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On some extensions of sequential topologies

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Resumen

Dado un ideal $\mathcal{I}$ y una topología τ sobre un conjunto X, existe una extensión natural $\tau_{\mathcal{I}}$ de τ asociada al ideal $\mathcal{I}$ que generaliza la extensión secuencial de τ. En este trabajo mostraremos algunas propiedades de $\tau_{\mathcal{I}}$. Uno de los resultados principales dice que esta extensión preserva primero numerabilidad cuando el ideal $\mathcal{I}$ es un p-ideal analítico.

Palabras clave: Espacios Frechet, espacios secuenciales, extensión secuencial, p-ideales.

Abstract

Given an ideal $\mathcal{I}$ and a topology τ over a set X, there is a natural extension $\tau_{\mathcal{I}}$ of τ associated to $\mathcal{I}$ which generalizes the sequential extension of τ. In this note we show some properties of $\tau_{\mathcal{I}}$. One the main results is that this extension preserves first countability when the ideal $\mathcal{I}$ is an analytic p-ideal.

key words. Frechet space, sequential space, sequential extension, p-ideal.

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1 Introduction

A topological space X is *Frechet* if for every $A \subseteq X$ and every $x \in \overline{A}$ there is a sequence $(x_n)_n$ in A converging to x. It is said to be *sequential* if every $A \subseteq X$ containing the limit of all its convergent sequence is actually closed. Every Frechet space is sequential but the converse is not true. The following generalization of sequentiality was introduced in [7]. Let $\mathcal{I}$ be an ideal over X containing all finite subsets of X and τ a topology over X. We say that τ is *weakly generated* by $\mathcal{I}$ if for all $A \subseteq X$ such that $cl_\tau(E) \subseteq A$ for all $E \subseteq A$ with $E \in \mathcal{I}$, then A is closed. If $\mathcal{I}$ is the ideal generated by the τ-convergent sequences, this notion corresponds to sequentiality. Analogously, we say that τ is *generated by* $\mathcal{I}$ if for every $A \subseteq X$, if $x \in cl_\tau(A)$, then there is $E \subseteq A$ in $\mathcal{I}$ such that $x \in cl_\tau(E)$. That is to say,

$$\overline{A} = \bigcup \{\overline{E} : \ E \subseteq A \text{ and } E \in \mathcal{I} \ \}.$$

In this case, one could say that X is $\mathcal{I}$-tight. If $\mathcal{I}$ is the ideal generated by the discrete subsets of X, then we to obtain the so called discretely generated spaces [2].

In this note we continue the work started in [7] where it was shown that for every topology τ in X and every ideal $\mathcal{I}$ there is a smallest topology $\tau_{\mathcal{I}}$ such that $\tau \subseteq \tau_{\mathcal{I}}$ and $\tau_{\mathcal{I}}$ is weakly generated by $\mathcal{I}$. Since the ideal $\mathcal{I}$ used will be clear from the context, we will denote $\tau_{\mathcal{I}}$ by $\overline{\tau}$. We call $\overline{\tau}$ the $\mathcal{I}$-closure of τ. This topology is characterized as follows:

$$A \text{ is } \overline{\tau}\text{-closed iff } \forall E\, [\, E \in \mathcal{I} \;\;\&\;\; E \subseteq A \Rightarrow cl_{\tau}(E) \subseteq A]. \tag{1}$$

One of the main result presented here says that if τ is a first countable T_1 topology over a countable set X and $\mathcal{I}$ is an analytic p-ideal, then $\overline{\tau}$ is first countable.

2 Terminology

A space is said to have the *diagonal sequence property* if whenever $(x_n^m)_n$ is a sequence converging to x for all m, there is a sequence of natural numbers k_n such that $(x_n^{k_n})_n$ converges to x. We recall that a subset $A \subseteq X$ is called *sequentially closed* if every convergent sequence of elements of A has the limit inside A. Thus a space is sequential iff every sequentially closed set is closed.

An *ideal* over a set X is a collection of subsets of X closed under finite unions and containing the subsets of its elements. An ideal $\mathcal{I}$ is called a *p-ideal* if for every sequence $E_n \in I$, with $n \in \mathbb{N}$, there is $E \in \mathcal{I}$ such that $E_n \setminus E$ is finite for all $n \in \mathbb{N}$. A typical example of a p-ideal over $\mathbb{N} \times \mathbb{N}$ is the collection $\emptyset \times \mathsf{Fin}$ of all $A \subseteq \mathbb{N} \times \mathbb{N}$ such that every vertical section of A is finite. And a typical non p-ideal is the collection $\mathsf{Fin} \times \emptyset$ of all $A \subseteq \mathbb{N} \times \mathbb{N}$ such that only finitely many vertical sections of A are not empty. An ideal $\mathcal{I}$ over X is called *tall* if for every infinite $A \subseteq X$, there is an infinite $E \subseteq A$ with $E \in \mathcal{I}$.

