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In search of traces of some holomorphic spaces on polyballs

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Resumen

Consideramos que una nueva traza problema para las funciones holomorfas en las pelotas de poli que completamente generalizar una conocido problema mapa diagonal para el disco poli y dar una descripción completa de las huellas de ciertas clases holomorfa en bolas producto definido con la ayuda del operador de la zona Luzin o una pelota de Bergman métricas.

Palabras claves: función holomorfa, bola Bergman métricas, el operador de la zona Luzin, derivado fraccionada, el problema de seguimiento.

Abstract

We consider a new trace problem for functions holomorphic on poly balls which completely generalize a known diagonal map problem for poly disk and give complete description of traces of certain holomorphic classes on product balls defined with the help of Luzin area operator or Bergman metric ball.

key words. holomorphic function, Bergman metric ball, Luzin area operator, fractional derivative, Trace problem

AMS(MOS) subject classifications. [2000]31B05, Primary 32A18

1 Introduction

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n. The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B} = \{z \in \mathbb{C}^n | |z| < 1\}$. The boundary of$\mathbf{B}$ will be denoted by $\mathbf{S}$, $\mathbf{S} = \{z \in \mathbb{C}^n | |z| = 1\}$.As usual, we denote by $H(\mathbf{B})$ the class of all holomorphic functions on $\mathbf{B}$.

For $0 < p < \infty$ we define the Hardy space $H^p(\mathbf{B}^n)$ consist of holomorphic functions f in $\mathbf{B}^n$ such that $\|f\|_p^p = \sup_{0<r<1} \int_{\mathbf{S}^n} |f(r\xi)|^p d\sigma(\xi) < \infty$. Here $d\sigma$ denotes the surface measure on $\mathbf{S}^n$

normalized so that $\sigma(\mathbf{S}^n)=1$. For every function $f\in H(\mathbf{B})$ having a series expansion$f(z)=\sum_{|k|\geq 0}a_kz^k$, we define the operator of fractional differentiation by

$$D^{\alpha}f(z)=\sum_{|k|\geq 0}(|k|+1)^{\alpha}a_kz^k,$$

where α is any real number.It is obvious that for any α, D^{α} is an operator acting from $H(\mathbf{B})$ to $H(\mathbf{B})$.For a fixed $\alpha>1$ let $\Gamma_{\alpha}(\xi)=\{z\in\mathbf{B}:|1-\overline{\xi}z|<\alpha(1-|z|)\}$ be the admissible approach region with vertex at $\xi\in\mathbf{S}$.Let dv denote the volume measure on $\mathbf{B}$, normalized so that $v(\mathbf{B})=1$, and let $d\sigma$ denote the surface measure on $\mathbf{S}$ normalized so that $\sigma(\mathbf{S})=1$. and let $d\mu$ denote the positive Borel measure.For $\alpha>-1$ the weighted Lebesgue measure dv_{α} is defined by

$$dv_{\alpha}=c_{\alpha}(1-|z|^2)^{\alpha}dv(z) \tag{1}$$

where

$$c_{\alpha}=\frac{\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} \tag{2}$$

is a normalizing constant so that $v_{\alpha}(\mathbf{B})=1$ (see[23]). Let also $d\widetilde{v}_{\beta}(z)=dv_{\beta}(z_1)\dots dv_{\beta}(z_m)=(1-|z_1|^2)^{\beta}\cdots(1-|z_n|^2)^{\beta}dv(z_1)\cdots dv(z_n)$. For $z\in\mathbf{B}$ and $r>0$ the set $\mathcal{D}(z,r)=\{w\in\mathbf{B}:\beta(z,w)<r\}$ where β is a Bergman metric on $\mathbf{B}$, $\beta(z,w)=\frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|}$ is called the Bergman metric ball at z (see[23]).

For $\xi\in\mathbf{S}^n$ and $r>0$ set $Q_r(\xi)=\{z\in\mathbf{B}^n:d(z,\xi)<r\}$, where d is a non-isotropic metric on $\mathbf{S}^n$, $d(z,w)=|1-\langle z,w\rangle|^{\frac{1}{2}}$ is called the Carles on tube at ξ (see [23]).For $\alpha>-1$ and $p>0$ the weighted Bergman space A^p_{α} consists of holomorphic functions f in $L^p(\mathbf{B},dv_{\alpha})$, that is,

$$A^p_{\alpha}=L^p(\mathbf{B},dv_{\alpha})\cap H(\mathbf{B}).$$

When the weight $\alpha=0$, we write A^p for A^p_{α}. These are the standard Bergman spaces. See [8] and [23] for more details of weighted Bergman spaces. Let $\mathbf{D}^n=\{z=(z_1,\dots,z_n)\in\mathbb{C}^n:|z_j|<1,\ 1\leq j\leq n\}$ be the unit poly disc in $\mathbb{C}^n$, and $\mathbf{T}^n=\{\xi=(\xi_1,\dots,\xi_n)\in\mathbb{C}^n:|\xi_j|=1,\ 1\leq j\leq n\}$ be the n-dimensional torus, the distinguished boundary of $\mathbf{D}^n$. If $f(z)=f(r\xi)$ is a measurable function in $\mathbf{D}^n$,then

$$M_p(f,r)=\left(\frac{1}{(2\pi)^n}\int_{\mathbf{T}^n}|f(r\xi)|^p dm_n(\xi)\right)^{\frac{1}{p}},\quad r=(r_1,\dots,r_n)\in\mathbf{I}^n, \tag{3}$$

where $0 < p < \infty$, $\mathbf{I}^n = (0,1)^n$, m_n is the n- dimensional Lebesgue measure on $\mathbf{T}^n$. Let $m,\ m > 1$ be a natural number, $M \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^{mn}, C^{mn} = C^n \times \cdots \times C^n$, be a hyper surface. Let $X(M)$ be a class of functions on M,$Y(K)$ the same. We say $TraceY(M^m) = X(M)$, $K = M^m$, $M^m = M \times \cdots \times M$, if for any $f \in Y(M^m)$, $f(w, \ldots, w) \in X(M)$, $w \in M$, and for any $g \in X(M)$, there exist a function$f \in Y(K)$ such that $f(w, \ldots, w) = g(w),\ w \in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in[13] and [22]. In polydisk this problem is also known as a problem of diagonal map (see [8] and references there).The intention of this paper is to consider the following natural Trace problem for polyballs. Let M be a unit ball and let Kbe a polyball (product of m balls) in definition we gave above.Let further $H(\mathbf{B} \times \cdots \times \mathbf{B})$ be a space of all holomorphic functions by each $z_j,\ \ z_j \in B$, $j = 1, \ldots, m : f(z_1, \ldots, z_m)$. Let further Y be a subspace of $H(\mathbf{B} \times \cdots \times \mathbf{B})$.

The question we would like to study and solve in this work is the following: Find the complete description of Trace Y in a sense of our definition for several concrete functional classes. We observe that for $n = 1$ this problem completely coincide with the well- known problem of diagonal map. The last problem of description of diagonal of various subspaces of $H(\mathbf{D}^n)$of spaces of all holomorphic functions in the polydisk was studied by many authors before (see [8], [11],[16], [17], [21] and references there). With the help of Luzin area operator and Bergman metric ball in$\mathbf{B}$ we introduce new holomorphic functional classes on polyballs and describe completely their Traces via classical Bergman spaces in the unit ball $\mathbf{B}$ of $\mathbb{C}^n$.In our previous paper (see [19]) we completely described traces of weighted Bergman classes on polyballs for all values of$p \in (0, \infty)$ and traces of some analytic Bloch type spaces on polyballs expanding known theorems on diagonal map in polydisk(see [8], [16], and references there). Some results of this paper are knew even for $n = 1$ (polydisk case). Main results of this paper will be proved in next section. In the final section we consider related estimates for spaces defined with the help of fractional derivatives, we describe traces of Hardy classes in polyballs and we consider similar constructions in polyhalfplane. Basic properties of a known so-called r-latticein the Bergman metric that can be found in [23] andes timates of expanded Bergman projection in the unit ball are essential for our proofs.Trace theorems even for $n = 1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see for example [3], [8], [18]).Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.We will write for two expressions $A \lesssim B$ if there is a positive constant C such that $A < CB$.

2 The description of Traces of analytic functional spaces in polyballs based on Luzin area operator and Bergman metric ball and the action of expanded Bergman projection.

