^2}\right)\tau \\ &\Rightarrow \quad \left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = -\left(\frac{\kappa^2}{\kappa_1^2}\right)\tau \\ &\Rightarrow \quad \kappa_1^2\left(\frac{\kappa_2}{\kappa_1}\right)^{\cdot} = -\kappa^2\tau \qquad (11) \end{aligned}$$

Obviously as a consequence of equations (10) and (11) we have the following:

$$\det \ddot{A} = -\kappa^4\tau^2 \tag{12}$$

As a consequence of equation of (12) we have the following:

$$\det \ddot{A} = 0 \Leftrightarrow \tau = 0$$

From this case we obtain, the solution systems of (6) are not unique in fixed space S_0 if and only if, at any time t, the curve $\alpha(t)$ is a plane. So that, the Bishop motion $Y = AX + C$ has not the 1^{st} acceleration pole point.
If $\det \ddot{A} \neq 0$ then $\alpha(t)$ is not plane. Thus we can give the following theorem:

Theorem 4.1 *$\forall t$, the curve $\alpha(t)$ is not plane in the moving space S $\Leftrightarrow$ the Bishop motion has a 1^{st} acceleration pole point; $X = -(\ddot{A})^{-1}\ddot{C}$,*

On the other hand, by means of $\det \ddot{A} = 0$ we have that $rank\ddot{A} < 3$ and since

$$\begin{vmatrix} -(\kappa_1^2+\kappa_2^2) & \dot{\kappa}_1 \\ -\dot{\kappa}_1 & -\kappa_1^2 \end{vmatrix} = \kappa_1^4 + \kappa_1^2\kappa_2^2 + \dot{\kappa_1}^2 > 0$$

Such being the case, $rank\ddot{A} = 2$ then, the solution of (6) is a line, at every instant t.
Therefore as a consequence of theorem 3.4 [5], for $n = 3$, we can write, the direction of the Instantaneous Screw Axis (ISA) is stationary. To sum up, we have proved the following theorem:

Theorem 4.2 *Structure of a Coned has a surface cylindrical $\Leftrightarrow$ the Bishop motion $Y = AX+C$, has not the 1^{st} acceleration pole points.*

4.2 2^{nd} acceleration pole points in Bishop motion

If T, N_1 and N_2 is an adapted Bishop frame, then we have;

$$\dddot{T} = -3(\kappa_1\dot{\kappa}_1 + \kappa_2\dot{\kappa}_2)T - (\kappa_1^3 + \kappa_1\kappa_2^2 - \ddot{\kappa}_1)N_1 - (\kappa_2^3 + \kappa_1^2\kappa_2 - \ddot{\kappa}_2)N_2$$
$$\dddot{N}_1 = (\kappa_1^3 + \kappa_1\kappa_2^2 - \ddot{\kappa}_1)T - (3\kappa_1\dot{\kappa}_1)N_1 - (\kappa_1\dot{\kappa}_2 + 2\dot{\kappa}_1\kappa_2)N_2$$
$$\dddot{N}_2 = (\kappa_2^3 + \kappa_1^2\kappa_2 - \ddot{\kappa}_2)T - (\kappa_2\dot{\kappa}_1 + 2\dot{\kappa}_2\kappa_1)N_1 - (3\kappa_2\dot{\kappa}_2)N_2$$
$$\begin{bmatrix} \dddot{T} \\ \dddot{N}_1 \\ \dddot{N}_2 \end{bmatrix} = \begin{bmatrix} -3(\kappa_1\dot{\kappa}_1 + \kappa_2\dot{\kappa}_2) & -(\kappa_1^3 + \kappa_1\kappa_2^2 - \ddot{\kappa}_1) & -(\kappa_2^3 + \kappa_1^2\kappa_2 - \ddot{\kappa}_2) \\ (\kappa_1^3 + \kappa_1\kappa_2^2 - \ddot{\kappa}_1) & -(3\kappa_1\dot{\kappa}_1) & -(\kappa_1\dot{\kappa}_2 + 2\dot{\kappa}_1\kappa_2) \\ (\kappa_2^3 + \kappa_1^2\kappa_2 - \ddot{\kappa}_2) & -(\kappa_2\dot{\kappa}_1 + 2\dot{\kappa}_2\kappa_1) & -(3\kappa_2\dot{\kappa}_2) \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}$$

$$\begin{aligned} \det \dddot{A} &= \det(\dddot{T}, \dddot{N}_1, \dddot{N}_2) \\ &= -3(\kappa_1^2 + \kappa_2^2)(\dot{\kappa}_1\kappa_2 - \kappa_1\dot{\kappa}_2)(\ddot{\kappa}_1\kappa_2 - \kappa_1\ddot{\kappa}_2) - 3(\ddot{\kappa}_1\dot{\kappa}_2 - \dot{\kappa}_1\ddot{\kappa}_2)(\ddot{\kappa}_1\kappa_2 - \kappa_1\ddot{\kappa}_2) \\ &+ 6(\dot{\kappa}_1\kappa_2 - \kappa_1\dot{\kappa}_2)^2(\kappa_1\dot{\kappa}_1 + \kappa_2\dot{\kappa}_2) \\ &= -3(2\kappa\dot{\kappa}\tau + \kappa^2\dot{\tau})(\kappa^4\tau + \ddot{\kappa}_1\dot{\kappa}_2 - \dot{\kappa}_1\ddot{\kappa}_2) + 6\kappa^5\dot{\kappa}\tau^2 \end{aligned} \tag{13}$$

As a consequence of equation of (13) we have the following:

$$\tau = 0 \Rightarrow \det \dddot{A} = 0$$

From this case we obtain, if at any time t, the curve $\alpha(t)$ is a plane, then the solution of system (8) are not unique in fixed space S_0 and the Bishop motion $Y = AX + C$ has not the 2^{nd} acceleration pole point.
Now we can give the following theorem:

Theorem 4.3 *$\forall t$, the curve $\alpha(t)$ is a plane $\Leftrightarrow$ in fixed space S_0, the Bishop motion hasnot a 2^{nd} acceleration pole point.*

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