

Minimal R_1 , minimal regular and minimal presober topologies

M.L. Colasante and D. Van der Zypen

Abstract

By means of filters, minimal R_1 and minimal regular topologies are characterized on suitable intervals consisting of non-trivial R_0 topologies.

key words. Alexandroff topology, R_0 , R_1 , regular and presober topologies, filters

AMS(MOS) subject classifications. 54A10, 54D10, 54D25

1 Introduction

The family $LT(X)$ of all topologies definable on a set X partially ordered by inclusion is a complete, atomic lattice in which the meet of a collection of topologies is their intersection, while the join is the topology with their union as a subbase. There has been a considerable amount of interest in topologies which are minimal in this lattice with respect to certain topological properties (see for instance [1], [2], [3], [4], [5], [8], [9], [10], [11], [12], [13], [15], [18]).

Given a topological property P (like a separation axiom) and given a family \mathcal{S} of members of $LT(X)$, then $\tau \in \mathcal{S}$ is said to be *minimal* P in \mathcal{S} if τ satisfies P but no member of \mathcal{S} which is strictly weaker than τ satisfies P . It is well known that a T_2 -topology on an infinite set X is minimal T_2 in $LT(X)$ iff every open filter on X with a unique adherent point is convergent ([3]). Also, a regular T_1 -topology is minimal regular in $LT(X)$ iff every regular filter on X with a unique adherent point is convergent ([4]). These are characterization of minimal topologies satisfying separation axioms above T_1 , and thus topologies in the lattice $\mathcal{L}_1 = \{\tau \in LT(X) : \mathcal{C} \leq \tau \leq 2^X\}$, where \mathcal{C} denotes the cofinite topology (*i.e* the minimal T_1 -topology on X) and 2^X denotes the powerset of X . Some separation axioms independent of T_1 (even independent of T_0) are vacuously satisfied by the indiscrete topology, thus the study of minimal topologies in $LT(X)$ satisfying such properties becomes trivial. This is the case of the R_1 and regularity (not necessarily T_1) separation axioms. The purpose of this paper is to show that, by restricting to suitable intervals

\mathcal{L}_ρ of $LT(X)$, associated each to a non-trivial R_0 -topology ρ , then minimal regular and minimal R_1 topologies in \mathcal{L}_ρ can be characterized in terms of filters. For instance, we prove in section 3 that an R_1 -topology in \mathcal{L}_ρ is minimal R_1 iff every open filter on X , for which the set of adherent points coincides with a point closure, is convergent, and that a regular topology in \mathcal{L}_ρ is minimal regular iff every regular filter on X , for which the set of adherent points coincides with a point closure, is convergent. The characterizations for minimal T_2 and minimal regular topologies mentioned at the beginning of this paragraph are immediate corollary of our results in case ρ is a T_1 -topology. Additionally, we consider in last section another topological property independent of T_0 , namely the presober property, and show that there are not minimal presober topologies in \mathcal{L}_ρ .

2 Preliminaries and notations

A topology $\tau \in LT(X)$ is said to be an *Alexandroff topology* if it is closed under arbitrary intersection. J. Steprans and S. Watson [16] attributed this notion to both Alexandroff and Tucker, and called them *AT* topologies. This class of topologies is specially relevant for the study of non- T_1 topologies. Note that the only T_1 Alexandroff topology is the discrete topology. Among the characterizations known for *AT* topologies, we recall the one related with the specialization preorder: $\tau \in LT(X)$ is *AT* iff it is the finest topology on X consistent with the specialization preorder, *i.e.* the finest topology giving the preorder \leq_τ satisfying $x \leq_\tau y$ iff x belongs to the τ -closure of $\{y\}$. This preorder characterizes the T_0 property (for every two points there is an open set containing one and only one of the points) in the sense that a topology τ is T_0 iff the preorder \leq_τ is a partial order.

By identifying a set with its characteristic function, 2^X can be endowed with the product topology of the Cantor cube $\{0, 1\}^X$. It was proved in [17] that a topology τ on X is *AT* iff it is closed when viewed as a subset of 2^X . Moreover, it was proved there that the closure $\bar{\tau}$ of τ in 2^X is also a topology, and therefore it is the smallest *AT* topology containing τ .

By $cl_\tau(A)$ we denote the τ -closure of a set A . If $A = \{x\}$, we use $cl_\tau(x)$ instead of $cl_\tau(\{x\})$, and refer to it as a point closure. The τ -kernel of a set $A \subseteq X$, denoted by $\ker_\tau(A)$, is the intersection of all open sets containing A . For any $x \in X$, we denote $\ker_\tau(\{x\}) = \ker_\tau(x)$. It is obvious that $x \in cl_\tau(y)$ iff $y \in \ker_\tau(x)$. A set A is said to be τ -kernelled (or just kernelled) if $A = \ker_\tau(A)$. Equivalently, A is kernelled iff $A = \bigcup_{x \in A} \ker_\tau(x)$. The family of all kernelled subsets of X is closed under arbitrary unions and intersections, so it is an *AT* topology. Moreover,

it coincides with $\bar{\tau}$. In fact, since every open set is kernelled and $\bar{\tau}$ is the smallest AT topology containing τ , then every member of $\bar{\tau}$ is kernelled. On the other hand, since $\bar{\tau}$ is closed under arbitrary intersections and it contains τ , then every kernelled set belongs to $\bar{\tau}$. Thus, $\bar{\tau}$ is the topology on X generated by the family $\{\ker_{\tau}(x) : x \in X\}$. In particular, $A \subseteq X$ is $\bar{\tau}$ -closed iff $A = \bigcup_{x \in A} cl_{\tau}(x)$. Note that, since τ is T_1 iff every subset of X is kernelled, then τ is T_1 iff $\bar{\tau} = 2^X$.