3 Preliminaries

Proofs of all our theorems in this section are heavily based on properties of r-lattice $\{a_k\}$ in a Bergman metric (see for example [23]). In particular we will use systematically the following lemmas.

Lemma A *[23]*

There exists a positive integer N such that for any $0 < r \leq 1$we can find a sequence $\{a_k\}$ in $\mathbf{B}$ with the following properties:

(1) $\mathbf{B} = \bigcup_k \mathcal{D}(a_k, r)$;

(2) The sets $\mathcal{D}(a_k, \frac{r}{4})$ are mutuallydisjoint;

(3) Each point $z \in \mathbf{B}$ belongs to at most N of thesets $\mathcal{D}(a_k, 4r)$

Lemma B *[23]*

For each $r > 0$ there exists a positive constant C_r such that

$$C_r^{-1} \leq \frac{1 - |a|^2}{1 - |z|^2} \leq C_r, \qquad C_r^{-1} \leq \frac{1 - |a|^2}{|1 - \langle z, a \rangle|} \leq C_r,$$

for all a and z such that $\beta(a, z) < r$. Moreover,if r is bounded above, then we may choose C_r independent ofr.

Lemma C *[23]*

Let $0 < p \leq 1$ and $\alpha > -1$. Then

$$\int_{\mathbf{B}^n} |f(z)|(1 - |z|^2)^{\frac{n+1+\alpha}{p} - (n+1)} dv(z) \leq \frac{\|f\|_{p,\alpha}}{c_\alpha} \tag{4}$$

for all $f \in A^p_\alpha$, where c_α is the constant defined in (2).

Lemma D *[23]*

Suppose $r > 0$, $p > 0$, and $\alpha > -1$. Then there exists a constant $C > 0$ such that

$$|f(z)|^p \leq \frac{C}{(1 - |z|^2)^{n+1+\alpha}} \int_{\mathcal{D}(z,r)} |f(w)|^p dv_\alpha(w) \tag{5}$$

for all $f \in H(\mathbf{B}^n)$ *and all* $z \in \mathbf{B}^n$.

We will need also:

Lemma E *[12]*

Let $\beta > 0$ *and* $p > 0$. *Let* $\{A_j\}_0^\infty$ *be a positive sequence and* $\sum_{n=1}^\infty 2^{-n\beta} A_n^p < \infty$. *Then*

$$\sum_{n=1}^\infty 2^{-n\beta} A_n^p \le C \left(A_0^p + \sum_{n=1}^\infty 2^{-n\beta} |A_n - A_{n-1}|^p \right).$$

Bergman classes on poly balls $A^p(\mathbf{B}^m, dv_{\alpha_1} \cdots dv_{\alpha_m})$ consists of functions f in $H(\mathbb{B}^m)$, such that,

$$\int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |f(z_1, \dots, z_m)|^p (1-|z_1|)^{\alpha_1} \cdots (1-|z_m|)^{\alpha_m} dv(z_1) \cdots dv(z_m) < \infty.$$

Let

$$A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m}) = \left\{ f \in H(\mathbb{B}^m) : \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |f(z_1, \dots, z_m)|^p \right.$$

$$\left. (1-|z_1|)^{\alpha_1} \cdots (1-|z_m|)^{\alpha_m} dv(z_1) \cdots dv(z_m) < \infty \right\}$$

be we need also the following theorem from our paper [19] where the description of traces of Bergman classes in poly balls were given.

Theorem A (1)

Suppose $1 \le p \le \infty$ *and* $s_1, \dots, s_m > -1$.*Put* $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. *Then there are bounded operators* $S : A^p(\mathbf{B}, dv_t) \to A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m})$, *and* $R : A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m}) \to A^p(\mathbf{B}, dv_t)$ *such that* $(Sf)(z, \dots, z) = f(z)$ *and*$(Rg)(z) = g(z, \dots, z)$ *for all* $f \in A^p(\mathbf{B}, dv_t)$, *all* $g \in A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m})$ *and all* $z \in \mathbf{B}$. *In other words, the* $Trace A^p(\mathbf{B}^m, dv_{s_1} \cdots dv_{s_m}) = A^p(\mathbf{B}, dv_t)$.(2) *Let* $0 < p \le 1$, $s_1, \dots, s_m > -1$, $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. *Then* $Trace A^p(\mathbf{B}^m, dv_{s_1}, \dots, dv_{s_m}) = A^p(\mathbf{B}, dv_t)$.

In this section we will give the complete description of traces of the following spaces of holomorphic functions on polyballs $\mathbb{B}^m = \mathbb{B} \times \cdots \times \mathbb{B}$ defined with the help of Luzin area operator and Bergman metric ball with some restrictions on parameters.

$M^p_\alpha = \{f \in H(\mathbb{B}^m) : \int_{\mathbb{S}} \int_{\Gamma_t(\xi)} \cdots \int_{\Gamma_t(\xi)} |f(z)|^p d\widetilde{v}_\alpha(z) d\sigma(\xi) < \infty\},$

$p \in (0, \infty)$, $\alpha > -1$.

$$K^{p,q}_{\alpha,\beta} = \{f \in H(\mathbb{B}^m) : \int_{\mathbb{B}} \cdots \int_{\mathbb{B}} \left(\int_{\mathcal{D}(z_1,r)} \cdots \int_{\mathcal{D}(z_m,r)} |f(z)|^p d\widetilde{v}_\alpha(z) \right)^{\frac{q}{p}} d\widetilde{v}_\beta(z) < \infty\},$$

$p, q \in (0, \infty)$, $\alpha > -1$, $\beta > -1$.

$$D^p_{\alpha,\beta} = \{f \in H(\mathbb{B}^m) : \int_0^1 (1-r)^\beta \left(\int_{|z_1|<1} \cdots \int_{|z_{m-1}|<1} \int_{|z_m|<r} |f(z)|^p \prod_{j=1}^m (1 - |z_j|^{\alpha_j}) dv(z_j) \right) dr < \infty\},$$

$|\widetilde{z}| < r$ $(|z_1| < r, \ \ldots, |z_m| < r)$, $p \in (0, \infty)$, $\beta > -1$, $\alpha_j > -1$, æ $= 1, \ldots m$. This classes were studied in unit disc before (see [21]).These classes were considered in the case of unit disk by manyauthors (see, for example, [2], [10], [21]).Functional classes defined with the help of Bergman metric balland Luzin area operator in the unit ball were studied in[4], [14], [15] and also in Chapter 5 andChapter 6 in [23]. Note that M^p_α coincide withusual weighted Bergman class in ball for $m = 1$ (see [14],[15]).The following Theorem for $n = 1$ was proved in [11].

Theorem 1 *Let* $m \in \mathbb{N}$, $n \in \mathbb{N}$, $0 < p < \infty$, $\alpha > -(n + 1)$ *and* $\gamma = (\alpha + n + 1)m - 1$. *Then* *Trace*$(M^p_\alpha(\mathbf{B}^m)) = A^p_\gamma(\mathbf{B})$.

Proof of Theorem 1.

In [5] it was shown

$$\int_0^1 |u(r\xi_1, \ldots, r\xi_n)|^p (1 - r)^\alpha dr \le C \int_{\Gamma_t(\xi)} |u(z)|^p (1 - |z|)^{\alpha - n} dv(z), \tag{6}$$

$\alpha > -1$, $0 < p < \infty$, $u \in H(\mathbf{B})$, $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{S}$. We use this estimate m times by each variable separately and get integrating both sides of obtained estimate by $\mathbf{S}$

$$\int_{\mathbf{S}} \int_0^1 \cdots \int_0^1 |u(r_1\xi_1, \ldots, r_1\xi_n, \ldots, r_m\xi_1, \ldots, r_m\xi_n)|^p (1 - r_1)^{\alpha_1} \ldots (1 - r_m)^{\alpha_m} dr_1 \ldots dr_m d\xi \tag{7}$$

$$\leq C \int_{\mathbf{S}} \int_{\Gamma_{t_1}(\xi)} \cdots \int_{\Gamma_{t_m}(\xi)} |u(z_1, \ldots, z_m)|^p (1-|z_1|)^{\alpha_1 - n} \ldots (1-|z_m|)^{\alpha_m - n} dv(z_1) \ldots dv(z_m) d\xi,$$

where $u \in H(\mathbf{B}^m), m \in \mathbb{N}, \alpha_j > -1, 0 < p < \infty$. The function $\widetilde{u}(z) = u(z, \ldots, z)$ is in $H(\mathbf{B})$. To prove the estimate we need we will use below so-called slice functions (see [23], page 125). Let Hence using standard slice function technique (see [23],page 125) we have the following chain of estimates $(\xi z) = (z\xi_1, \ldots, z\xi_n), u_\xi(z, \ldots, z) = u(\xi z, \ldots, \xi z), u_\xi \in H(\mathbf{B}^m), (\widetilde{u}_\xi)(z) = \widetilde{u}(\xi z), z \in \mathbf{D}, z = |z|\tau, \mathbf{D} = \{w : |w| < 1\}, \xi \in \mathbf{S}$,

$$u_\xi(z_1, \ldots, z_m) = u(\xi_1 z_1, \ldots, \xi_n z_1, \ldots, \xi_1 z_m, \ldots, \xi_n z_m),$$

$z_j = r_j \phi,\ z_j \in \mathbf{D}, j = 1, \ldots, m, \widetilde{u}_\xi(z) = u(\xi_1 z, \ldots, \xi_n z, \ldots, \xi_1 z, \ldots, \xi_n z),\ z \in \mathbf{D}$.