A collection $\mathcal{A}$ of subsets of a countable set X (like a topology or an ideal) is called *analytic* if there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to 2^X$ such $f[\mathbb{N}^{\mathbb{N}}] = \mathcal{A}$ [3]. Analytic topologies were studied in [5, 6]. $\emptyset \times \mathsf{Fin}$ and $\mathsf{Fin} \times \emptyset$ are example of analytic (in fact, Borel) ideals. We follow the notation for Borel as in [3].

3 Some properties of $\mathcal{I}$-closures

In this section we will analyze the $\mathcal{I}$-closure of sequential spaces. We recall from [7] that $cl_{\tau}(E) = cl_{\overline{\tau}}(E)$ for all $E \in \mathcal{I}$. Another property of $\overline{\tau}$ that will be used is that if $(x_n)_n$ is a sequence with range in $\mathcal{I}$, then $(x_n)_n$ is τ-convergent to x iff it is $\overline{\tau}$-convergent to x.

We start showing that for a certain class of ideal the $\mathcal{I}$-closure of a sequential topology does not change the topology.

Theorem 3.1 *Let (X, τ) be a sequential space. The following are equivalent*

(i) (X, τ) is a Frechet space.

(ii) τ is generated by any tall ideal.

Proof: Suppose (X, τ) is Frechet and $\mathcal{I}$ is a tall ideal. Let $A \subseteq X$ and $x \in cl_\tau(A) \setminus A$. Then there is a sequence $x_n \in A$ converging to x. Let B be the range of $(x_n)_n$. Since $\mathcal{I}$ is tall, there is $E \subseteq B$ infinite with $E \in \mathcal{I}$. Then $x \in cl_\tau(E)$ and thus τ is generated by $\mathcal{I}$.

Suppose now that (X, τ) is not Frechet. Let $A \subseteq X$ and $x \in cl_\tau(A)$ such that no sequence in A converges to x. Let $\mathcal{I}$ be the ideal generated by all τ-convergent sequences together with the τ-closed discrete subsets of X. Then $\mathcal{I}$ is tall. In fact, let $B \subseteq X$ be infinite. Since τ is sequential, then either B contains a (non eventually constant) convergent sequence or it is closed discrete. It is clear that there is no $E \in \mathcal{I}$ such that $E \subseteq A$ and $x \in cl_\tau(E)$. Hence τ is not generated by $\mathcal{I}$. $\square$

The following result can be analogously proved.

Theorem 3.2 *Let τ be a sequential topology, then τ is weakly generated by any tall ideal.*

Example 3.3 Let X be a convergent sequence and $\mathcal{I}$ be (an isomorphic copy) of $\mathsf{Fin} \times \emptyset$. Clearly $\mathcal{I}$ is not tall. X is Frechet but it is not generated by $\mathcal{I}$ as $\overline{\tau}$ is the sequential fan.

Theorem 3.4 *Suppose τ is sequential and $cl_\tau(E) \in \mathcal{I}$ for all $E \in \mathcal{I}$, then $\overline{\tau}$ es sequential.*

Proof: For a topology ρ over X, let $[A]_\rho$ denote the ρ-sequential closure of A, that is to say, the smallest ρ-sequential closed set containing A. Since $\tau \subseteq \overline{\tau}$, then for every $A \subseteq X$,

$$A \subseteq [A]_{\overline{\tau}} \subseteq [A]_\tau.$$

We claim

$$[E]_{\overline{\tau}} = [E]_\tau$$

for every $E \in \mathcal{I}$. In fact, it suffices to show that $[E]_{\overline{\tau}}$ is τ-sequentially closed. To see this, notice first that $[E]_{\overline{\tau}} \in \mathcal{I}$ because $[E]_{\overline{\tau}} \subseteq cl_{\overline{\tau}}(E) = cl_\tau(E)$ and we are assuming that $cl_\tau(E) \in \mathcal{I}$.

Therefore every sequence in $[E]_{\overline{\tau}}$ which is τ-convergent, is also $\overline{\tau}$-convergent. Thus $[E]_{\overline{\tau}}$ is τ-sequentially closed.