In what follows $\mathcal{N}_{\tau}(x)$ denotes the filter base of τ -neighborhoods of $x \in X$. A filter \mathcal{F} on X is said to be τ -convergent to a point $x \in X$ if $\mathcal{F} \supseteq \mathcal{N}_{\tau}(x)$. By $adh_{\tau}\mathcal{F}$ we denote the set of adherent points of \mathcal{F} (i.e. $adh_{\tau}\mathcal{F} = \bigcap_{F \in \mathcal{F}} cl_{\tau}(F)$). Since $adh_{\tau}\mathcal{F}$ is a closed set, then it contains the τ -closure of all its points. It is immediate that if \mathcal{F} is τ -convergent to x , then \mathcal{F} is τ -convergent to every $y \in cl_{\tau}(x)$. A filter \mathcal{F} is said to be τ -open if $F \in \tau$ for all $F \in \mathcal{F}$, and \mathcal{F} is said to be τ -regular if it is τ -open and for every $F \in \mathcal{F}$ there exists $F' \in \mathcal{F}$ such that $cl_{\tau}(F') \subseteq F$. Thus, a τ -regular filter is equivalent to a τ -closed filter. A filter on X is said to be ultrafilter if it is a maximal filter.

For definitions and notations not given here, we refer the reader to [19].

3 Minimal R_1 and minimal regular topologies in \mathcal{L}_{ρ}

In this section, we restrict our attention to suitable intervals consisting of R_0 topologies, and give characterizations of minimal R_1 and minimal regular topologies on those intervals. Recall that a topology $\tau \in LT(X)$ is said to be:

(R_0) if for all $x, y \in X$, $x \in cl_{\tau}(y)$ iff $y \in cl_{\tau}(x)$, thus τ is R_0 iff the point closures form a partition of X . [14]

(R_1) if for all $x, y \in X$ with $cl_{\tau}(x) \neq cl_{\tau}(y)$, there are disjoint open sets separating $cl_{\tau}(x)$ and $cl_{\tau}(y)$. [7]

(*Regular*) if for each $V \in \tau$ and each $x \in V$ there exists $U \in \tau$ such that $x \in U \subseteq cl_{\tau}(U) \subseteq V$.

The separation axioms R_0 and R_1 are also denoted as S_1 and S_2 , respectively ([6]). We use in this paper the most common notations R_0 and R_1 . It is easy to show that *Regularity* $\Rightarrow R_1 \Rightarrow R_0$, and that none of the implications can be reversed. Moreover, τ is T_1 iff τ is R_0 and T_0 , and τ is T_2 iff τ is R_1 and T_0 .

Examples of topologies which are regular non- T_0 (thus, regular non- T_1) abound. For instance, if \mathcal{P} denotes any non-trivial partition of a set X , then the associated partition topology $\tau_{\mathcal{P}}$, defined

as the topology having as open sets the unions of elements of \mathcal{P} together with the empty set, is a regular topology which is not T_0 . On the other hand, if a topological space satisfies any of the (R) properties stated above and one doubles the space by taking the product of X with the two point indiscrete space, then the resulting space is not longer T_0 but it satisfies the same (R) properties as did the original space.

The following characterizations, which are straightforward to prove, are used throughout the paper without explicitly mentioning them.

Lemma 3.1 *Let $\tau \in LT(X)$. Then*

(i) τ is R_0 iff $cl_\tau(x) = \ker_\tau(x)$ for all $x \in X$, iff $cl_\tau(x) \subseteq V$, for all $V \in \tau$ and $x \in V$.

(ii) τ is R_1 iff τ is R_0 and for all $x, y \in X$ such that $y \notin cl_\tau(x)$ there are disjoint open sets separating x and y .

(iii) τ is R_1 iff τ is R_0 and $adh_\tau \mathcal{N}_\tau(x) = cl(x)$, for all $x \in X$.

To each $\rho \in LT(X)$ we associate the interval

$$\mathcal{L}_\rho = \{\tau \in LT(X) : at(\rho) \leq \tau \leq \bar{\rho}\}$$

where $at(\rho)$ denotes the topology on X generated by the sets $\{X \setminus cl_\rho(H) : H \text{ is a finite subset of } X\}$, and $\bar{\rho}$ is the closure of ρ in 2^X .

Note that, if ρ is any T_1 -topology, then $at(\rho) = \mathcal{C}$ and $\bar{\rho} = 2^X$. In this case, \mathcal{L}_ρ is precisely the lattice \mathcal{L}_1 of all T_1 topologies on X .

Lemma 3.2 *Let $\rho \in LT(X)$. Then $cl_{at(\rho)}(x) = cl_\rho(x) = cl_{\bar{\rho}}(x)$, for every $x \in X$.*

Proof. Let $x \in X$. Since a set is $\bar{\rho}$ -closed iff it is union of ρ -closed sets, then $cl_\rho(x) \subseteq cl_{\bar{\rho}}(x)$. On the other hand, $cl_\rho(x)$ is an $at(\rho)$ -closed set, and thus $cl_{at(\rho)}(x) \subseteq cl_\rho(x)$. Since $at(\rho) \subseteq \rho \subseteq \bar{\rho}$, then $cl_{\bar{\rho}}(x) \subseteq cl_\rho(x) \subseteq cl_{at(\rho)}(x)$. From this we have the result. ■

Corollary 3.3 *Let $\tau, \rho \in LT(X)$. Then $\tau \in \mathcal{L}_\rho$ iff $cl_\rho(x) = cl_\tau(x)$, for every $x \in X$.*

