

Interior Controllability of a Broad Class of Reaction Diffusion Equations

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Abstract

In this paper we prove the interior approximate controllability of the following broad class of reaction diffusion equation in the Hilbert spaces $Z = L^2(\Omega)$ given by

$$z' = -Az + 1_\omega u(t), \quad t \in [0, \tau],$$

where Ω is a domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u \in L^2(0, t_1; L^2(\Omega))$ and $A : D(A) \subset Z \rightarrow Z$ is an unbounded linear operator with the following spectral decomposition:

$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}$. The eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \dots < \lambda_n \rightarrow \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A . The operator $-A$ generates a strongly continuous semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}.$$

Our result can be apply to the n D heat equation, the equation modelling the damped flexible beam, the Ornstein-Uhlenbeck equation, the Laguerre equation and the Jacobi equation.

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1 Introduction.

In this paper we prove the interior approximate controllability of the following broad class of reaction diffusion equation in the Hilbert spaces $Z = L^2(\Omega)$ given by

$$\begin{cases} z' = -Az + 1_\omega u(t), & t \in [0, \tau], \\ z(0) = z_0 \end{cases} \quad (1.1)$$

where Ω is a domain in \mathbb{R}^n , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω and the distributed control $u \in L^2(0, t_1; L^2(\Omega))$ and $A : D(A) \subset Z \rightarrow Z$ is an unbounded linear operator with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \quad (1.2)$$

with $\langle \cdot, \cdot \rangle$ denoting an inner product in Z , and

$$E_j z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}.$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \dots < \lambda_j < \dots < \lambda_n \rightarrow \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors of A . So, $\{E_j\}$ is a complete family of orthogonal projections in Z and $z = \sum_{j=1}^{\infty} E_j z$, $z \in Z$. The operator $-A$ generates a strongly continuous semigroup $\{T(t)\}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z.$$

Systems of the form (1.1) are thoroughly studied in [6] and [7], but the interior controllability is not considered here.

At this point we must mention the works done by others authors to complete the exposure of this paper. To complement this requirement, we present some observations to the works of others authors, showing the difference between our results and those of them.

The interior approximate controllability is very well known fascinate and important subject in systems theory; there are some works done by [9], [10], [11], [12] and [13].

Particularly, Zuazua in [13], where the author proves the interior approximate controllability of the system (1.4) in two different ways. In the first one, he uses the Hanh-Banach theorem, integrating by parts, the adjoint equation

$$\begin{cases} \varphi_t + \Delta\varphi = 0, & \text{in } (0, \tau) \times \Omega, \\ \varphi = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ \varphi(\tau) = \varphi_\tau, & \text{in } \Omega, \end{cases} \quad (1.3)$$

the Carleman estimates and the following result from [8]:

Holmgren Uniqueness Theorem: Let P be a differential operator with constant coefficient in \mathbb{R}^n . Let z be a solution of the equation $Pz = 0$ in Q_1 where Q_1 is an open set of \mathbb{R}^n . Suppose

that $z = 0$ in Q_2 , where Q_2 is an open nonempty subset of Q_1 . Then $z = 0$ on Q_3 , where Q_3 is the open subset of Q_1 which contains Q_2 and such that any characteristic hyperplane of the operator P which intersects Q_3 also intersects Q_1 . Here we find some differences; it is good to mention that the Carleman estimates depend on the Laplacian operator Δ , so it may not be applied to those equations that do not involve the Laplacian operator, like the Ornstein-Uhlenbeck equation, the Laguerre equation and the Jacobi equation.

The second method is constructive and uses a variational technique: let us fix the control time $\tau > 0$, the initial and final state $z_0 = 0$, $z_1 \in L^2(\Omega)$ respectively and $\epsilon > 0$. The control steering the initial state z_0 to a ball of radius $\epsilon > 0$ and center z_1 is given by the point in which the following functional achieves its minimum value

$$J_\epsilon(\varphi_\tau) = \frac{1}{2} \int_0^\tau \int_\omega \varphi^2 dx dt + \epsilon \|\varphi_\tau\|_{L^2(\Omega)} - \int_\Omega z_1 \varphi_\tau,$$

where φ is the solution of the adjoint equation (1.3) with initial data φ_τ .

Also in [11], X. Zhang proved that the null exact controllability property for system (1.4) may be obtained as a singular limit of the exact controllability properties of singularly perturbed damped wave equation with a changing controller.