To show that $\overline{\tau}$ is sequential, let $A \subseteq X$ be a $\overline{\tau}$-sequentially closed set. To see that A is $\overline{\tau}$-closed, fix $E \subseteq A$ in $\mathcal{I}$. We need to show that $cl_\tau(E) \subseteq A$. Since τ is sequential, then $[E]_\tau = cl_\tau(E)$. Thus $cl_\tau(E) = [E]_{\overline{\tau}}$. Since A is $\overline{\tau}$-sequentially closed, then $[E]_{\overline{\tau}} \subseteq A$ and we are done. □

Theorem 3.5 *Let τ be a T_1 topology and $\mathcal{I}$ be an ideal over X. If τ is Frechet, then $\overline{\tau}$ is sequential. Moreover, if $\mathcal{I}$ is a p-ideal, then $\overline{\tau}$ is Frechet and generated by $\mathcal{I}$.*

Proof: Suppose τ is Frechet and let $A \subseteq X$ be $\overline{\tau}$-sequentially closed. Let $E \subseteq A$, $E \in \mathcal{I}$ and $x \in cl_\tau(E)$. Since τ is Frechet, there is a sequence $(x_n)_n$ in E τ-converging to x. Since the range of $(x_n)_n$ belongs to $\mathcal{I}$, then $(x_n)_n$ converges to x also with respect to $\overline{\tau}$. Hence $x \in A$ and therefore $cl_\tau(E) \subseteq A$. Thus A is $\overline{\tau}$-closed.

Suppose now that $\mathcal{I}$ is a p-ideal. Let $A \subseteq X$. To show that $\overline{\tau}$ is generated by $\mathcal{I}$, it suffices to show that the following set is $\overline{\tau}$-closed:

$$B = \bigcup \{cl_\tau(E) : \; E \subseteq A \;\&\; E \in \mathcal{I}\}.$$

Since $\overline{\tau}$ is sequential, it suffices to show that B is $\overline{\tau}$-sequentially closed. Let $x_n \in B$ be a sequence $\overline{\tau}$-converging to x. For each n, fix $E_n \in \mathcal{I}$ such that $E_n \subseteq A$ and $x_n \in cl_\tau(E_n)$. Since $\mathcal{I}$ is a p-ideal, there is $E \in \mathcal{I}$ such that $E_n \setminus E$ is finite for all n. It is clear that we can assume without lost of generality that $E \subseteq A$. As $x_n \in cl_\tau(E)$ for all n, then $x \in cl_\tau(E)$. Since $E \in \mathcal{I}$, we conclude that $x \in B$.

To see that $\overline{\tau}$ is Frechet, let $x \in cl_{\overline{\tau}}(A)$. Since $\overline{\tau}$ is generated by $\mathcal{I}$, there is $E \subseteq A$ such that $x \in cl_\tau(E)$. As τ is Frechet, there is $(x_n)_n$ in E which is τ-convergent to x. Since $E \in \mathcal{I}$, then $(x_n)_n$ is also $\overline{\tau}$-convergent. □

The proof of theorem 3.5 also says the following:

Theorem 3.6 *Let $\mathcal{I}$ be a p-ideal and τ be a T_1 topology over X. If τ is weakly generated by $\mathcal{I}$, then τ is generated by $\mathcal{I}$.*

Proof: It suffices to show that the following set is τ-closed:

$$B = \bigcup \{cl_\tau(E) : \; E \subseteq A \;\&\; E \in \mathcal{I}\}.$$

We have to show that if $E \subseteq B$ is in $\mathcal{I}$, then $cl_\tau(E) \subseteq B$. In fact, let $y \in cl_\tau(E)$ and let $(y_n)_n$ be an enumeration of E. For each n, fix $E_n \subseteq A$ in $\mathcal{I}$ such that $y_n \in cl_\tau(E_n)$. Since $\mathcal{I}$ is a p-ideal, there is F in $\mathcal{I}$ such that $E_n \subseteq^* F$. It is clear that $E_n \subseteq^* F \cap A$ and thus we assume that $F \subseteq A$. It is also clear that $y_n \in cl_\tau(F)$ (here we use that τ is T_1) for all n and thus $y \in cl_\tau(F)$ and hence $y \in B$.

□

We will say that an ideal $\mathcal{I}$ is a *s-ideal with respect to τ*, if each $E \in \mathcal{I}$ is a subset of a finite union of (the ranges of) τ-convergent sequences. The following result shows that $\overline{\tau}$ is a generalization of the sequential extension of a topology.