Proof. If $\tau \in \mathcal{L}_\rho$ and $x \in X$, then Lemma 3.2 implies that $cl_\tau(x) = cl_\rho(x)$. Conversely, suppose $cl_\rho(x) = cl_\tau(x)$, for every $x \in X$. It is immediate that $at(\rho) \leq \tau$. Note that $\ker_\rho(x) = \ker_\tau(x)$, thus if $V \in \tau$ then $V = \bigcup_{x \in V} \ker_\rho(x) = \bigcup_{x \in V} \ker_\tau(x)$ is a $\bar{\rho}$ -open set. Therefore $at(\rho) \leq \tau \leq \bar{\rho}$. ■

Corollary 3.3 can be stated as follows: $\tau \in \mathcal{L}_\rho$ iff τ has the same preorder of specialization as ρ . Thus, when one refers to the τ -closure of $x \in X$, for any $\tau \in \mathcal{L}_\rho$, there is not need for specifying the topology. We will often write $cl(x)$ without further comment. It is clear that the topologies on \mathcal{L}_ρ share the topological properties defined in terms of point closures. In particular $\tau \in \mathcal{L}_\rho$ is R_0 iff ρ is R_0 . Note that the property R_1 is expansive in \mathcal{L}_ρ (i.e. if $\tau \in \mathcal{L}_\rho$ is R_1 , then τ' is R_1 for all $\tau' \in \mathcal{L}_\rho$ finer than τ).

In ([6]) it was proved that the properties R_0 , R_1 and regularity coincide for AT topologies. Thus, $\bar{\rho}$ is R_0 iff $\bar{\rho}$ is R_1 iff $\bar{\rho}$ is regular. If we start with an R_0 -topology ρ on X , it is immediate that there exists at least a regular topology (so at least an R_1 -topology) in \mathcal{L}_ρ . Our goal is to characterize minimal R_1 and minimal regular topologies in \mathcal{L}_ρ . Note that, if ρ is R_0 and X can be written as a finite union of disjoint point closures then, for each $x \in X$, the set $cl(x)$ is the complement of finite union of point closures, thus $cl(x) \in at(\rho)$. It follows that $at(\rho) = \rho = \bar{\rho}$ and therefore $\mathcal{L}_\rho = \{\rho\}$. To avoid triviality, from now on we assume that $\rho \in LT(X)$ is an R_0 -topology such that X can be written as infinite union of disjoint point closures (in particular, this is the case for any T_1 -topology on an infinite set). It is worth notice that $at(\rho)$ can not be R_1 , thus it can not be regular, since any pair of non-empty $at(\rho)$ -open sets intersect. We give an example of an R_0 (not T_0) topology satisfying the above conditions.

Example 3.4 Let X be the set of all positive integers N , and let ρ be the topology generated by the subbase $\{\emptyset, N \setminus \{1\}, N \setminus \{2n, 2n+1\}, n \geq 1\}$. It is easy to see that ρ is an R_0 -topology which is not T_0 , and that N can be written as the infinite disjoint union of the odd integers point closure. Note that $at(\rho) = \rho$, and $\bar{\rho}$ is the topology generated by the sets $\{1\}, \{2n, 2n+1\}, n \geq 1$.

For $x \in X$, let $\mathcal{E}(x)$ denote the family of all the subsets of X not containing x . If \mathcal{F} is any filter on X , then $\mathcal{E}(x) \cup \mathcal{F}$ is a topology on X . Given $\tau \in LT(X)$, we consider the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Note that $\beta \leq \tau$, and $\beta = \tau$ iff $\mathcal{F} = \mathcal{N}_\tau(x)$.

Now, if ρ is R_0 and $\tau \in \mathcal{L}_\rho$, a local base for the topology β can be describe as follows:

$$\mathcal{N}_\beta(y) = \mathcal{N}_\tau(y) \cap \mathcal{E}(x), \text{ for every } y \notin cl(x);$$

$$\mathcal{N}_\beta(y) = \mathcal{N}_\tau(x) \cap \mathcal{F}, \text{ for every } y \in cl(x).$$

A set $A \subseteq X$ is β -closed iff A is τ -closed and either $x \in A$ or $X \setminus A \in \mathcal{F}$. Thus, $cl_\tau(A) \subseteq cl_\beta(A) \subseteq cl_\tau(A) \cup cl(x)$ for all $A \subseteq X$. In particular $cl_\beta(x) = cl(x)$.

Lemma 3.5 Let $\tau \in \mathcal{L}_\rho$. Given $x \in X$ and a filter \mathcal{F} on X , let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Then

(i) β is R_0 iff $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$ iff $\beta \in \mathcal{L}_\rho$.

(ii) If $\text{adh}_\tau \mathcal{F} = \text{cl}(x)$, then $\beta \in \mathcal{L}_\rho$.

Proof. (i) It is immediate that $\mathcal{F} \supseteq \mathcal{N}_{\text{at}(\rho)}(x)$ iff $\beta \in \mathcal{L}_\rho$, and that if $\beta \in \mathcal{L}_\rho$ then β is R_0 . On the other hand, if β is R_0 and $y \notin \text{cl}(x) = \text{cl}_\beta(x)$, then $x \notin \text{cl}_\beta(y)$. Thus $X \setminus \text{cl}_\beta(y) \in \mathcal{F}$, and this implies that $X \setminus \text{cl}(y) \in \mathcal{F}$. Since this holds for every $y \notin \text{cl}(x)$, it follows that $\mathcal{F} \supseteq \mathcal{N}_{\text{at}(\rho)}(x)$.