The technique given here in this work is not trivial, but it is so simple that, those young mathematicians who live in remote and inhospitable places, far from major research centers in the world, can also understand and enjoy the interior controllability with a minor effort. That is one of the novelties of this work. Our proof is based in the following results:

THEOREM 1.1 (see Theorem 1.23 from [1], pg. 20) *Suppose $\Omega \subset \mathbb{R}^n$ is open, non-empty and connected set, and f is real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω . Then, $f = 0$ in Ω .*

LEMMA 1.1 (see Lemma 3.14 from [6], pg. 62) *Let $\{\alpha_j\}_{j \geq 1}$ and $\{\beta_{i,j} : i = 1, 2, \dots, m\}_{j \geq 1}$ be two sequences of real numbers such that: $\alpha_1 > \alpha_2 > \alpha_3 \dots$. Then*

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \dots, m$$

iff

$$\beta_{i,j} = 0, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, \infty.$$

Examples of this class are the following well known partial differential equations:

EXAMPLE 1.1 *The interior controllability of the heat equation*

$$\begin{cases} z_t = \Delta z + 1_\omega u(t, x), & \text{in } (0, \tau) \times \Omega, \\ z = 0, & \text{on } (0, \tau) \times \partial\Omega, \\ z(0, x) = z_0(x), & \text{in } \Omega, \end{cases} \quad (1.4)$$

where Ω is a bounded domain in \mathbb{R}^n of class C^2 , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , $z_0 \in L^2(\Omega)$ and the distributed control $u \in L^2(0, \tau; L^2(\Omega))$.

EXAMPLE 1.2 (see [3] and [4])

1. *The interior controllability of the Ornstein-Uhlenbeck equation*

$$z_t = \sum_{i=1}^d \left[x_i \frac{\partial^2 z}{\partial x_i^2} + -x_i \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t, x) \quad t > 0, \quad x \in \mathbb{R}^d, \quad (1.5)$$

where $u \in L^2(0, \tau; L^2(\mathbb{R}^d, \mu))$, $\mu(x) = \frac{1}{\pi^{d/2}} \prod_{i=1}^d e^{-|x_i|^2} dx$ is the Gaussian measure in \mathbb{R}^d and ω is an open nonempty subset of \mathbb{R}^d .

2. *The interior controllability of the Laguerre equation*

$$z_t = \sum_{i=1}^d \left[x_i \frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t, x), \quad t > 0, \quad x \in \mathbb{R}_+^d, \quad (1.6)$$

where $u \in L^2(0, \tau; L^2(\mathbb{R}_+^d, \mu_\alpha))$, $\mu_\alpha(x) = \prod_{i=1}^d \frac{x_i^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i + 1)} dx$ is the Gamma measure in \mathbb{R}_+^d and ω is an open nonempty subset of \mathbb{R}_+^d .

3. *The interior controllability of the Jacobi equation*

$$z_t = \sum_{i=1}^d \left[(1 - x_i^2) \frac{\partial^2 z}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)x_i) \frac{\partial z}{\partial x_i} \right] + 1_\omega u(t, x), \quad (1.7)$$

where $t > 0$, $x \in [-1, 1]^d$, $u \in L^2(0, \tau; L^2([-1, 1]^d, \mu_{\alpha, \beta}))$, $\mu_{\alpha, \beta}(x) = \prod_{i=1}^d (1 - x_i)^{\alpha_i} (1 + x_i)^{\beta_i} dx$ is the Jacobi measure in $[-1, 1]^d$ and ω is an open nonempty subset of $[-1, 1]^d$.

2 Proof of the Main Theorem

In this section we shall prove the main result of this paper on the controllability of the linear system (1.1). But before that, we shall give the definition of approximate controllability for this

system. To this end, the system (1.1) can be written as follows

$$\begin{cases} z' = -Az + B_\omega u(t), z \in Z \\ z(0) = z_0 \end{cases} \quad (2.8)$$

where the operator $B_\omega : Z \rightarrow Z$ is defined by $B_\omega f = 1_\omega f$. For all $z_0 \in Z$ and $u \in L^2(0, \tau; Z)$ the initial value problem (2.8) admits only one mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)B_\omega u(s)ds, \quad t \in [0, \tau]. \quad (2.9)$$

DEFINITION 2.1 (Exact Controllability) *The system (2.8) is said to be exactly controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$ there exists $u \in L^2(0, \tau; Z)$ such that the solution $z(t)$ of (2.9) corresponding to u verifies: $z(\tau) = z_1$.*

DEFINITION 2.2 (Approximate Controllability) *The system (2.8) is said to be approximately controllable on $[0, \tau]$ if for every $z_0, z_1 \in Z$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; Z)$ such that the solution $z(t)$ of (2.9) corresponding to u verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

REMARK 2.1 *The following result was prove in [2]: If the semigroup $\{T(t)\}$ is compact, then the system $z' = -Az + B_\omega u(t)$ can never be exactly controllable on time $\tau > 0$, which is the case of the heat equations, the equation modelling the damped flexible beam, the Ornstein-Uhlenbeck equation, the Laguerre equation, the Jacobi equation and many others partial differential equations.*

The following theorem can be found in a general form for evolution equation in [7].

