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Some results on $T-$zamfirescu operators

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Abstract

The purpose of this paper is to obtain sufficient conditions for the existence of unique fixed point of $T-$zamficescu operators in complete metric spaces.

1 Introduction

Reciently, A. Beiranvand, S. Moradi, M. Omid and H. Pazadeh [1], introduced the notions of $T-$Banach contraction and $T-$contractive mapping, and then they extended the Banach contraction principle, (see [2]) and Edelstein's fixed point theorem [3], S. Moradi [4] introduced the $T-$Kannan contractive mapping and extended the Kannan's fixed point theorem [5]. Inspired and motived by the above said facts, the authors have introduced the motions of $T-$chatterjea contractive mapping [6] and the $T-$operator of Banach [8].

The purpose of this paper is to study the existence of fixed points for mapping S defined on a complete metric space such that is a $T-$zamfirescu operator.

2 Preliminaries

In the first, we recall some definitions.

Definition 2.1 *([1])*

Let (M,d) be a metric space and $T,S : M \longrightarrow M$ be two functions. A mappings S is said to be $T-$Banach contraction, (TB-Contraction), if there is $a \in [0,1)$ such that

$$d(TSx,TSy) \leq ad(Tx,Ty) \tag{2.1}$$

for all $x,y \in M$.

If we take $T = Id =$ Identity in (2.1) then we obtain the definition of Banach's contraction.

Definition 2.2 *([2])*

Let (M, d) be a metric space and $T, S : M \longrightarrow M$ be two functions. We say that S is a $T-$Kannan contraction, (TK - Contraction), if there is $b \in [0, 1/2)$ such that

$$d(TSx, TSy) \leq b[d(Tx, TSx) + d(Ty, TSy)] \tag{2.2}$$

for all $x, y \in M$.

If we take $T = Id$ then we get the definition given by Kannan [5].

Definition 2.3 *([6])*

Let (M, d) be a metric space and $T, S : M \longrightarrow M$ be two mappings. We say that S is a $T-$Chatterjea contraction, (TC - Contraction), if there is $c \in [0, 1/2)$ such that

$$d(TSx, TSy) \leq c[d(Tx, TSy) + d(Ty, TSx)] \tag{2.3}$$

for all $x, y \in M$.

If we take $T = Id$ then we obtain the chatterjea's definition [9].

Definition 2.4 *([7])*

Let (M, d) be a metric space and $T, S : M \longrightarrow M$ be two mappings. The function S is called a $T-$zamfirescu operator, (TZ - operator), if and only if there are real numbers, $0 \leq a < 1,\ 0 \leq b,\ c < 1/2$ such that for all $x, y \in M$ at least one condition is true:

$$\left.\begin{array}{rcl} TZ_1. - d(TSx, TSy) & \leq & ad(Tx, Ty) \\ TZ_2. - d(TSx, TSy) & \leq & b[d(Tx, TSx) + d(Ty, TSy)] \\ TZ_3. - d(TSx, TSy) & \leq & c[d(Tx, TSy) + d(Ty, TSx)] \end{array}\right\} \tag{2.4}$$

If we take $T = Id$ then we obtain the zamfirescu's definition [10].

Lemma 2.5 *Let (M,d) be a metric space and $T,S : M \longrightarrow M$ be two functions. Is S is a TZ - operator then there is $0 \leq \delta < 1$ such that*

$$d(TSx,TSy) \leq \delta d(Tx,Ty) + 2\delta d(Tx,TSx) \tag{2.5}$$

for all $x,y \in M$.

Proof: If S is a TZ - operator then at least one of (TZ_1), (TZ_2) or (TZ_3) is true. If (TZ_2) holds then

$$\begin{aligned} d(TSx,TSy) &\leq b[d(Tx,TSx) + d(Ty,TSy)] \\ &\leq b[d(Tx,TSx) + d(Ty,Tx) + d(Tx,TSx) + d(TSx,Ty)] \end{aligned}$$