(ii) If $\text{adh}_\tau \mathcal{F} = \text{cl}(x)$ and $y \notin \text{cl}(x)$ then $y \notin \text{adh}_\tau \mathcal{F}$, and thus there exist $F \in \mathcal{F}$ and $V \in \mathcal{N}_\tau(y)$ such that $V \cap F = \emptyset$. Since $\text{cl}(y) \subseteq V$, then $F \subseteq X \setminus \text{cl}(y)$ and thus $X \setminus \text{cl}(y) \in \mathcal{F}$. Hence $\mathcal{F} \supseteq \mathcal{N}_{\text{at}(\rho)}(x)$. ■

Proposition 3.6 *Let $\tau \in \mathcal{L}_\rho$ be R_1 . Given $x \in X$ and a filter \mathcal{F} on X , then the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1 iff there exists a τ -open filter $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\text{adh}_\tau \mathcal{F}_0 = \text{cl}(x)$.*

Proof. (\Rightarrow) If $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1 , then $\text{adh}_\beta \mathcal{N}_\beta(x) = \text{cl}(x)$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$. Now, since $\beta \leq \tau$, then $\text{cl}(x) \subseteq \text{adh}_\tau \mathcal{N}_\beta(x) \subseteq \text{adh}_\beta \mathcal{N}_\beta(x) = \text{cl}(x)$. Let $\mathcal{F}_0 = \mathcal{N}_\beta(x) = \mathcal{N}_\tau(x) \cap \mathcal{F}$. It is clear that \mathcal{F}_0 is a τ -open filter contained in \mathcal{F} such that $\text{adh}_\tau \mathcal{F}_0 = \text{cl}(x)$.

(\Leftarrow) Suppose there exists a τ -open filter $\mathcal{F}_0 \subseteq \mathcal{F}$ such that $\text{adh}_\tau \mathcal{F}_0 = \text{cl}(x)$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$. To prove that $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is R_1 , let $y, z \in X$ such that $y \notin \text{cl}(z)$. We will show that y and z can be separated by β -open sets. Since τ is R_1 , there exist $W_y \in \mathcal{N}_\tau(y)$ and $W_z \in \mathcal{N}_\tau(z)$ such that $W_y \cap W_z = \emptyset$. We consider two possible cases.

Case (i). If $x \notin \text{cl}(y)$ and $x \notin \text{cl}(z)$, then $y, z \notin \text{cl}(x)$. Choose $V_y \in \mathcal{N}_\tau(y)$ and $V_z \in \mathcal{N}_\tau(z)$ such that $x \notin V_y$ and $x \notin V_z$. Let $O_y = W_y \cap V_y$ and $O_z = W_z \cap V_z$. Then $O_y, O_z \in \tau \cap \mathcal{E}(x) \leq \beta$ and $O_y \cap O_z = \emptyset$.

Case (ii). If $x \in \text{cl}(y)$, then $\text{cl}(y) = \text{cl}(x) = \text{adh}_\tau \mathcal{F}_0$. Since $z \notin \text{cl}(y)$, there exists $U \in \mathcal{N}_\tau(z)$ and $F \in \mathcal{F}_0$ such that $U \cap F = \emptyset$. Take $O_y = W_y \cup F$ and $O_z = W_z \cap U$. Then it is immediate that $O_y \in \tau \cap \mathcal{F}$ and $O_z \in \tau \cap \mathcal{E}(x)$. Thus O_y and O_z are disjoint β -neighborhoods of y and z , respectively. ■

Remark 3.7 *For any $x \in X$, the open filter $\mathcal{F} = \mathcal{N}_{\text{at}(\rho)}(x)$ satisfies $\text{adh}_{\bar{\rho}} \mathcal{F} = \text{cl}(x)$. In fact, $\text{cl}(y) = \ker(y) \in \mathcal{N}_{\bar{\rho}}(y)$ for each $y \in X$. Then, $y \in \text{cl}(x)$ implies that $x \in V$, for all $V \in \mathcal{N}_{\bar{\rho}}(y)$, and thus $y \in \text{adh}_{\bar{\rho}} \mathcal{N}_{\text{at}(\rho)}(x)$. On the other hand, if $y \notin \text{cl}(x)$ then the disjoint sets $\text{cl}(y) \in \mathcal{N}_{\bar{\rho}}(y)$ and $X \setminus \text{cl}(y) \in \mathcal{N}_{\text{at}(\rho)}(x)$ witness that $y \notin \text{adh}_{\bar{\rho}} \mathcal{N}_{\text{at}(\rho)}(x)$. Since $\bar{\rho}$ is R_1 , above proposition implies that $\beta = \bar{\rho} \cap (\mathcal{E}(x) \cup \mathcal{N}_{\text{at}(\rho)}(x))$ is R_1 and hence $\beta \in \mathcal{L}_\rho$. Moreover, β is strictly weaker than $\bar{\rho}$ since $\text{cl}(x) \in \bar{\rho}$ but $\text{cl}(x) \notin \mathcal{N}_{\text{at}(\rho)}(x)$. Therefore $\bar{\rho}$ is not the minimal R_1 topology in \mathcal{L}_ρ .*

We are now ready to prove a characterization of minimal R_1 in \mathcal{L}_ρ .

Theorem 3.8 *Let $\tau \in \mathcal{L}_\rho$ be R_1 . Then τ is minimal R_1 iff given any open filter \mathcal{F} on X such that $\text{adh}_\tau \mathcal{F} = \text{cl}(x)$ for some $x \in X$, then \mathcal{F} is convergent (necessarily to every point of $\text{cl}(x)$).*

Proof. Suppose τ is minimal R_1 and let \mathcal{F} be an open filter on X such that $\text{adh}_\tau \mathcal{F} = \text{cl}(x)$, for some $x \in X$. Let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. By Lemma 3.5(i), $\beta \in \mathcal{L}_\rho$ and, by Proposition 3.6, β is R_1 . Since τ is minimal R_1 in \mathcal{L}_ρ , it must be that $\beta = \tau$, and thus $\mathcal{F} \supseteq \mathcal{N}_\tau(x)$.