Then $\widetilde{u}_\xi \in H(\mathbf{D})$, and

$$\int_0^1 \int_0^{2\pi} |\widetilde{u}_\xi(z)|^p (1-|z|)^\gamma d\tau d|z| \leq C \sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} \int_0^{2\pi} |\widetilde{u}_\xi(z)|^p (1-|z|)^\gamma d\tau d|z| \qquad (8)$$

$$\lesssim C_1 \sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left(2^{\frac{-k_1\gamma}{m}} \cdots 2^{\frac{-k_m\gamma}{m}}\right) \left(2^{\frac{-k_1}{m}} \cdots 2^{\frac{-k_m}{m}}\right) \int_{1-2^{-(k_1+1)}}^{1-2^{-(k_1+2)}} \cdots \int_{1-2^{-(k_m+1)}}^{1-2^{-(k_m+2)}} \times$$

$$\times \int_0^{2\pi} |(u_\xi)(z_1, \ldots, z_m)|^p (2^{k_1} \cdots 2^{k_m}) d\tau d|z_1| \cdots d|z_m|$$

$$\leq C_2 \int_0^1 \cdots \int_0^1 \int_0^{2\pi} |(u_\xi)(z_1, \ldots, z_m)|^p \prod_{k=1}^m (1-|z_k|)^{\frac{\gamma+1}{m}-1} d\tau d|z_1| \cdots d|z_m|.$$

Using Lemma 1.10 from [23] we haveUsing that for $f \in L^1(\mathbf{S}^n, d\sigma)$

$$\int_{\mathbf{S}^n} f d\sigma = \int_{\mathbf{S}^n} d\sigma(\zeta) \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}\zeta) d\theta, \qquad (9)$$

and if $1 < k < n$, then

$$\int_{\mathbf{S}^n} f d\sigma = c \int_{\mathbf{B}^k} (1-|z|^2)^\alpha dv_k(z) \int_{\mathbf{S}^{n-k}} f(z, \sqrt{1-|z|^2}\eta) d\sigma_{n-k}(\eta),$$

where $c = \begin{pmatrix} n-1 \\ k \end{pmatrix}$ and $\alpha = n-k-1$,

$$f(t\xi) = f(t\xi_1, \dots, t\xi_n, \dots, t\xi_1, \dots, t\xi_n), \xi \in \mathbf{S},\ t \in (0, 2\pi), \tag{10}$$

(see [23],Lemma 1.10), we have from (8) integrating both sides of it by sphere $\mathbf{S}$

$$\int_{\mathbf{B}} |\widetilde{u}(z)|^p (1-|z|)^\gamma dv(z) = \sum_{k=0}^{\infty} \int_{1-2^{-k}}^{1-2^{-(k+1)}} \int_{\mathbf{S}} |\widetilde{u}(z)|^p (1-|z|)^\gamma d\sigma(\xi) d|z| \tag{11}$$

$$\leq \ C \textstyle\sum_{k=0}^{\infty} \left(\int_{\mathbf{S}} |\widetilde{u}(r_0\xi)|^p d\sigma(\xi) \right) 2^{-k\gamma} 2^{-k}$$

$$\leq \ C_1 \textstyle\sum_{k_1=0}^{\infty} \cdots \sum_{k_m=0}^{\infty} \left(2^{\frac{-k_1\gamma}{m}} \dots 2^{\frac{-k_m\gamma}{m}} \right) \cdots \left(2^{\frac{-k_1}{m}} \dots 2^{\frac{-k_m}{m}} \right) \times$$

$$\times \ \left(\textstyle\int_{1-2^{-(k_1+1)}}^{1-2^{-(k_1+2)}} \cdots \int_{1-2^{-(k_m+1)}}^{1-2^{-(k_m+2)}} \int_{\mathbf{S}} |u(z_1, \dots, z_m)|^p d\sigma(\xi) \right) \left(2^{k_1} \cdots 2^{k_m} \right)$$

$$\leq \ C \textstyle\int_0^1 \dots \int_0^1 \int_{\mathbf{S}} |u(z)|^p d\sigma(\xi) (1-|z_1|)^{\frac{\gamma+1}{m}-1} \dots (1-|z_m|)^{\frac{\gamma+1}{m}-1} d|z_1| \dots d|z_m|,$$

$\widetilde{u}(z) = u(z, \dots, z)$. Combining (7) and (11) we have for $0 < p < \infty, \gamma > -1$

$$\int_{\mathbf{B}} |u(z, \dots, z)|^p (1-|z|)^\gamma dv(z) \leq C \int_{\mathbf{S}} \int_{\Gamma_\delta(\xi)} \cdots \int_{\Gamma_\delta(\xi)} |u(z_1, \dots, z_m)|^p \times$$

$$\times (1-|z_1|)^{\frac{\gamma+1}{m}-(n+1)} \dots (1-|z_m|)^{\frac{\gamma+1}{m}-(n+1)} dv(z_1) \dots dv(z_m) d\sigma(\xi).$$

Let $\beta = \frac{\gamma+1}{m} - (n+1)$, then $\gamma = (\beta+n+1)m - 1, \beta > -(n+1)$.

So we proved completely one part of our assertion in Theorem1.Now we prove the reverse to the last inequality. Let $p \leq 1$. We use systematically properties of, so-called, r-lattice $\{a_k\}$ or sampling sequences (see Lemma Fand Lemma G, [23]).For every $s > -1$ we have $F(z, \dots, z) = f(z)$, where

$$F(z_1, \dots, z_m) = C \int_{\mathbf{B}} \frac{f(w)(1-|w|)^s}{\prod_{j=1}^m (1 - \langle \overline{w}, z_j \rangle)^{\frac{s+1+n}{m}}} dv(w)$$

by Bergman representation formula (see [23], Theorem 2.11).So we get the following chain estimates ($p \leq 1$, s is large enough) using known properties of $\{a_k\}$ from LemmaF and Lemma G(see also [23], Chapter 2) and the fact that

$$\left(\sum_{k=1}^{\infty} a_k\right)^p \leq \sum_{k=1}^{\infty} a_k^p, \quad a_k \geq 0, \quad p \leq 1,$$

$$|F(z_1,\dots,z_m)|^p \lesssim \sum_{k\geq 0} \max_{\mathcal{D}(a_k,r)} |f(w)|^p \left(\int_{\mathcal{D}(a_k,r)} \frac{(1-|w|)^s}{\prod_{j=1}^m |1-\langle \overline{w}, z_j\rangle|^{\frac{s+1+n}{m}}} dv(w)\right)^p$$

$$\lesssim \sum_{k\geq 0} \max_{\mathcal{D}(a_k,r)} |f(w)|^p \frac{(1-|a_k|)^{ps}\,(v(\mathcal{D}(a_k,r)))^p}{\prod_{j=1}^m |1-\langle \overline{a_k}, z_j\rangle|^{\frac{s+1+n}{m}p}}.$$

Then using again the relation

$$|1-\langle w,z\rangle| \asymp |1-\langle a_k,z\rangle|, \quad w\in\mathcal{D}(a_k,r), \quad z\in\mathbf{B},$$