Proposition 3.7 *Let τ be a topology and $\mathcal{I}$ a s-ideal with respect to τ. Then $\overline{\tau}$ is sequential.*

Proof: Let A be a non $\overline{\tau}$-closed set. Then there is $E \subseteq A$ with $E \in \mathcal{I}$ such that $cl_\tau(E) \not\subseteq A$. Since E is a union of finite many τ-convergent sequences then there is a sequence $(x_n)_n$ in E τ-converging outside A. Since the range of $(x_n)_n$ belongs to $\mathcal{I}$, then $(x_n)_n$ is $\overline{\tau}$-convergent.

□

Example 3.8 Given an ideal $\mathcal{I}$, let $s(\mathcal{I})$ be the ideal generated by the τ-convergent sequences with range in $\mathcal{I}$. Thus $s(\mathcal{I}) \subseteq \mathcal{I}$. It is obvious that $s(\mathcal{I})$ is a s-ideal with respect to τ. Therefore the $s(\mathcal{I})$-closure of τ is sequential.

Example 3.9 Let $\mathcal{I}$ be $\mathsf{Fin} \times \emptyset$ and (X, τ) a first countable space. By 3.5 we know $\overline{\tau}$ is sequential. We will present two examples. One showing that $\overline{\tau}$ can be non Frechet and another showing that it can have sequential rank ω_1.

Let

$$X = \{1/n + 1/m : \ n, m \geq 1\} \cup \{1/n : \ n \geq 1\} \cup \{0\}$$

as a subspace of $\mathbb{R}$. Let $S_n = \{1/n + 1/m : \ m \geq 1\} \cup \{1/n\}$ for $n \geq 1$ and $S_0 = \{1/n : \ n \geq 1\} \cup \{0\}$. We will view X as $\mathbb{N} \times \mathbb{N}$ by identifying S_n with $\{n\} \times \mathbb{N}$. Then $(X, \overline{\tau})$ is homeomorphic to the Arens' space. The fact that the Arens' space is not Frechet says that $(X, \overline{\tau})$ is not generated by $\mathcal{I}$.

Now consider $\mathbb{Q}$ the rationals with the usual order topology. For each $r \in \mathbb{Q}$, fix a sequence S_r in $\mathbb{Q}$ converging to r and w.l.o.g assume that the S_r's are pairwise disjoint. Identify $\mathbb{N} \times \mathbb{N}$ with $\mathbb{Q}$ in such a way that each vertical line $\{n\} \times \mathbb{N}$ corresponds to one of the convergent sequences

S_r (put also r in $\{n\} \times \mathbb{N}$ if r does not belong to a line $\{m\} \times \mathbb{N}$ with $m < n$). Notice $\mathcal{I}$ is isomorphic to the ideal generated by S_r with $r \in \mathbb{Q}$. Then $(\mathbb{Q}, \overline{\tau})$ is homeomorphic to the Arkhanglel'skiĩ-Franklin space S_ω [1].

□

4 Closure of first countable spaces

In this section we show the following result.

Theorem 4.1 *Let τ be a first countable T_1 topology over a countable set X. If $\mathcal{I}$ is an analytic p-ideal, then $\overline{\tau}$ is first countable.*

We show first a lemma.

Lemma 4.2 *Let τ be a topology with the diagonal sequence property and $\mathcal{I}$ a p-ideal. Then $\overline{\tau}$ has the diagonal sequence property.*

Proof: Let $S_n = (x_{nk}) \to_k x$ respect to $\overline{\tau}$ for all n. Then each S_n has a subsequence with range in $\mathcal{I}$, so we can assume without loss of generality that $E_n = \{x_{nk} : k \geq 1\} \in \mathcal{I}$. Consider now the sequences $S_n^m = \{x_{nk} : k \geq m\}$ with $n, m \in \mathbb{N}$. Since τ has the diagonal sequence property, then there is a sequence S τ-converging to x such that $S \cap S_n^m \neq \emptyset$ for all $n, m \in \mathbb{N}$. Observe that $S \cap S_n$ is infinite for all n. Since $\mathcal{I}$ is a p-ideal, there is $E \in \mathcal{I}$ such that $E_n \setminus E$ is finite for all n. Therefore $S \cap S_n \cap E$ is infinite. Finally, let T be $S \cap E$. Then T is a sequence converging to x with range in $\mathcal{I}$ and $T \cap S_n \neq \emptyset$ for all n.