Conversely, suppose every open filter \mathcal{F} on X such that $\text{adh}_\tau \mathcal{F} = \text{cl}(x)$, for some $x \in X$, is τ -convergent and let $\tau' \in \mathcal{L}_\rho$ be an R_1 -topology such that $\tau' \leq \tau$. Let $V \in \tau$ and $x \in V$. Since $\text{adh}_{\tau'} \mathcal{N}_{\tau'}(x) = \text{cl}(x)$, the hypothesis implies that the τ -open filter $\mathcal{N}_{\tau'}(x)$ is τ -convergent to x . Thus $\mathcal{N}_{\tau'}(x) \supseteq \mathcal{N}_\tau(x)$, and hence $V \in \mathcal{N}_{\tau'}(x)$. Since this happens for all $x \in V$, then $V \in \tau'$. Therefore $\tau = \tau'$, and this implies that τ is minimal R_1 . ■

Since τ is minimal T_2 iff $\tau \in \mathcal{L}_1$ and is minimal R_1 , then Theorem 3.8 applied to any T_1 -topology ρ yields the following well known result on minimal T_2 .

Corollary 3.9 *Let X be an infinite set, and let $\tau \in LT(X)$ be T_2 . Then τ is minimal T_2 iff every open filter on X with a unique adherent point is convergent (to that point).*

Recall that $\tau \in LT(X)$ is said to be compact if every open cover of X has a finite subcover. Equivalently, τ is compact iff every filter on X has an adherent point iff every ultrafilter on X converges. ([19]). It is known that if τ is minimal T_2 then τ is regular iff it is compact ([19]). We will show that this last equivalence holds for minimal R_1 topologies in \mathcal{L}_ρ . The results given in the following lemma are well known. For the sake of completeness, we include the proofs.

Lemma 3.10 *Let $\tau \in LT(X)$.*

- (i) *If τ is R_1 and compact, then τ is regular.*
- (ii) *If τ is regular and every open filter on X has an adherent point, then τ is compact.*

Proof. (i) Let τ be R_1 and compact, and let $x \in V \in \tau$. Then for each $y \in X \setminus V$, there exist $U^y \in \mathcal{N}_\tau(x)$ and $V_y \in \mathcal{N}_\tau(y)$ such that $U^y \cap V_y = \emptyset$. Now, the family $\{V_y\}_{y \in X \setminus V}$ is an open cover of $X \setminus V$, a closed set and hence a compact set. Thus, $X \setminus V \subseteq \bigcup_{i=1}^n V_{y_i}$ for some finite collection $\{y_1, \dots, y_n\}$ of points in $X \setminus V$. Let $U = \bigcap_{i=1}^n U^{y_i}$. It is immediate that $U \in \mathcal{N}_\tau(x)$ and $\text{cl}(U) \subseteq V$, which shows that τ is regular.

(ii) Let τ be regular and such that every open filter on X has an adherent point. Given an ultrafilter \mathcal{R} on X , consider the open filter $\mathcal{F} = \mathcal{R} \cap \tau$. Then \mathcal{F} has an adherent point $x \in X$. Now, if \mathcal{R} does not converge to x , there exists $V \in \mathcal{N}_\tau(x)$ such that $V \notin \mathcal{R}$ and hence $X \setminus V \in \mathcal{R}$,

since \mathcal{R} is ultrafilter. By regularity of τ , one can choose $U \in \mathcal{N}_\tau(x)$ with $cl_\tau(U) \subseteq V$. Then $X \setminus cl_\tau(U) \supseteq X \setminus V$ and thus $X \setminus cl_\tau(U) \in \mathcal{R} \cap \tau = \mathcal{F}$. But, since $x \in adh_\tau \mathcal{F}$, it must be that $U \cap X \setminus cl_\tau(U) \neq \emptyset$, a contradiction. Thus \mathcal{R} converges to x , and therefore τ is compact. ■

Proposition 3.11 *Let $\tau \in \mathcal{L}_\rho$. If τ is minimal R_1 , then every open filter on X has an adherent point.*

Proof. Suppose there is an open filter \mathcal{F} on X such that $adh_\tau \mathcal{F} = \emptyset$. For each $x \in X$ there exist $V \in \mathcal{N}_\tau(x)$ and $F \in \mathcal{F}$ such that $V \cap F = \emptyset$. In particular $V \notin \mathcal{F}$. On the other hand, since $cl(x) \subseteq V$ then $X \setminus cl(x) \supseteq X \setminus V \supseteq F$, and thus $X \setminus cl(x) \in \mathcal{F}$. This shows that $\mathcal{F} \supseteq \mathcal{N}_{at(\rho)}(x)$ and $\mathcal{F} \not\supseteq \mathcal{N}_\tau(x)$, for each $x \in X$. Now, fix $x \in X$ and let $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$. Then β is a topology in \mathcal{L}_ρ which is strictly weaker than τ . We will prove that β is R_1 and thus τ is not minimal R_1 .