(see [23],page 63)

and Lemma 2.24 from [23] and Lemma F we finally get

$$|F(z_1,\dots,z_m)|^p \leq C\int_{\mathbf{B}} \frac{|f(\widetilde{w})|^p(1-|\widetilde{w}|)^t dv(\widetilde{w})}{\prod_{k=1}^m |1-\langle \overline{z_k}, \widetilde{w}\rangle|^{\frac{s+1+n}{m}p}}, \tag{12}$$

where $t=(n+1+s)p-(n+1)$, $t>-1$. Integrating both sides of the last inequality by sphere we have the following estimate

$$\begin{aligned}\int_{\mathbf{S}}\int_{\Gamma_{t_1}(\xi)}\cdots\int_{\Gamma_{t_m}(\xi)} |F(z_1,\dots,z_m)|^p \quad &\times \quad (1-|z_1|)^{\frac{\gamma+1}{m}-(n+1)}\dots(1-|z_m|)^{\frac{\gamma+1}{m}-(n+1)}dv(z_1)\dots dv(z_m)d\sigma(\xi)\\ &\leq \quad C\int_{\mathbf{B}} |f(\widetilde{w})|^p(1-|\widetilde{w}|)^\gamma dv(\widetilde{w}); \qquad \gamma>-1, \quad p\in(0,1].\end{aligned}$$

We used m times the following known estimate

$$\int_{\Gamma_t(\xi)} \frac{(1-|z|)^\nu}{|1-\langle \overline{w}, z\rangle|^{s_1}} dv(z) \leq \frac{C}{|1-\langle \overline{\xi}, w\rangle|^{s_1-n-1-\nu}}, \qquad \text{(see[13] [14])} \tag{13}$$

$\xi\in\mathbf{S}$, $w\in\mathbf{B}$, $\nu>-n-1$, $s_1>\nu+n+1$. It is easy to see that our Theorem 1 is proved completely for all $p\in(0,1]$.For $p>1$ we use the following estimate, which can be easily obtained from Hölder's inequality applied twice and the estimate (see [23], Theorem 1.12):

$$\int_{\mathbf{B}} \frac{(1-|z|)^{\nu}}{|1-\langle w,z\rangle|^{s_1}} dv(z) \le \frac{C}{(1-|w|)^{s_1-n-1-\nu}},$$

$w \in \mathbf{B},\ \nu > -1,\ s_1 > \nu + n + 1$, applied m times for $s_1 = \tau_2 p' m$

$(\tau = p\left(\frac{\nu+n+1}{mp'} - \tau_2\right), \frac{1}{p} + \frac{1}{p'} = 1)$

$$|F(z_1,\dots,z_m)|^p \lesssim \int_{\mathbf{B}} \frac{|f(w)|^p (1-|w|)^s (1-|z_1|^2)^{\tau} \cdots (1-|z_m|^2)^{\tau}}{\prod_{k=1}^m |1-\langle z_k, \overline{w}\rangle|^{pr_1}} dv(w), \tag{14}$$

where $z_j \in \mathbf{B},\ j=1,\dots,m; r_1 + r_2 = \frac{s+n+1}{m}, \tau = p\left(\frac{s+n+1}{mp} - r_2\right), r_1, r_2 > 0$, and continue as for the case $p \le 1$, using appropriate τ_1, τ_2. The proof of theorem is complete. □ Let us turn to the proof of the theorem:

Remark 1 *For $m = n = 1$ the statement of Theorem 1 is obvious. Let us note that for $m = n = 1$ after definition of M^p_α spaces it is obvious that for $m = 1$ these classes coincide with classical Bergman spaces in the unit ball.*

Theorem 2 *Let $n \in \mathbb{N}, 0 < p < \infty, t_j > -1, \beta_j > -1, j = 1,\dots,m, \alpha > -1$ and $\alpha = \sum_{j=1}^m (\beta_j + 2(n+1) + t_j) - (n+1)$, then*

$$\textit{Trace}\left(K^{p,p}_{t,\beta}(\mathbf{B}^m)\right) = A^p_\alpha(\mathbf{B}).$$

The proof needs small modification of above argument and is based on some embed dings obtained in [16].

Proof of Theorem 2. Obviously by LemmaF and G

$$\int_{\mathbf{B}} |\widetilde{f}(z)|^p (1-|z|)^{\alpha} dv(z) \lesssim \sum_{k\ge 0} \left(\max_{z\in\mathcal{D}(a_k,r)} |\widetilde{f}(z)|^p\right) \left(v_\alpha(\mathcal{D}(a_k,r))\right), \tag{15}$$

where $\widetilde{f}(z) = f(z,\dots,z), 0 < p < \infty, \alpha > -1$. We have by Lemma 2.24 from [23] and Lemma F and G

$$\sum_{k\ge 0} \max_{z\in\mathcal{D}(a_k,r)} |\widetilde{f}(z)|^p C_{k,\alpha+n+1} \lesssim \sum_{k_1\ge 0,\dots,k_m\ge 0} \left(\max_{z_j\in\mathcal{D}(a_{k_j},r)} |f(z_1,\dots,z_m)|^p\right) C_{k_1,\dots,k_m,\alpha+n+1}, \tag{16}$$

$$C_{k,\alpha+n+1} = v_{\alpha+n+1}(\mathcal{D}(a_k,r)), C_{k_1,\dots,k_n,\alpha} = v_\alpha\left(\mathcal{D}(a_{k_1},r)\right)^{\frac{1}{m}}\cdots v_\alpha\left(\mathcal{D}(a_{k_n},r)\right)^{\frac{1}{m}},$$

$$\alpha = \sum_{j=1}^{m}\widetilde{\beta}_j + \sum_{j=1}^{m} t_j - n - 1, \quad t_j > -1, \ j = 1,\dots,m,$$

$$\widetilde{\beta}_j = \beta_j + 2(n+1), \ j = 1,\dots,m.$$

$$\begin{aligned}\|f\|^p_{A^p_\alpha} \quad &\leq \quad \textstyle\int_{\mathbf{B}}\cdots\int_{\mathbf{B}}(1-|z_1|)^{t_1}\cdots(1-|z_m|)^{t_m}\times \\ &\times \quad \left(\textstyle\int_{\mathcal{D}(z_1,r)}\cdots\int_{\mathcal{D}(z_m,r)}|f(w_1,\dots,w_m)|^p d\widetilde{v}_{\overrightarrow{\beta}}(\widetilde{w})\right) d\widetilde{v}(z),\end{aligned}$$

where $d\widetilde{v}_{\overrightarrow{\beta}}(\widetilde{w}) = \prod_{k=1}^m(1-|w|^2)^{\beta_k}dv(w)$. At the final step we used Lemma F again and the fact that for $z\in\mathcal{D}(a_k,r)$ $(1-|z|)^t \asymp (1-|a_k|)^t,\ t\in\mathbb{R}$. Let us prove the reverse to the estimate we obtained above. We again use properties of expanded Bergman projection and we have the following chain of estimates. (First we consider the case $p=q$) We have as before for positive large enough integer s,

$$F(z_1,\dots,z_m) = C\int_{\mathbf{B}}\frac{f(w)(1-|w|)^s dv(w)}{\prod_{j=1}^m(1-\langle\overline{w},z_j\rangle)^{\frac{s+1+n}{m}}};$$

$$F(z,\dots,z) = f(z).$$

Hence by Lemma 2.15 from [23], for $p\leq 1$, we have

$$\begin{aligned}&\textstyle\int_{\mathbf{B}}\cdots\int_{\mathbf{B}}\prod_{k=1}^m(1-|\widetilde{z_k}|)^{t_k}\left(\int_{\mathcal{D}(\widetilde{z}_1,r)}\cdots\int_{\mathcal{D}(\widetilde{z_m},r)}|F(z_1,\dots,z_m)|^p d\widetilde{v}_\beta(z)\right)d\widetilde{v}(\widetilde{z})\\ &\lesssim \textstyle\int_{\mathbf{B}}\cdots\int_{\mathcal{D}(\widetilde{z}_1,r)}\cdots\int_{\mathcal{D}(\widetilde{z_m},r)}\int_{\mathbf{B}}\frac{f(w)(1-|w|)^{p(n+1+s)-(n+1)}}{|\prod_{j=1}^m(1-\langle\overline{w},z_j\rangle)^{\frac{s+1+n}{m}}|^p}\times\\ &\times\textstyle\prod_{k=1}^m(1-|\widetilde{z_k}|)^{t_k}\prod_{k=1}^m(1-|z_k|)^{\beta_k}d\widetilde{v}(\widetilde{z})d\widetilde{v}(\widetilde{w})d\widetilde{v}(z),\end{aligned}$$