□

Remark 4.3 The previous result is not true for an arbitrary ideal. In fact, the closure of a first countable space does not necessarily have the diagonal sequence property: For example, the closure of a convergent sequence with respect to the ideal $\mathsf{Fin} \times \emptyset$ is the sequential fan. A natural question is to determine when the weak diagonal sequence property is preserved under the $\mathcal{I}$-closure operation.

Proof of 4.1: We recall some terminology from [4]. The orthogonal $\mathcal{I}^{\perp}$ of an ideal $\mathcal{I}$ is the collection of those $B \subseteq X$ such that $B \cap E$ is finite for all $E \in \mathcal{I}$.

From 3.5 and 4.2 we know that $\overline{\tau}$ is Frechet and has the diagonal sequence property. By theorem 6.6 of [5] it suffices to show that $\overline{\tau}$ is analytic. Fix $x \in X$ and let V_n be a local base at x for τ such that V_n is decreasing. Let $A \subseteq X$ with $x \in cl_\tau(A) \setminus A$. We claim

$$x \in cl_{\overline{\tau}}(A) \iff \forall n\, (A \cap V_n \notin \mathcal{I}^\perp) \tag{2}$$

To see (2), first suppose that $x \in cl_{\overline{\tau}}(A)$. By 3.5 $\overline{\tau}$ is generated by $\mathcal{I}$, therefore there is $E \subseteq A$ in $\mathcal{I}$ such that $x \in cl_\tau(E)$. Thus $E \cap V_n$ is infinite for all n and thus $A \cap V_n \notin \mathcal{I}^\perp$. Conversely, suppose that for all n there is an infinite $E_n \subseteq A \cap V_n$ in $\mathcal{I}$. Since $\mathcal{I}$ is a p-ideal, there is $E \in \mathcal{I}$ such that $E_n \setminus E$ is finite for all n. As $E_n \subseteq A$, we can assume that $E \subseteq A$. Since $E \cap V_n$ is infinite for all n then $x \in cl_\tau(E)$. Hence $x \in cl_{\overline{\tau}}(A)$.

Since $\mathcal{I}$ is an analytic p-ideal, then its orthogonal is countable generated [4], in particular, $\mathcal{I}^\perp$ is F_σ. It follows from (2) that $\{A \subseteq X : \; x \in cl_{\overline{\tau}}(A)\}$ is Borel for all $x \in X$ and therefore $\overline{\tau}$ is also Borel.

□

In general the $\mathcal{I}$-closure of a metrizable space is not metrizable even when $\mathcal{I}$ is a p-ideal (see example 4.5 below). In the case of a countable space the question reduces to know when the closure of a metrizable space is regular. The following proposition is related to this problem.

Proposition 4.4 *Suppose $\mathcal{I}$ contains a τ-dense set. Then*

(i) τ and $\overline{\tau}$ have the same clopen sets.

(ii) If in addition, X is countable and $\overline{\tau}$ is regular, then $\tau = \overline{\tau}$.

Proof: (i) Let $E \in \mathcal{I}$ be a τ-dense set. Since $cl_\tau(E) = cl_{\overline{\tau}}(E)$ then E is also $\overline{\tau}$-dense. It suffices to show that $cl_\tau(W) = cl_{\overline{\tau}}(W)$ for all $W \in \overline{\tau}$. Fix $W \in \overline{\tau}$, then

$$cl_{\overline{\tau}}(W) = cl_{\overline{\tau}}(W \cap E) = cl_\tau(W \cap E) = cl_\tau(W).$$

To see the last equality observe: if $x \in cl_\tau(W)$ and $x \in V \in \tau$, then $\emptyset \neq V \cap W \in \overline{\tau}$. Therefore $V \cap W \cap E \neq \emptyset$, as E is $\overline{\tau}$-dense. Hence $x \in cl_\tau(W \cap E)$.

Clearly (ii) follows from (i) as every countable regular space is necessarily zero-dimensional.

□

Example 4.5 Suppose $(\mathbb{N} \times \mathbb{N}, \tau)$ is a copy of $\mathbb{Q}$ in $\mathbb{N} \times \mathbb{N}$ such that at least one vertical line is not τ-closed discrete and one horizontal line is τ-dense. Let $\mathcal{I}$ be $\emptyset \times \mathsf{Fin}$. Notice that every vertical line is $\overline{\tau}$-closed discrete and every horizontal line belongs to $\mathcal{I}$. Therefore $\tau \neq \overline{\tau}$ and by the previous proposition $\overline{\tau}$ is not regular.

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