By Proposition 3.6, it is enough to show that \mathcal{F} contains an open filter \mathcal{F}_0 such that $adh_\tau \mathcal{F}_0 = cl(x)$. Let $\mathcal{F}_0 = \{F \in \mathcal{F} : F \cap V \neq \emptyset, \text{ for all } V \in \mathcal{N}_\tau(x)\}$. It is clear that \mathcal{F}_0 is an open non-empty proper sub-filter of \mathcal{F} and that $cl(x) \subseteq adh_\tau \mathcal{F}_0$. Now, let $y \notin cl(x)$. Since τ is R_1 , there exist $V \in \mathcal{N}_\tau(x)$ and $W \in \mathcal{N}_\tau(y)$ such that $V \cap W = \emptyset$. On the other hand, since $y \notin adh_\tau \mathcal{F}$, there exist $U \in \mathcal{N}_\tau(y)$ and $F \in \mathcal{F}$ such that $U \cap F = \emptyset$. If $O = W \cap U$ and $G = V \cup F$, then $O \in \mathcal{N}_\tau(y)$, $G \in \mathcal{F}_0$ and $O \cap G = \emptyset$. Thus $y \notin adh_\tau \mathcal{F}_0$ and therefore $adh_\tau \mathcal{F}_0 = cl(x)$. ■

Next result follows immediately from Lemma 3.10 and Proposition 3.11.

Theorem 3.12 *Let $\tau \in \mathcal{L}_\rho$ be minimal R_1 . Then τ is compact iff it is regular.*

We end this section with the characterization of minimal regular topologies in \mathcal{L}_ρ announced over the introduction of this paper.

Theorem 3.13 *Let $\tau \in \mathcal{L}_\rho$ be regular. Then τ is minimal regular iff every regular filter \mathcal{F} on X such that $adh_\tau \mathcal{F} = cl(x)$, for some $x \in X$, is convergent (necessarily to every point of $cl(x)$).*

Proof. (\Rightarrow) Let \mathcal{F} be a τ -regular filter on X such that $adh_\tau \mathcal{F} = cl(x)$, for some $x \in X$, and suppose \mathcal{F} does not converge. Then, there exists $U \in \mathcal{N}_\tau(x)$ such that $U \notin \mathcal{F}$, and hence $\beta = \tau \cap ((\mathcal{E}(x) \cup \mathcal{F})) \in \mathcal{L}_\rho$ is strictly weaker than τ . Note that $x \in F$ for all $F \in \mathcal{F}$. Otherwise $x \notin cl_\tau(F')$ for some $F' \in \mathcal{F}$ and hence $x \notin adh_\tau \mathcal{F}$, which contradicts the hypothesis that $adh_\tau \mathcal{F} = cl(x)$. We prove that β is regular and therefore τ is not minimal regular.

Let $V \in \beta$ and $y \in V$. If $y \in cl(x)$, then $V \in \mathcal{N}_\tau(x) \cap \mathcal{F} = \mathcal{F}$, a regular filter, and thus there exists $U \in \mathcal{F}$ such that $cl_\tau(U) \subseteq V$. Since $x \in U$, then $cl_\beta(U) = cl_\tau(U) \subseteq V$. Now, if $y \notin cl(x) = adh_\tau \mathcal{F}$, there exist $U' \in \mathcal{N}_\tau(y)$ and $F \in \mathcal{F}$ such that $U' \cap F = \emptyset$. Choose $U \in \mathcal{N}_\tau(y)$

such that $cl_\tau(U) \subseteq V$ (this is possible since τ is regular). If $W = U \cap U'$, then $cl_\tau(W) \cap F = \emptyset$ and thus $X \setminus cl_\tau(W) \in \mathcal{F}$. It follows that $cl_\beta(W) = cl_\tau(W) \subseteq cl_\tau(U) \subseteq V$.

(\Leftarrow) Suppose that every τ -regular filter on X for which the set of adherent points coincides with a point closure, is τ -convergent. Let $\tau' \in \mathcal{L}_\rho$ be a regular topology such that $\tau' \leq \tau$. Fix $V \in \tau$ and $x \in V$. It is clear that $cl(x) = adh_\tau \mathcal{N}_\tau(x) \subseteq adh_\tau \mathcal{N}_{\tau'}(x) \subseteq adh_{\tau'} \mathcal{N}_{\tau'}(x) = cl(x)$. Since $\mathcal{N}_{\tau'}(x)$ is a τ' -regular filter, then $\mathcal{N}_{\tau'}(x)$ is a τ -regular filter. By hypothesis, $\mathcal{N}_{\tau'}(x)$ is τ -convergent, *i.e.* $\mathcal{N}_\tau(x) \subseteq \mathcal{N}_{\tau'}(x)$. Since this holds for every $x \in V$, then $V \in \tau'$ and thus $\tau' = \tau$. Therefore τ is minimal regular in \mathcal{L}_ρ . ■

Corollary 3.14 *A regular and T_1 -topology on X is minimal regular iff every regular filter on X with a unique adherent point is convergent.*

Proof. Apply Theorem 3.13 to any T_1 -topology ρ . ■

4 Presober topologies in \mathcal{L}_ρ

In this last section we consider a topological property known as presoberty, which is strictly weaker than R_1 , and show that there are not minimal presober topologies in \mathcal{L}_ρ . As in previous section, we assume $\rho \in LT(X)$ is any R_0 -topology such that X can be written as infinite union of disjoint point closures.

Definition 4.1 *A non-empty closed subset C of X is said reducible if there are non-empty, proper closed subsets C_1, C_2 of C , such that $C = C_1 \cup C_2$. Otherwise C is irreducible. By convention, \emptyset is neither reducible nor irreducible.*

Every point closure is irreducible. If C is an irreducible closed set then it may be the case that it is the point closure of some point x . If so, x is called a generic point of C

Definition 4.2 *A topology is said to be presober iff each irreducible closed set has at least one generic point.*

In case that every irreducible closed subset of a space has a unique generic point, the topology is said to be *sobber*. Soberty is thus a combination of two properties: the existence of generic points and their uniqueness. It is straightforward to see that the generic points in a topological space are unique iff the space satisfies the T_0 separation axiom. Thus, a topology is sober precisely when it is T_0 and presober.