$d\widetilde{v}(\widetilde{z}) = dv(\widetilde{z}_1)\dots dv(\widetilde{z}_m)$. Using the fact that s is large and using the inequality known estimates for integrals on $\mathcal{D}(z_j,r), j=1,\dots,m; z_j\in\mathbf{B}$ based on lemmas F and G, (see [23])

$$\int_{\mathcal{D}(z_1,r)}\cdots\int_{\mathcal{D}(z_m,r)}\frac{(1-|w_1|)^{\beta_1}\cdots(1-|w_m|)^{\beta_m}}{\prod_{j=1}^m|1-\langle w_j,z\rangle|^{\frac{s+1+n}{m}p}}dv(w_1)\cdots dv(w_m)$$

$$\leq C\prod_{j=1}^m|1-\langle z,z_j\rangle|^{\frac{-(s+1+n)p}{m}+\beta_j+n+1}, \quad z\in\mathbf{B},\ z_j\in\mathbf{B},$$

we finally get the estimate we need:

$$\|F\|^p_{K^{p,p}_{t,\beta}} \lesssim \|f\|^p_{A^p_\alpha}, \tag{17}$$

where is

$$\alpha = (n+1)\left(\sum_{j=1}^m \beta_j + \sum_{j=1}^m t_j - 1\right),$$

$\alpha > -1, t_j > -1, \beta_j > -1, j = 1, \dots, m$. when α, t, β were defined above. Theorem 2 is proved completely for $p \le 1$. To finish the proof we need estimate (17) for $p > 1$.
To finish the proof we need the estimate (17) for $p > 1$.For that reason we apply estimate (14). Then repeat arguments we provided above for $p \le 1$ using the fact that $s$is large enough in Bergman representation formula. The proof of theorem is complete.

Remark 2 *Let us note that for $m = 1$ and $n = 1$ the Theorem 2 is obvious. Our Theorem 2 is new even for $m > 1$, $n = 1$(poly disk case).for $n = 1$ this theorem was obtained recently in*

Let us note that for $m = n = 1$ after definition of M^p_α spaces it is obvious that for $m = 1$ these classes coincide with classical Bergman spaces in the unit ball.

Remark 3 *The general case of $K^{p,q}_{\alpha,\beta}$ spaces can be considered almost similarly for $0 < p \le q \le 1$.*

Remark 4 *Traces of spaces of type $\int_0^1 \left(\int_{|\tilde{z}|<r} |f(z_1, \dots, z_m)|^q d\tilde{v}_\alpha(z_1, \dots, z_n)\right) dr$; $|\tilde{z}| < r$ i.e. $(|z_1| < r, \% \dots, |z_n| < r)$, $q \in (0, \infty)$ can be obtained similarly.*

Theorem 3 *Let $0 < p < \infty, \beta > -1, \alpha_j > -1, j = 1, \dots, m$. Then $Trace(D^p_{\alpha,\beta}(\mathbf{B}^m)) = Trace(A^p_{\overrightarrow{\alpha}}(\mathbf{B}^m)) = A^p_{\sum_{j=1}^m \alpha_j + \beta + 1 + (n+1)(m-1)}(\mathbf{B})$.*

Proof 1 *We obtain first a characterization of $D^p_{\alpha,\beta}$ classes via weighted Bergman spaces on poly balls and then we will apply Theorem A and thus we will calculate completely traces of $D^p_{\alpha,\beta}$ classes in poly bals. Let $0 < t < 1, f_t(z) = f(tz), z \in \mathbf{B} \times \dots \times \mathbf{B}$. Let $r_n = 1 - 2^{-n}, n = 0, 1, \dots$. Using decomposition*

$$\int_0^1 P(r)dr = \sum_{k=0}^\infty \int_{r_k}^{r_{k+1}} P(r)dr,$$

where $P(r)$ is any measurable function on $(0, 1)$, it is easy to check

$$\begin{aligned}\|f\|^p_{D^p_{\alpha,\beta}} &= \int_0^1 (1-r)^\beta \left(\int_{|z_1|<1}\cdots\int_{|z_{m-1}|<1}\int_{|z_m|<r}|f_t(z_1,\dots,z_m)|^p\prod_{j=1}^m(1-|z_j|)^{\alpha_j}dv(z_j)\right)dr\\ &\leq C\sum_{n=1}^\infty 2^{-n(\beta+1)}\left(\int_{|z_1|<1}\cdots\int_{|z_{m-1}|<1}\int_{|z_m|<r_n}|f_t(z)|^p\prod_{j=1}^m(1-|z_j|)^{\alpha_j}dv(z_j)\right)\\ &= C\sum_{n=1}^\infty 2^{-n(\beta+1)}A_n^p = K(p,f,\alpha).\end{aligned}$$

We have by Lemma E

$$K \leq C_1\sum_{n=1}^\infty 2^{-n(\beta+1)}\int_{|z_1|<1}\cdots\int_{|z_{m-1}|<1}\int_{r_{n-1}\leq z_m<r_n}|f_t(z)|^p\prod_{j=1}^m(1-|z_j|)^{\alpha_j}dv(z_j)$$

$$\leq C_2\|f_t\|_{A^p(\mathbf{B}^m,dv_{\alpha_1},\dots,dv_{\alpha_{m-1}},dv_{\beta+1+\alpha_m})} = \sum_{n=1}^\infty 2^{-n(\beta+1)}\left(\int_{\mathbf{S}}\cdots\int_{\mathbf{S}}|f_t(r_n\xi)|^p d\widetilde{\sigma}(\xi)\right)^{\frac{q}{p}}2^{-n(\alpha+1)\frac{q}{p}}\cdots 2^{-n(\alpha+1)\frac{q}{p}} \tag{18}$$

$$\leq C\int_0^1\cdots\int_0^1(1-r_1)^{\frac{\beta+\frac{q}{p}(\alpha+1)m}{m}}\cdots(1-r_m)^{\frac{\beta+\frac{q}{p}(\alpha+1)m}{m}}\times$$

$$\times\left(\int_{\mathbf{S}}\cdots\int_{\mathbf{S}}|f_t(r_1\xi_1,\dots,r_m\xi_m)|^p d\sigma(\xi_1,\dots,\xi_m)\right)^{\frac{q}{p}}.$$

$$= C_2\int_{|z_1|<1}\cdots\int_{|z_m|<1}|f_t(z_1,\dots,z_m)|^p dv_{\alpha_1}(z_1)\cdots dv_{\alpha_{m-1}}(z_{m-1})dv_{\beta+1+\alpha_m}(z_m).$$

Finally we have

$$\|f_t\|^p_{D^p_{\alpha,\beta}} < C\|f_t\|_{A^p(\mathbf{B}^m,dv_{\alpha_1},\dots,dv_{\alpha_{m-1}},dv_{\beta+1+\alpha_m})}. \tag{19}$$

We tend $t\to 1$ and apply Theorem A. So we get

$$Trace(D^p_{\alpha,\beta}(\mathbf{B}^m)) \subset A^p_{\sum_{j=1}^m\alpha_j+\beta+1+(n+1)(m-1)}(\mathbf{B}).$$

The reverse to estimate (18) can be obtained by similar argument.The reverse to (19) can be obtained by very similar arguments. So using again Theorem A we get there verse inclusion

$$A^p_{\sum_{j=1}^m\alpha_j+\beta+1+(n+1)(m-1)}(\mathbf{B}) \subset Trace(D^p_{\alpha,\beta}(\mathbf{B}^m)).$$

The proof of theorem is complete.

4 On traces of Hardy spaces and some functional classes defined with the help of fractional derivatives in poly balls and related estimates in poly half plane

In this section we will give estimates for traces of classes of holomorphic functions of Besov-type in poly balls defined with the help of fractional derivatives, obtain the description of traces of Hardy classes $H^p(\mathbf{B}^m)$ in poly balls and present similar assertions and estimates for polyhalfplane.Let

$$H^p_{\alpha,\beta} = \{f \in H(\mathbf{B}^m) : \sup_{r<1} \left(\widetilde{M}_p(D^\beta f, r)\right)(1-r)^\alpha < \infty\},$$

$$\alpha \geq 0, \quad p \in (0,\infty), \quad \beta \in \mathbb{R}, \quad D^\beta f = D^\beta_{z_1} \cdots D^\beta_{z_m} f,$$

$$\widetilde{M}^p_p(f,r) = \int_{\mathbf{S}} \cdots \int_{\mathbf{S}} |f(r\xi_1, \dots r\xi_m)|^p d\sigma(\xi_1) \cdots d\sigma(\xi_n), \ r \in (0,1).$$

We define Hardy class H^p in poly balls as $H^p(\mathbf{B}^m) = H^p_{0,0}$ for $p \in (0,\infty)$. As usual, we denote by $\overrightarrow{\alpha}$ the vector $(\alpha_1, \dots, \alpha_n)$. Let

$$A^p_{t,\overrightarrow{\alpha}} = \{f \in H(\mathbf{B}^m) : \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |D^{\alpha_1} \cdots D^{\alpha_m} f|^p (1-|\widetilde{z}|)^t d\widetilde{v}(z) < \infty\},$$

$\alpha_j \in \mathbb{R}, j = 1, \dots, m, t > -1, 0 < p < \infty$ and $(1-|\widetilde{z}|) = \prod_{k=1}^m (1-|z_k|)$.