THEOREM 2.1 *The system (2.8) is approximately controllable on $[0, \tau]$ if, and only if,*

$$B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau] \Rightarrow z = 0. \quad (2.10)$$

Now, we are ready to formulate and prove the main theorem of this work.

THEOREM 2.2 *If for an open non-empty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^\omega = \phi_{j,k}|_\omega$ to ω are linearly independent functions on ω , then for all $\tau > 0$ the system (2.8) is approximately controllable on $[0, \tau]$.*

Proof. We shall apply Theorem 2.1 to prove the approximate controllability of system (2.8). To this end, we observe that $B_\omega = B_\omega^*$ and $T^*(t) = T(t)$. Suppose that $B_\omega^* T^*(t)z = 0, \quad \forall t \in [0, \tau]$.

Then,

$$\begin{aligned}
 B_{\omega}^* T^*(t)z &= \sum_{j=1}^{\infty} e^{-\lambda_j t} B_{\omega}^* E_j z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_{\omega} \phi_{j,k} = 0. \\
 &\iff \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle 1_{\omega} \phi_{j,k}(x) = 0, \quad \forall x \in \Omega. \\
 &\iff \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega.
 \end{aligned}$$

Hence, from Lemma 1.1, we obtain that

$$\sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \dots$$

Since $\phi_{j,k}$ are linearly independent on ω , we obtain that

$$\langle z, \phi_{j,k} \rangle = 0, \quad j = 1, 2, 3, \dots$$

Therefore, $E_j z = 0$, $j = 1, 2, 3, \dots$, which implies that $z = 0$. So, the system (2.8) is approximately controllable on $[0, \tau]$. \square

COROLLARY 2.1 *If $\phi_{j,k}$ are analytic functions on Ω , then for all open non-empty set $\omega \subset \Omega$ and $\tau > 0$ the system (2.8) is approximately controllable on $[0, \tau]$.*

Proof . It is enough to prove that, for all open non-empty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^{\omega} = \phi_{j,k}|_{\omega}$ to ω are linearly independent functions on ω , which follows directly from Theorem 1.1. \square

3 Applications.

As an application of our result we shall prove the controllability of the n D heat equation, the Ornstein-Uhlenbeck equation, the Laguerre equation and the Jacobi equation.

3.1 The interior controllability of the heat equation (1.4)

In this subsection we shall prove the controllability of system (1.4), but before that, we shall prove the following Theorem

THEOREM 3.1 *The eigenfunctions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω .*

To this end, first, we shall consider the following definition and results from [5].

DEFINITION 3.1 *A Differential operator L is say to be hypoelliptic -Analytic if for each open subset Ω of \mathbb{R}^n and each distribution $u \in \dot{D}(\Omega)$, we have that: If $L(u)$ is an analytic function in Ω , then u is an analytic function in Ω .*

COROLLARY 3.1 *(see [5] pg. 15.) Every second order elliptic operator with constant coefficients is hypoelliptic-Analytic*

Proof of Theorem 3.1. Let ϕ be an eigenfunction of $-\Delta$ with corresponding eigenvalue $\lambda > 0$. Then, the second order differential operator $L = \Delta + \lambda$ is an elliptic operator according with definiton 7.2 from [5], pg. 97. Therefore, applying the foregoing Corollary we get that $L = \Delta + \lambda$ hypoelliptic-Analytic.

On the other hand, we know that $L\phi = \Delta\phi + \lambda\phi = 0$, which is trivially an analytic function, then ϕ is an analytic function in Ω .

□

Now, we shall make the abstract formulation of the problem, and to this end, let us consider $Z = L^2(\Omega)$ and the linear unbounded operator $A : D(A) \subset Z \rightarrow Z$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H_0^1(\Omega) \cap H^2(\Omega). \quad (3.11)$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \dots < \lambda_j \rightarrow \infty,$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors of A .

b) For all $z \in D(A)$ we have

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j z, \quad (3.12)$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X and

$$E_n z = \sum_{k=1}^{\gamma_j} \langle z, \phi_{j,k} \rangle \phi_{j,k}. \quad (3.13)$$

So, $\{E_j\}$ is a family of complete orthogonal projections in Z and

$$z = \sum_{j=1}^{\infty} E_j z, \quad z \in Z. \quad (3.14)$$

c) $-A$ generates an analytic semigroup $\{T(t)\}_{t \geq 0}$ given by

$$T(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j z. \quad (3.15)$$

The system (1.4) can be written as an abstract equation in the space $Z = L^2(\Omega)$

$$\begin{cases} z' = -Az + B_\omega u(t), z \in Z \\ z(0) = z_0 \end{cases} \quad (3.16)$$

where the control function u belong to $L^2(0, \tau; Z)$ and the operator $B_\omega : Z \rightarrow Z$ is define by $B_\omega f = 1_\omega f$.