since $0 \leq b < 1/2$ we obtain

$$d(TSx,TSy) \leq \frac{b}{1-b} d(Tx,Ty) + \frac{2b}{1-b} d(Tx,TSx) \tag{2.6}$$

If (TZ_3) holds, then similarly we get

$$d(TSx,TSy) \leq \frac{c}{1-c} d(Tx,Ty) + \frac{2c}{1-c} d(Tx,Tsx) \tag{2.7}$$

Therefore by denoting,

$$\delta = \max \left\{ a, \frac{b}{1-b}, \frac{c}{1-c} \right\}$$

we have $0 \leq \delta < 1$ and for all $x,y \in M$ the following inequality

$$d(Tsx,TSy) \leq \delta d(Tx,Ty) + 2\delta d(Tx,TSx) \tag{2.8}$$

holds. □

In a similar "maner" we obtain

$$d(TSx,TSy) \leq \delta d(Tx,Ty) + 2\delta d(Tx,TSy) \tag{2.9}$$

holds for all $x,y \in M$.

Example 2.6 1. *Let $M = [1, \infty) \subset \mathbb{R}$ be with the metric induced by $\mathbb{R} : d(x, y) = |x - y|$. We consider the functions $T, S : M \longrightarrow M$ defined by $Tx = \frac{1}{x} + 1,\ Sx = 2x,\ x \in M$. Then*

1.1.- *It is clear that S is not a Banach contraction.*

1.2.- *S is TB - Contraction because:*

$$\begin{aligned} d(TSx, TSy) &= |TSx - TSy| = \left|\frac{1}{2x} + 1 - \frac{1}{2y} - 1\right| \\ &= \frac{1}{2}\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{1}{2}\left|\frac{1}{x} - 1 - \frac{1}{y} - 1\right| = \frac{1}{2}|Tx - Ty| \leq ad(Tx, Ty) \end{aligned}$$

where $a = 1/2 < 1$.

2. *Let $M = [0, 1] \subset \mathbb{R}$ be with the metric induced by $\mathbb{R} : d(x, y) = |x - y|$. We consider the functions $T, S : M \longrightarrow M$ defined by $Tx = x^2$ and $Sx = \frac{x}{2},\ x \in M$. Then*

2.1.- *It is clear that S is a Banach contraction.*

2.2.- *S is not a $K-$contraction since*

$$\begin{aligned} d(Sx, S0) &= |Sx - S0| = \frac{x}{2} \\ d(x, Sx) &= |x - Sx| = \frac{x}{2} \\ d(0, S0) &= 0. \end{aligned}$$

Therefore,

$$\begin{aligned} d(Tx, S0) &= \frac{x}{2} = |x - Sx| + |0 - S0| \\ &= d(x, Sx) + d(0, S0). \end{aligned}$$

2.3.- *S is a TK - contraction because:*

$$\begin{aligned} d(TSx, TSy) &= |TSx - TSy| = \left|\frac{x^2}{4} - \frac{y^2}{4}\right| \\ &\leq \frac{1}{3}\Big[|Tx - TSx| + |y - TSy|\Big] \\ &= \frac{1}{3}\Big(d(Tx, TSx) + d(y, TSy)\Big). \end{aligned}$$

2.4.- *In a similar maner we see that S is a TC - Contraction.*

3. *Let $M = [0,1]$ be with the metric induced by $\mathbb{R} : d(x,y) = |x-y|$. We consider the functions $T, S : M \longrightarrow M$ defined by $Tx = \sqrt{x}$ and $Sx = x^2,\ x \in M$. Then*

3.1.- *It is clear that S is not a Banach contraction.*

3.2.- *S satisfies the condition (2.8) with $\delta \in (1/3, 1)$ since*

$$d(TSx, TSy) = |TSx - TSy| = |x-y| \leq \delta|x-y| + 2\delta|x-y|$$

□

Definition 2.7 *([1])*

Let (M,d) be a metric space and $T : M \longrightarrow M$.

1. *The function T is said sequentially convergent if we have, for every sequence (y_n), if $T(y_n)$ is convergent then (y_n) is also convergent.*
2. *The function T is said subsequentially convergent if we have, for every sequence (y_n), if $T(y_n)$ is convergent then (y_n) has a convergent subsequence.*

3 Main results

We introduce the following, let (M,d) be a metric space, $x_0 \in M$ and $T, S : M \longrightarrow M$ two mappings. The sequence $(Tx_n) \subset M$ defined by

$$Tx_{n+1} = TSx_n = TS^n x_0, \qquad n = 0, 1, \ldots \tag{3.10}$$

is called the $T-$Picard iteration associated to S.