Then the following result can be formulated as a direct corollary of estimates for expanded Bergman projection obtained during the proof of Theorem 1 (see estimate (12)) and some known calculations with fractional derivatives on Bergman kernels that can be found also in [14] and [15].

Theorem 4 (1) *Let* $p \leq 1$, $\alpha \geq \beta \geq 0$, *then* $A^p_{nm+(\alpha-\beta)pm-(n+1)}(\mathbf{B}) \subset Trace(H^p_{\alpha,\beta}(\mathbf{B}^m))$;

(2) *Let* $p \leq 1, \alpha_j \geq 0, t \geq \frac{\sum_{j=1}^m \alpha_j}{m}$, *then* $A^p_{mt+m(n+1)-(n+1)-\sum_{j=1}^m \alpha_j}(\mathbf{B}) \subset Trace(A^p_{t,\overrightarrow{\alpha}}(\mathbf{B}^m))$.

Remark 5 *Note that for* $n = 1$ *(poly disk case) and* $\alpha = 0, \beta = 0$ *the first inclusion can be found in [8]. The second inclusion for* $\alpha_1, \dots, \alpha_n = 0$ *can be also found in [8].*

It is not difficult to see as a consequence of Cauchy formula (as in case of poly disk) that the expansion of any function $f, f \in H(\mathbf{B}^m)$ can be defined as follows

$$f(z_1,\dots,z_m)=\sum_{n_1\geq 0}\cdots\sum_{n_m\geq 0}a_{n_1,\dots,n_m}z_1^{n_1}\cdots z_m^{n_m},$$

$$z_j^{n_j}=z_1^{n_j^1}\cdots z_n^{n_j^n},\quad j=1,\dots,m.$$

$z_k^{n_j^i}$ is in unit disk in $\mathbb{C}$. And hence the corresponding homogeneous expansion of will be defined as follows:

$$f(z_1,\dots,z_m)=\sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}f_{k_1,\dots,k_m}(z_1,\dots,z_m);$$

$$f_{k_1,\dots,k_m}(z_1,\dots,z_m)=\sum_{|s_1|=k_1}\cdots\sum_{|s_m|=k_m}a_{s_1,\dots,s_m}z_1^{\overrightarrow{s_1}}\cdots z_m^{\overrightarrow{s_m}},$$

$$|s_j|=\sum_{i=1}^{n}s_i^j,\quad z_j\in\mathbf{B},$$

and the action of the fractional derivative is given by

$$\left(D^{t_1,\dots,t_m}f\right)(z)=\sum_{k_1=0}^{\infty}\cdots\sum_{k_m=0}^{\infty}\prod_{j=1}^{m}(k_j+1)^{t_j}f_{k_1,\dots,k_m}(z_1,\dots,z_m),$$

$$t_j\in\mathbb{R},\quad j=1,\dots,m,\quad \left(D^{\overrightarrow{t}}f\right):H(\mathbf{B}^m)\to H(\mathbf{B}^m).$$

We now provide some estimates for traces of Hardy H^p classes in poly balls.

Let $p\geq 2$. Then $\mathrm{Trace}(H^p(\mathbf{B}^m))\subset A^p_{n(m-1)-1}, m\in\mathbb{N}, m>1, n\in\mathbb{N}$.

The proof follows directly from binomial formula. We will give as hort proof for $n=1$ poly disk case. The general case needs smallmodification.Let $\mathbf{D}^n$ be unit poly disk, $dm_{2n}(z)$ Lebesgue measure on $\mathbf{D}^n$. Then we have the following chain of estimates. First

$$(k_1+\ldots+k_n+1)^s=\sum_{\alpha_j\geq 0;\sum_{j=1}^{n+1}\alpha_j=s}C_s(\alpha_1,\dots,\alpha_n)\left(\prod_{j=1}^{n}(k_j+1)^{\alpha_j}\right)(-1)^{\alpha_{n+1}}(n-1)^{\alpha_{n+1}},$$

$\sum_{j=1}^{n}\alpha_j=s,\alpha_j\geq 0$, we have $\alpha_1,\dots,\alpha_n s\in\mathbb{N}$. Hence

$$\begin{aligned} M &= \int_{\mathbf{D}^n} |\sum_{k_1,\dots,k_n}(k_1+\dots+k_n+1)^s a_{k_1,\dots,k_n} z_1^{k_1}\cdots z_n^{k_n}|^p \times \\ &\times (1-|z_1|)^{\alpha_1 p-1}\cdots(1-|z_n|)^{\alpha_n p-1} dm_{2n}(z) \\ &\le C\int_{\mathbf{D}^n} |D^{\alpha_1\cdots\alpha_n} f|^p \prod_{k=1}^n (1-|z_k|)^{\alpha_k p-1} dm_{2n}(z) \le C\|f\|_{H^p}. \end{aligned}$$

The last estimate follows directly from Theorem 4.41 from[23] for $H^p, p\ge 2$ applied n times.On the other hand obviously by Theorem A we have

$$\begin{aligned} M &\ge C_1\int_{\mathbf{D}} |\sum_{k_1,\dots,k_n}(k_1+\dots+k_n+1)^s a_{k_1,\dots,k_n} z^{k_1+\dots+k_n}|^p (1-|z|)^{sp+n-2} dm_2(z) \\ &\ge C_2\int_{\mathbf{D}} \left|\sum_{m\ge 0}(m+1)^s \left(\sum_{k_1+\dots+k_n=m} a_{k_1,\dots,k_n}\right) z^m\right|^p (1-|z|)^{sp+n-2} dm_2(z) \\ &\ge C_3\|\widetilde{f}\|^p_{A^p_{n-2}}, \quad p\ge 2, \end{aligned}$$

$widetildef = f(z,\dots,z)$.

Remark 6 *It is easy to notice that our appendix will immediately give results for „smooth classes“ $X=\{D^s f\in H^p_\alpha\}$ or $Y=\{D^s f\in H^p\}$. where D^s is a fractional derivative.*

Remark 7 *Various assertions for traces of different spaces can be obtained using estimates for slice functions from chapter 1 [23]. We give one example.????????*

Remark 8 *The complete description of traces of classes of $Z_{p,s}$- type cane be obtained by methods outdated above using $F(z_1,\dots,z_n)$ functions.*

Lemma F *[23] There exists a positive integer N such that for any $0<r\le 1$ we can find a sequence $\{a_k\}$ in $\mathbf{B}^n$ with the following properties:*

(1) $\mathbf{B}^n=\bigcup_k \mathcal{D}(a_k,r)$;

(2) The sets $\mathcal{D}(a_k,\frac{r}{4})$ are mutually disjoint;

(3) Each point $z\in\mathbf{B}^n$ belongs to at most N of the sets $\mathcal{D}(a_k,4r)$

Lemma G *[23] For each $r>0$ there exists a positive constant C_r such that*

$$C_r^{-1}\le\frac{1-|a|^2}{1-|z|^2}\le C_r,\qquad C_r^{-1}\le\frac{1-|a|^2}{|1-\langle z,a\rangle|}\le C_r,$$

for all a and z such that $\beta(a,z) < r$. Moreover,if r is bounded above, then we may choose C_r independent of r.