THEOREM 3.2 *For all open non-empty set $\omega \subset \Omega$ and $\tau > 0$ the system (3.16) is approximately controllable on $[0, \tau]$.*

3.2 The interior controllability of equations (1.5), (1.6) and (1.7)

THEOREM 3.3 *The systems (1.5), (1.6) and (1.7) are approximately controllable.*

Proof It is enough to prove that the operators

i) Ornstein-Uhlenbeck operator: $-A = \frac{1}{2}\nabla - \langle x, \Delta_x \rangle$, defined on $\Omega = \mathbb{R}^d$,
with $\Delta_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ in the space $Z = L^2(\mathbb{R}^d, \mu)$.

ii) Laguerre operator: $A = -\sum_{i=1}^d \left[x_i \frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \right]$, defined on $\Omega = (0, \infty)^d$,
with $\alpha_i > -1$, $i = 1, \dots, d$ in the space $Z = L^2(\mathbb{R}_+^d, \mu_\alpha)$

ii) Jacobi operator: $A = -\sum_{i=1}^d \left[(1 - x_i^2) \frac{\partial^2 z}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2) x_i) \frac{\partial z}{\partial x_i} \right]$,
 $\Omega = (-1, 1)^d$, with $\alpha_i, \beta_i > -1$, $i = 1, \dots, d$ in the space $Z = L^2([-1, 1]^d, \mu_{\alpha, \beta})$,

can be represented in the form of (1.2). It was done in [3] and [4], where they prove that the eigenfunctions in these cases are polynomial functions in multiple variables, which are trivially analytic functions.

4 Conclusion

As one can see, this new method to study the interior controllability can be applied to a broad class of reaction diffusion equation like the following well known systems of partial differential equations:

EXAMPLE 4.1 *The Thermoelastic Plate Equation*

$$\begin{cases} w_{tt} + \Delta^2 w + \alpha \Delta w = 1_\omega u_1(t, x), & \text{in } (0, \tau) \times \Omega, \\ \theta_t - \beta \Delta \theta - \alpha \Delta w_t = 1_\omega u_2(t, x), & \text{in } (0, \tau) \times \Omega, \\ \theta = w = \Delta w = 0, & \text{on } (0, \tau) \times \partial\Omega, \end{cases} \quad (4.17)$$

where $\alpha \neq 0$, $\beta > 0$, Ω is a sufficiently regular bounded domain in \mathbb{R}^3 , ω is an open nonempty subset of Ω , 1_ω denotes the characteristic function of the set ω , the distributed control $u_i \in L^2([0, \tau]; L^2(\Omega))$, $i = 1, 2$. and w, θ denote the vertical deflection and the temperature of the plate respectively.

EXAMPLE 4.2 *The equation modelling the damped flexible beam:*

$$\begin{cases} \frac{\partial^2 z}{\partial t^2} = -\frac{\partial^4 z}{\partial x^4} + 2\alpha \frac{\partial^3 z}{\partial t \partial x^2} + 1_\omega u(t, x) & t \geq 0, \quad 0 \leq x \leq 1 \\ z(t, 1) = z(t, 0) = \frac{\partial^2 z}{\partial x^2}(0, t) = \frac{\partial^2 z}{\partial x^2}(1, t) = 0, \\ z(0, x) = \phi_0(x), \quad \frac{\partial z}{\partial t}(0, x) = \psi_0(x), \quad 0 \leq x \leq 1 \end{cases} \quad (4.18)$$

where $\alpha > 0$, $u \in L^2([0, r]; L^2[0, 1])$, ω is an open nonempty subset of $[0, 1]$ and $\phi_0, \psi_0 \in L^2[0, 1]$.

EXAMPLE 4.3 *The strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} \frac{\partial^2 w}{\partial t^2} + \eta(-\Delta)^{1/2} \frac{\partial w}{\partial t} + \gamma(-\Delta)w = 1_\omega u(t, x), & t \geq 0, \quad x \in \Omega, \\ w(t, x) = 0, & t \geq 0, \quad x \in \partial\Omega. \\ w(0, x) = \phi_0(x), \quad \frac{\partial w}{\partial t}(0, x) = \psi_0(x), & x \in \Omega, \end{cases} \quad (4.19)$$

where Ω is a sufficiently smooth bounded domain in \mathbb{R}^N , $u \in L^2([0, r]; L^2(\Omega))$, ω is an open nonempty subset of Ω and $\phi_0, \psi_0 \in L^2(\Omega)$.

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