Theorem 3.1 *Let (M,d) be a complete metric space and $T, S : M \longrightarrow M$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is a TZ - operator then S has a unique fixed point. Moreover, if T is sequentially convergent then for every $x_0 \in M$ the $T-$Picard iteration associated to S, $(TS^n x_0)$ converges to $T(z_0)$, where z_0 is the fixed point of S.*

Proof: Since S is a TZ - operator then by lemma 2.5 there is $0 < \delta < 1/3 < 1$ such that

$$d(TSx, TSy) \leq \delta d(Tx, Ty) + 2\delta d(Tx, TSx) \tag{3.11}$$

for all $x, y \in M$.

Now let $(Tx_n) \subset M$ the $T-$Picard iteration associated to S defined by (3.11) and $x_0 \in M$ arbitrary. Then

$$d(TS^{n+1}x_0, TS^n x_0) \leq hd(TS^n x_0, TS^{n-1}x_0) \tag{3.12}$$

where $\quad h = \dfrac{\delta}{1-2\delta} < 1$. Therefore, for all n,

$$d(TS^{n+1}x_0, TS^n x_0) \leq h^n d(TSx_0, Tx_0) \tag{3.13}$$

From (3.13) we get

$$\lim_{n\to\infty} d(TS^{n+1}x_0, TS^n x_0) = 0 \tag{3.14}$$

Now, for $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} d(TS^m x_0, TS^n x_0) &\leq (h^n + \ldots + h^{m-1}) d(TSx_0, Tx_0) \\ &\leq \frac{h^n}{1-h} d(TSx_0, Tx_0) \end{aligned} \tag{3.15}$$

From (3.15) we obtain,

$$\lim_{n,m\to\infty} d(TS^m x_0, TS^n x_0) = 0 \tag{3.16}$$

and hence $(TS^n x_0) \subset M$ is a Cauchy sequence in M and thus there is $y_0 \in M$ such that

$$\lim_{n\to\infty} TS^n x_0 = y_0 \tag{3.17}$$

since T is subsequentially convergent, $(S^n x_0)$ has a convergent subsequence, so there is a $z_0 \in M$ such that

$$\lim_{k\to\infty} S^{n_k}x_0 = z_0 \tag{3.18}$$

Now, using the continuity of T and (3.18) we have,

$$\lim_{k\to 0} TS^{n_k}x_0 = Tz_0 \tag{3.19}$$

From (3.17) and (3.19) we obtain $y_0 = Tz_0$. So

$$\begin{aligned} d(TSz_0, Tz_0) &\leq d(TSz_0, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) \\ &+ d(TS^{n(k)+1}x_0, Tz_0) \longrightarrow 0, \qquad (k \to \infty). \end{aligned}$$

Therefore, $d(TSz_0, Tz_0) = 0$. Since T is one to one $Sz_0 = z_0$ so S has a fixed point.

Since (3.11) holds and T is one to one, S has a unique fixed point.

Finally, if T is sequentially convergent, by replacing (h) with (n_k) we conclude that

$$\lim_{n\to\infty} S^n x_0 = z_0 \qquad \text{and} \qquad \lim_{n\to\infty} TS^n x_0 = Tz_0.$$

□

If we take $T = Id$ then we obtain the zamfirescu's fixed point [10]. Since the $T-$Kannan's and $T-$Chatterjea's contractive conditions are both included in the class of TZ - operators, by theorem 3.1, we obtain inmediately the following results,

Corollary 3.2 *Let (M, d) be a complete metric space and $T, S : M \longrightarrow M$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is TK - contraction then S has a unique fixed point, said $z_0 \in M$. Moreover, if T is sequentially convergent then for every $x_0 \in M$, $(S^n x_0)$ converges to z_0 and $TS^n x_0 \longrightarrow Tz_0$. If we take $T = Id$ we get the Kannan's fixed point theorem [5].*

Corollary 3.3 *Let (M, d) be a complete metric space and $T, S : M \longrightarrow M$ be two mappings such that T is continuous, one to one and subsequentially convergent. If S is TC - contraction then S has a unique fixed point, said $z_0 \in M$. Moreover, if T is sequentially convergent then for every $x_0 \in M$, $(S^n x_0)$ converges to z_0 and $TS^n x_0 \longrightarrow Tz_0$.*

If we take $T = Id$ we obtain the chtterjea's fixed point theorem [9].

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