For $\xi \in \mathbf{S}^n$ and $r > 0$ set $Q_r(\xi) = \{z \in \mathbf{B}^n : d(z,\xi) < r\}$, where d is a non-isotropic metric on $\mathbf{S}^n$, $d(z,w) = |1 - \langle z,w\rangle|^{\frac{1}{2}}$ is called the Carles on tube at ξ (see [23]).Let us note also that according to well known estimate for Poisson integral of functions from Hardy classes in the unit ball (see[23], page 154) applied $(m-2)$ times we have Since according to page 154 of [23]

$$|f(z,\dots,z)|^p \le C \int_{\mathbf{S}^n} \cdots \int_{\mathbf{S}^n} |f(\xi_1,\dots,\xi_{m-2},z,z)|^p \times$$

$$\times \frac{(1-|z|^2)^n \cdots (1-|z|^2)^n}{\prod_{k=1}^{m-2} |1-\langle \xi_k, z\rangle|^{2n}} d\sigma(\xi_1)\cdots d\sigma(\xi_{m-2}),$$

$z \in \mathbf{B}, f \in H^p(\mathbf{B}\times\cdots\times\mathbf{B}) = H^p(\mathbf{B}^m), 0 < p < \infty$. From last estimate now it is easy to see that if

$$\int_{\mathbf{B}} |f(z,z)|^p (1-|z|)^{n-1} dv(z) \le C\|f\|_{H^p(\mathbf{B}^2)}, \tag{20}$$

$$0 < p < \infty,\ n \ge 1,\ f \in H^p(\mathbf{B}^2),$$

then for any $m > 1\, m \in \mathbb{N},\ f \in H^p(\mathbf{B}^m)$, $0 < p < \infty$,

$$\int_{\mathbf{B}} |f(z,\dots,z)|^p (1-|z|)^{n(m-1)-1} dv(z) \le C\|f\|^p_{H^p(\mathbf{B}^m)}. \tag{21}$$

So $Trace(H^p(\mathbf{B}^m)) \subset A^p_{n(m-1)-1}(\mathbf{B}), 0 < p < \infty$. Let us note for (20) it is enough to show that

$$\int_{\mathbf{B}} (u(z,z))^2 (1-|z|)^{n-1} dv(z) \le C\|u\|_{L^2(\mathbf{B}^2)}$$

for any non negative subharmonic (by z_1 and z_2) function $u = u(z_1,z_2)$. It follows from the fact that all functions $|f(z_1,z_2)|^p,\ p \in (0,\infty),\ f \in H(\mathbf{B}^2)$ are subharmonic by z_1 and z_2. Note

$$u(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} a_{n_1 n_2} z_1^{n_1} z_2^{n_2} = \sum_{n_2=-\infty}^{\infty} v_{n_2}(z_1) z_2^{n_2}\ z_1, z_2 \in \mathbf{B}.$$

Hence by Hölder inequality

$$(u(z,z))^2 \leq \left(\sum_{k_2=-\infty}^{\infty}(|k_2|+1)^{\frac{n}{2}}|z|^{|k_2|}|v_{k_2}(z)|^2\right)\left(\sum_{k_2=-\infty}^{\infty}(|k_2|+1)^{\frac{-n}{2}}|z|^{|k_2|}\right);$$

Hence

$$\int_{\mathbf{B}}(u(z,z))^2(1-|z|)^{n-2}dv(z) \leq C\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}|a_{n_1,n_2}|^2 = C\|u\|_{L^2}.$$

Theorem 5 *Let* $0<p<\infty\, m\, n\in\mathbb{N}\, m>1$. *Let also* $\int_{\mathbf{B}}|f(z,z)|^p(1-|z|)^{n-1}dv(z) \leq C\|f\|_{H^p(\mathbf{B}^2)}$. *Then* $Trace(H^p(\mathbf{B}^m)) = A^p_{n(m-1)-1}(\mathbf{B})$.

Proof 2 *The proof for* $p\leq 1$ *follows from (21) and Theorem 4.Let* $f\in A^p_{n(m-1)-1}(\mathbf{B})$, $p>1$, *and let*

$$g(z_1,\dots,z_n) = C\int_{\mathbf{B}}\frac{f(w)(1-|w|^2)^{2mn-n-1}dv(w)}{(1-\langle\overline{w},z_1\rangle)^{2n}\cdots(1-\langle\overline{w},z_m\rangle)^{2n}},$$

$$g(z,\dots,z) = f(z).$$

So we must show $g\in H^p(\mathbf{B}^m)$. *Let* $\Phi\in L^q(\mathbf{S}^m), \|\Phi\|_{L^q(\mathbf{S}^m)} = 1, \frac{1}{p}+\frac{1}{q}=1$. *Then by standard duality argument inserting the integral representation for g function and using Fubini's theorem we have*

$$\|g_r\|_{H^p(\mathbf{S}^m)} \leq C\int_{\mathbf{B}}|f(w)|(1-|w|^2)^{nm-n-1}dv(w)\int_{\mathbf{S}^m}\frac{|\Phi(\xi)|(1-|w|^2r^2)^{nm}}{\prod_{k=1}^m|1-\langle\overline{w}r,\xi_k\rangle|^{2n}}d\sigma(\xi_1,\dots,\xi_m)$$

$$= C\int_{\mathbf{B}}|f(w)|(1-|w|^2)^{nm-n-1}P(\Phi)(wr,\dots,wr)dv(w),$$

where $P(\Phi)$ *is a Poisson integral of* Φ *in the ball* $\mathbf{B}$ *(see [23], Chapter 4).It remains to use Hölder's inequality with* p, $q\left(\frac{1}{p}+\frac{1}{q}=1\right)$ *and estimate (21) and known properties of Poisson integral in the ball (see[23], page 154).*

$$\|g_r\|_{L^p(\mathbf{S}^m)} \leq C\left(\int_{\mathbf{B}}|f(w)|^p(1-|w|^2)^{nm-n-1}dv(w)\right)^{\frac{1}{p}}.$$

The proof of theorem is complete.

Remark 9 *Arguments we used above in the proof of Theorem 5 were partially based on corresponding arguments for $n = 1$ case (case of poly disk, see [8]).*

Remark 10 *Note that for $n = 1$ Theorem 5 is well-known (see[8], [9], [20] and references there).*

We mention at the end of paper some similar constructions concerning poly half planes. Ideas we present below are based one stimates and results from [7] and [6] for A^p_α classes in a half plane. We need basic definitions. Let $\mathbb{C}_+$ is a upper half plane in $\mathbb{C}$, and $\mathbb{C}^m_+ = \mathbb{C}_+ \times \cdots \times \mathbb{C}_+, m \in \mathbb{N}, m > 1$, be a poly half plane.$H(\mathbb{C}_+)$ is a space of all analytic functions in upper half plane, and $H(\mathbb{C}^m_+)$ is a space of all functions of the type $F(z_1, \dots, z_m)$ analytic in$H(\mathbb{C}_+)$ by each z_j variable.

We define Bergman classes in half plane. Let

$$A^p_\alpha(\mathbb{C}_+) = \{f \in H(\mathbb{C}_+) : \|f\|_{p,\alpha} = \left(\int_0^\infty \int_{-\infty}^\infty |f(x+iy)|^p y^\alpha dxdy\right)^{\frac{1}{p}} < \infty,$$

$\alpha > -1,\ 0 < p < \infty\}$.

We also define Bergman classes in poly half plane. Let

$$A^p_\alpha(\mathbb{C}^m_+) = \{f \in H(\mathbb{C}^m_+) : \|f\|_{p,\alpha} = (\int_0^\infty \int_{-\infty}^\infty \cdots \int_0^\infty \int_{-\infty}^\infty |f(x_1+iy_1, \dots, x_m+iy_m)|^p \times$$

$$\times \prod_{j=1}^m y_j^{\alpha_j} dx_j dy_j)^{\frac{1}{p}} < \infty, \quad \alpha_j > -1,\ j = 1, \dots, m, 0 < p < \infty\}.$$

According to result from [6] the following proposition is true. Let $d\widetilde{m}_2(w)$ denote a Lebegues measure on $\mathbb{C}_+, d\widetilde{m}_2(w) = dxdy$.

Proposition A *If $f \in A^p_\alpha(\mathbb{C}_+), 0 < p < \infty, \alpha > -1$ then*

$$f(z) = C_\beta \int_0^\infty \int_{-\infty}^\infty \frac{f(w)(Im(w))^\beta}{(\overline{w} - z)^{2+\beta}} d\widetilde{m}_2(w),$$

where $0 < p \le 1, \beta \ge \frac{2+\alpha}{p} - 2,$ *or* $1 \le p < \infty, \beta \ge \frac{1+\alpha}{p} - 1.$

We need also a lemma from [7].