In any T_2 -topology, the irreducible closed sets are the singleton, so T_2 implies sobriety. The cofinite topology on an infinite set is an example of a T_1 -topology which is not sober, so it is also an example of a T_0 and not presober topology.

Proposition 4.3 *Every R_1 -topology $\tau \in LT(X)$ is presober.*

Proof. Let $\tau \in LT(X)$ be R_1 , and let $C \subseteq X$ be closed. Let $x, y \in C$ with $x \neq y$. Then $cl_\tau(x), cl_\tau(y) \subseteq C$. If $y \notin cl_\tau(x)$, there exist disjoint open sets $U \in \mathcal{N}_\tau(x)$ and $V \in \mathcal{N}_\tau(y)$ such that $cl_\tau(x) \subseteq U$ and $cl_\tau(y) \subseteq V$. Let $C_1 = C \cap X \setminus U$ and $C_2 = C \cap X \setminus V$. Then C_1, C_2 are non-empty proper closed subsets of C such that $C_1 \cup C_2 = C$, and thus C is reducible. It follows that an irreducible closed set must be a point closure, and hence τ is presober. ■

Presobriety does not imply R_1 , as the next example shows.

Example 4.4 *Let X be a set with cardinality ≥ 3 , and let $a, b \in X$ with $a \neq b$. Let τ be the topology $\{G \subseteq X : \{a, b\} \subseteq G\} \cup \{\emptyset\}$. So, a set C is closed iff $C \cap \{a, b\} = \emptyset$ or $C = X$. It is clear that every $x \notin \{a, b\}$ is closed. If C is non-empty, closed proper subset of X , then C is irreducible iff it is a singleton $x \notin \{a, b\}$, since otherwise $C = \{x\} \cup (C - \{x\})$ for any $x \in C$, and both $\{x\}$ and $C - \{x\}$ are closed and non-empty. Also X is itself irreducible since it is a point closure, $X = cl_\tau(a) = cl_\tau(b)$. Thus the irreducible closed sets are the point closures, and so τ is presober. But τ is not R_1 since given any $x \notin \{a, b\}$, then $cl_\tau(x)$ and $cl_\tau(a)$ can not be separated by disjoint open sets. Note that τ is an Alexandroff not T_0 topology on X .*

Proposition 4.5 *The presober property is expansive in \mathcal{L}_ρ (i.e. if $\tau \in \mathcal{L}_\rho$ is presober, then τ' is presober for all $\tau' \in \mathcal{L}_\rho$ finer than τ).*

Proof. Let $\tau \in \mathcal{L}_\rho$ be presober, and let $\tau' \in \mathcal{L}_\rho$ with $\tau' \leq \tau$. Given a not empty τ' -closed subset A of X , let $B = cl_\tau(A)$. If B is τ -reducible and F and G are two not empty, τ -closed, proper subsets of B such that $B = F \cup G$, then $F_1 = (A \cap F)$ and $G_1 = (A \cap G)$ are two not empty, τ' -closed, proper subsets of A such that $A = F_1 \cup G_1$. Hence A is τ' -reducible. On the other hand, if B is τ -irreducible there exists $b \in B$ such that $B = cl(b)$, since τ is presober. Note that $b \in A$. Otherwise, if a is any point of A , then $b \notin cl(a)$. Since τ is R_0 , one has that $cl(a) \cap cl(b) = \emptyset$ and hence $cl(a) \cap B = \emptyset$, which contradicts the fact that $A \subseteq B$. Thus $cl(b) = B \subseteq A$, and it follows that A is τ -closed. We have proved that the τ' -irreducible subsets of X are τ -irreducible, and thus a point closure. Therefore τ' is presober. ■

Since ρ is R_0 , then $\bar{\rho}$ is R_1 and thus it is presober. Therefore, there exists at least a presober member of \mathcal{L}_ρ . On the other hand, $at(\rho)$ is not presober since a proper subset of X is $at(\rho)$ -closed

iff it is finite union of disjoint point closure sets, and thus X is $at(\rho)$ -irreducible, but X is not a point closure. Thus, $at(\rho)$ is an example of an R_0 -topology which is not presober. We will prove that there are not minimal presober topologies in \mathcal{L}_ρ .

Given $\tau \in \mathcal{L}_\rho$, $x \in X$ and \mathcal{F} a filter on X , lets consider the topology $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$.

Lemma 4.6 *Let $\tau \in \mathcal{L}_\rho$ be presober, and let $A \subseteq X$ be β -closed. If A is τ -reducible then it is also β -reducible.*

Proof. Let $A \subseteq X$ be β -closed and τ -reducible, and let F and G be non-empty τ -closed proper subsets of A such that $A = F \cup G$. Then either $x \in A$ or $X \setminus A \in \mathcal{F}$. If $X \setminus A \in \mathcal{F}$ or $x \in F \cap G$ then F and G are β -closed and therefore A is β -reducible. Thus, we just need to consider the case when x belongs to only one of the sets F or G .