Proposition B *[7]*

Let $\mathbb{C}_+$ *is a covered by dyadic cubes* $\mathbb{C}_+ = \bigcup_{k=1}^{\infty} \triangle_k$. *Let* $(\triangle_k^*)$ *be enlarged cube (see[7]) and* $u \in H(\mathbb{C}_+)$. *Then*

$$\sup_{z\in\triangle_k} |u(z)|^p (Imz)^\alpha \leq \frac{C}{|\triangle_k^*|} \int_{\triangle_k^*} |u(w)|^p (Im(w))^\alpha d\widetilde{m}_2(w),$$

$0 < p < \infty, \alpha > -1.$

We use these assertions to obtain an estimate for expanded Bergman projections in the poly half plane an analogue of the estimate we had for poly ball above and using it as we did above we describe traces of Bergman classes in poly half plane.

Theorem 6 *Let* $p \leq 1, \alpha_j > -1, j = 1, \dots, m,$ *then* β *is large enough positive number, then* $Trace(A^p_\alpha(\mathbb{C}^m_+))$ *contain* $A^p_\gamma(\mathbb{C}_+)$ *that is* $Trace(A^p_{\overrightarrow{\alpha}}(\mathbb{C}^m_+)) = A^p_\gamma(\mathbb{C}_+)$, *for* $\gamma = (\sum_{k=1}^m \alpha_k) + 2m - 2$. *In other words for every function* f, $f \in A^p_{\overrightarrow{\alpha}}(\mathbb{C}^m_+), f(z, \dots, z)$ *is in* $A^p_\gamma(\mathbb{C}_+)$ *and the converse is also true.*

Proof 3 *Let* $f \in A^p_\gamma(\mathbb{C}_+)$. *Then for large enough* β *according to Proposition A we have* $f(z) = C_\beta \int_0^\infty \int_{-\infty}^\infty \frac{f(w)(Im(w))^\beta}{(\overline{w}-z)^{2+\beta}} d\widetilde{m}_2(w)$. *Now we consider*

$$F(z_1, \dots, z_m) = C_\beta \int_0^\infty \int_{-\infty}^\infty \frac{f(w)(Im(w))^\beta}{\prod_{k=1}^m (\overline{w} - z_k)^{\frac{2+\beta}{m}}} d\widetilde{m}_2(w).$$

Then obviously $F(z, \dots, z) = f(z)$ *and* $F(z_1, \dots, z_m) \in H(\mathbb{C}^m_+)$. *We will show that* $F \in A^p_{\overrightarrow{\alpha}}(\mathbb{C}^m_+)$ *for some* α. *This will prove the half of the theorem. We have the following chain of estimates by Proposition B and by the following estimates that can checked without difficulties (see also [7]). If* w_k *is a center of diadic cube* $\{\triangle_j\}$ *from Proposition B then*

$$(Im(w_k))^2 \asymp |\triangle_k| = \widetilde{m}_2(\triangle_k), \ \ (see\,[5],\,[6]) \tag{22}$$

$$(Im(w_k))^2 \asymp |\triangle_k^*| = \widetilde{m}_2(\triangle_k^*)$$

$$Im(w) \asymp Im(w_k), \ w \in \triangle_k,$$

$$\int_{\mathbb{C}_+} \frac{(Im(z))^\alpha d\widetilde{m}_2(z)}{|\overline{w} - z|^{(2+\beta)p}} \leq C(Im(\overline{w}))^{\alpha+2-(\beta+2)p}, \ \overline{w} \in \mathbb{C}_+, \ \ (see\ [5],\ [6]) \tag{23}$$

for all $\beta, (\beta+2)p-2>\alpha$. *Since* $\mathbb{C}_+ = \bigcup_{k=1}^{\infty} \triangle_k, p \le 1$ *we have*

$$\mathcal{J} = \int_{\mathbb{C}_+} \cdots \int_{\mathbb{C}_+} |F(z_1,\dots,z_m)|^p (Im(z_1))^{\alpha_1} \cdots (Im(z_m))^{\alpha_m} d\widetilde{m}_2(z_1)\cdots d\widetilde{m}_2(z_m)$$

$$\le C \int_{\mathbb{C}_+} \cdots \int_{\mathbb{C}_+} \prod_{k=1}^{m} (Im(z_k))^{\alpha_k} \sum_k \left(\int_{\triangle_k} \frac{|f(w)|(Im(w))^{\beta} d\widetilde{m}_2(w)}{|\prod_{k=1}^{m} (\overline{w}-z_k)^{\frac{\beta+2}{m}}|} \right)^p d\widetilde{m}_2(z_1)\cdots d\widetilde{m}_2(z_m).$$

Hence using (22) and (23)we get the following chain of estimates

$$\mathcal{J} \lesssim C \int_{\mathbb{C}_+} \cdots \int_{\mathbb{C}_+} \prod_{k=1}^{m} (Im(z_k))^{\alpha_k} \sum_k \max_{\triangle_k} |f(w)|^p (Im(w_k))^{\beta p} \frac{|\triangle_k|^p d\widetilde{m}_2(z_1)\cdots d\widetilde{m}_2(z_m)}{(\prod_{s=1}^{m} |\overline{w}_k - z_s|^{\frac{\beta+2}{m}p})}$$

$$\le C \left(\sum_k \max_{\triangle_k} |f(w)|^p \right) (Im(w_k))^{\sum_k \alpha_k + 2m - (\beta+2)p + \beta p} |\triangle_k|^p \le C \left(\sum_k \max_{\triangle_k} |f(w)|^p \right) (Im(w_k))^{\sum_k \alpha_k + 2m}$$

$$\le C \sum_k \int_{\triangle_k^*} |f(w)|^p (Im(w))^{\sum_k \alpha_k + 2m - 2} d\widetilde{m}_2(w) \lesssim C \int_{\mathbb{C}_+} |f(w)|^p (Im(w))^{\sum_k \alpha_k + 2m - 2} d\widetilde{m}_2(w).$$

In last estimate we used the fact that $\{\triangle_j\}$ *is a finite covering of* $\mathbb{C}_+$ *(see [6],[7] and Proposition B). So one part of the Theorem is obtained. To finish the proof we have to prove the following estimate*

$$K = \int_{\mathbb{C}_+} |f(w,\dots,w)|^p (Im(w))^{\sum_k \alpha_k + 2m - 2} d\widetilde{m}_2(w)$$
$$\le C \int_{\mathbb{C}_+} \cdots \int_{\mathbb{C}_+} |f(w_1,\dots,w_m)|^p (Im(w_1))^{\alpha_1} \cdots (Im(w_m))^{\alpha_m} d\widetilde{m}_2(w_1)\cdots d\widetilde{m}_2(w_m) = CK_1,$$

$0<p<\infty$. *We have by Lemma F and GThis last estimate follows from the following chain of estimates based on dyadic decomposition of* $\mathbb{C}_+$, *(22) and Proposition B*

$$K = \int_{\mathbb{C}_+} |f(w,\dots,w)|^p (Im(w))^{\sum_k \alpha_k + 2m - 2} d\widetilde{m}_2(w)$$

$$\lesssim C \sum_{k\ge 0} \max_{w\in\triangle_k} |f(w,\dots,w)|^p (Im(w_k))^{\sum_k \alpha_k + 2m}$$

$$\lesssim C \sum_{k_1,\dots,k_m \geq 0} \max_{\triangle_{k_1},\dots,\triangle_{k_m}} |f(w_1,\dots,w_m)|^p (Im(w_{k_1}))^{\alpha_{k_1}+2} \cdots (Im(w_{k_m}))^{\alpha_{k_m}+2}$$

$$\lesssim C \sum_{k_1,\dots,k_m} \int_{\triangle_{k_1}} \cdots \int_{\triangle_{k_m}} |f(w_1,\dots,w_m)|^p \prod_{k=1}^{m} Im(w_k)^{\alpha_k} d\widetilde{m}_2(w_k) \lesssim CK_1.$$

$$\lesssim C \sum_{k_1,\dots,k_m} \max_{\triangle_{k_1},\dots,\triangle_{k_m}} |f(w_1,\dots,w_m)|^p \left(Im(w_{k_1})^{\alpha_1+2} \cdots Im(w_{k_m})^{\alpha_m+2}\right) \leq CK_1.$$

The proof of theorem is complete.

Remark 11 *The description of traces of Bergman classes in poly half plane for $p > 1$ can be also obtained with the help of technique we developed above for $p \leq 1$ case and by some modification of arguments we provided in our paper [19] for the case of Bergman classes in poly balls $\mathbf{B}^m$ for* $p > 1$.

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