Suppose $x \in F \setminus G$ (the case $x \in G \setminus F$ is similar). Then, it is clear that F is β -closed. Moreover, since $x \notin G$ and since τ is R_0 , it must be that $cl(x) \cap G = \emptyset$ (if $y \in cl(x) \cap G$ then $x \in cl(y) \subseteq G$). Write $A = F \cup \{cl(x) \cup G\}$. If $F \setminus \{cl(x) \cup G\} \neq \emptyset$, then F and $cl(x) \cup G$ are non-empty β -closed proper subsets of A , and thus A is β -reducible. If $F \setminus \{cl(x) \cup G\} = \emptyset$, we distinguish the following cases:

(i) G is τ -irreducible. In this case $G = cl(g)$ for some $g \in G$, since τ is presober. Thus $A = cl(x) \cup cl(g)$ and therefore A is β -reducible.

(ii) G is τ -reducible. Then there exist G_1 and G_2 , non empty τ -closed proper subsets of G , such that $G = G_1 \cup G_2$. Write $A = (cl(x) \cup G_1) \cup (cl(x) \cup G_2)$. It is clear that A is β -reducible. ■

The following result is immediate consequence of Lemma 4.6.

Corollary 4.7 *Let $\tau \in \mathcal{L}_\rho$ be presober. Then every β -irreducible subset of X is also τ -irreducible.*

Proposition 4.8 *Let $\tau \in \mathcal{L}_\rho$ be presober, $x \in X$ and \mathcal{F} a filter on X . If $\mathcal{F} \supseteq N_{at(\rho)}(x)$ then $\beta = \tau \cap (\mathcal{E}(x) \cup \mathcal{F})$ is presober.*

Proof. If $\mathcal{F} \supseteq N_{at(\rho)}(x)$, then $\beta \in \mathcal{L}_\rho$ (Lemma 3.5(i)). Given a β -irreducible set A , then A is τ -irreducible (Corollary 4.7), and hence A is the τ -closure of a point, and thus the β -closure of a point. Therefore β is presober. ■

Proposition 4.9 *There are not minimal presober members of \mathcal{L}_ρ .*

Proof. Let $\tau \in \mathcal{L}_\rho$ be a presober topology. Since $at(\rho)$ can not be presober, there is $V \in \tau \setminus at(\rho)$. Let $y \in V$ and let $\beta = \tau \cap (\mathcal{E}(y) \cup N_{at(\rho)}(y))$. By Proposition 4.8, β is a presober topology which is obviously strictly weaker than τ . Therefore τ is not minimal presober. ■

Corollary 4.10 *There are not minimal (sober and T_1) topologies on an infinite set.*

Proof. Follows from Proposition 4.9 with ρ any T_1 -topology. ■

References

- [1] O.T. Alas, R.G. Wilson, *Minimal properties between T_1 and T_2* , Houston J. Math., 32(2)(2006), 493-504.
- [2] A. Bella, C. Constantini, *Minimal KC spaces are compact*, Topology Appl., 155 (13)(2008), 1426-1429.
- [3] M.P. Berri, *Minimal topological spaces*, Trans. Amer. Mat. Soc., Vol 108 (1)(1963), 97-105.
- [4] M.P. Berri, R.H. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc., Vol. 14 (3)(1963), 454-458.
- [5] M.P. Berri, Hon-Fei Lai, D.H. Pettey, *Non compact, minimal regular spaces*, Pacific J. Math., Vol. 69 (1)(1977), 19-23.
- [6] M.L. Colasante, C. Uzcátegui, J. Vielma, *Boolean algebras and low separation axioms*. Top. Proc., Vol. 34(2009), 1-15.
- [7] A.S. Davis, *Indexed systems and neighborhoods for general topological spaces*, Amer. Math. Monthly, (68)(1961), 886-893.
- [8] J. Dontchev, M. Ganster, G.J. Kennedy, S.D. McCartan, *On minimal Door, minimal anti-compact and minimal $T_{3/4}$ -spaces*, Math. Proc. Royal Irish Acad.. Vol. 98A (2)(1998), 209-215.
- [9] J.E. Joseph, *Characterizations of minimal Hausdorff Spaces*, Proc. Amer. Math. Soc., Vol. 61 (1)(1976), 145-148.
- [10] R.L. Larson, *Minimal T_0 -spaces and Minimal T_D -spaces*, Pacific J. Math., Vol. 31 (2)(1969), 451-458.
- [11] A.E. McCluskey, S.D. McCartan, *Minmality with respect to Young's axiom*, Houston J. Math. 21(2)(1995), 413-428.

- [12] J. Porter, J. Thomas, *On H-closed minimal Hausdorff spaces*, Trans. Amer. Math. Soc., Vol. 138 (1969), 159-170.
- [13] C.T. Scarborough, *Minimal Urysohn spaces*, Pacific J. Math., 3(1968), 611-617.
- [14] A.N. Shanin, *On separation in topological spaces*, Doklady SSSR, (38)(1943), 110-113.
- [15] M.K. Singal, Asha Mathur, *A note on minimal spaces*, Israel J. Math., Vol. 12(2)(1972), 204-206.
- [16] J. Stepráns, S. Watson, *Mutually complementary families of T_1 topologies, equivalence relations and partial orders*, Proc. Amer. Math. Soc., Vol. 123 (7)(1995), 2237-2249.
- [17] C. Uzcátegui, J. Vielma, *Alexandroff topologies viewed as closed sets in the Cantor cube*, Divulg. Mat., (13)(2005), 45-53.
- [18] G. Viglino, *A co-topological application to minimal spaces*, Pacific J. Math., Vol. 27(1)(1968), 197-200.
- [19] S. Willard, *General Topology*, Addison-Wesley Publishing Co., 1970.

MARÍA L. COLASANTE

Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: marucola@ula.ve

D. Van der ZYPEN

Allianz Suisse Insurance, Bleicherweg 19
CH-8022 Zurich, Switzerland
e-mail: dominic.zypen@gmail.com