Revista Notas de Matemática
Vol.4(1), No. 260, 2008, pp.1-24
http://www.saber.ula.ve/notasdematematica/
Comisión de Publicaciones
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

A Lemma on Evolution Operators and Applications to Parabolic-Delay Equations

Alexander Carrasco and Hugo Leiva

Resumen

En este trabajo se caracterizan a una familia de operadores de la evolución que se pueden aplicar para demostrar la existencia y unicidad de las soluciones de una clave general de ecuación diferenciales parciales no - autónomos con retraso. Con una aplicación se considera el siguiente sistema de la ecuación parabólica de retardo:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} &= D\Delta u(t,x) + B(t)u_t(t,x) + f(t,x), \ t > s, \ u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} &= 0, \ t > s, \ x \in \partial\Omega, \\ u(s,x) &= \psi(x), \ x \in \Omega, \\ u_s(\tau,x) &= \phi(\tau,x), \ \tau \in [-r,0), \ x \in \Omega, \end{cases}$$

donde $s \geq 0$, Ω es un dominio limitado en $\mathbb{R}^N (N \geq 1)$, D es una matriz diagonal $n \times n$ cuyos valores propios son semisimple con términos parte real negativa y $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ es una buena función. La notación estandar $u_s(x)$ define una función de $[-r,0]$ a $\mathbb{R}^n$ (con x fijo) por $u_s(x)(\tau) = u(s+\tau,x)$, $-r \leq \tau \leq 0$. Está $r \geq 0$ es la máxima demora, que se supone es finito. Suponemos que, el operador $B \in L^\infty([0,\infty); \mathcal{L}(Z_1,Z))$ con $Z_1 = L^2([-r,0],Z)$, $Z = L^2(\Omega)$. El principal objetivo de este trabajo es generalizar el Lema 1.1 para una familia de operadores de la evolución y utilizarla para obtener una variación de las constantes de la fórmula para las soluciones de este sistema (Ecuación parabólica con retardo).

Palabras claves: Operador fuertemente continuo de evolución, operador de evolución del Lema, ecuación parabólica de retraso.

Resumen

In this paper we characterize a family of evolution operators that can be applied to prove the existence and uniqueness of the solutions of a general class of non-autonomous partial differential equations with delay. As application we consider the following system of Parabolic-Delay Equations:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = & D\Delta u(t,x) + B(t)u_t(t,x) + f(t,x),\ t > s,\ u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} & = & 0,\ t > s,\ x \in \partial\Omega, \\ u(s,x) & = & \psi(x),\ x \in \Omega, \\ u_s(\tau,x) & = & \phi(\tau,x),\ \tau \in [-r,0),\ x \in \Omega, \end{cases}$$

where $s \geq 0$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, D is a $n \times n$ diagonal matrix whose eigenvalues are semisimple with non negative real part and $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ is a smooth function. The standard notation $u_s(x)$ define a function from $[-r,0]$ to (with x fixed) by $u_s(x)(\tau) = u(s+\tau,x),\ -r \leq \tau \leq 0$. Here $r \geq 0$ is the maximum delay, which is supposed to be finite. We assume that, the operator $B \in L^\infty([0,\infty);\mathcal{L}(Z_1,Z))$ with $Z_1 = L^2([-r,0],Z)$, $Z = L^2(\Omega)$. The main objective of this work is to generalize Lemma 1.1 for a family of evolution operators and use it to derive a variation of constants formula for the solutions of this system(Parabolic-Delay Equations).

key words. strongly continuous evolution operator, lemma evolution operator, parabolic-delay equations.

AMS(MOS) subject classifications. primary: 34G10, 93B05; secondary: 35B40,93C25. **Running Title: A Lemma on Evolution Operator.**

1 Introduction

The Lemma 2.1 from [9] has been generalized in Lemma 3.1 from [1] which states that:

Lemma 1.1 *Let Z be a separable Hilbert space, $\{S_n(t)\}_{n\geq 1}$ a family of strongly continuous semigroups and $\{P_n\}_{n\geq 1}$ a family of complete orthogonal projection in Z such that:*

$$\Lambda_n P_n = P_n \Lambda_n, \ \ n \geq 1, 2, \ldots$$

where Λ_n is the infinitesimal generator of S_n.

Define the following family of linear operators

$$S(t)z = \sum_{n=1}^{\infty} S_n(t) P_n z, \qquad t \geq 0.$$

Then:

(a) $S(t)$ is a bounded linear bounded operator if $\| S_n(t) \| \leq g(t), n = 1, 2, ...,$ with $g(t) \geq 0$, continuous for $t \geq 0$.

(b) $\{S(t)\}_{t\geq 0}$ *is a strongly continuous semigroup in the Hilbert space* Z *whose infinitesimal generator* Λ *is given by*

$$\Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z, \qquad z \in D(\Lambda).$$

with

$$D(\Lambda) = \left\{ z \in Z \,/\, \sum_{n=1}^{\infty} \| \Lambda_n P_n z \|^2 < \infty \right\}.$$

(c) The spectrum $\sigma(\Lambda)$ *of* Λ *is given by*

$$\sigma(\Lambda) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}, \tag{1.1}$$

where $\bar{\Lambda}_n = \Lambda_n P_n : \mathcal{R}(P_n) \to \mathcal{R}(P_n)$.

Those Lemmas have been used in [1], [2], [3], [6],[9], [10],[11] and [12] in order to prove existence of solutions for partial and functional partial differential equations, stability and controllability as well. However, these Lemmas can not be applied directly to time dependent differential equations or evolutionary differential equations in general like the following system of time varying functional partial parabolic equation:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} &=& D\Delta u(t,x) + B(t)u_t(t,x) + f(t,x),\ t > s,\ u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} &=& 0,\ t > s,\ x \in \partial\Omega, \\ u(s,x) &=& \psi(x),\ x \in \Omega, \\ u_s(\tau,x) &=& \phi(\tau,x),\ \tau \in [-r,0),\ x \in \Omega, \end{cases} \tag{1.2}$$

where $s \geq 0$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, D is a $n \times n$ diagonal matrix whose eigenvalues are semisimple with non negative real part and $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ is a smooth function. The standard notation $u_s(x)$ define a function from $[-r, 0]$ to $\mathbb{R}^n$(with x fixed) by $u_s(x)(\tau) = u(s+\tau, x),\ -r \leq \tau \leq 0$. Here $r \geq 0$ is the maximum delay, which is supposed to be finite. We assume that, the operator $B \in L^\infty([0,\infty); \mathcal{L}(Z_1, Z))$ with $Z_1 = L^2([-r,0], Z)$, $Z = L^2(\Omega)$.

This system is motivated by the following Open Problem: Suppose the next functional ordinary differential equation admits an asymptotically stable periodic orbit $\mathcal{P}$

$$\dot{y}(t) = h(y_t), \quad t \in I\!R, \quad y \in I\!R^n, \tag{1.3}$$

where $h \in C^2(C([-r,0];I\!R^n);I\!R^n)$, $y_t : [-r,0] \longrightarrow I\!R^n$ defined by $y_t(s) = y(t+s), \quad s \in [-r,0]$ and $r \geq 0$ is the maximum delay, which is supposed to be finite.

Now, if we add diffusion to the system (1.3) with Neumann boundary condition we get the following functional partial differential system

$$\left\{\begin{array}{lcl} \dfrac{\partial w(t,x)}{\partial t} & = & D\Delta w(t,x) + h(w_t(t,x)),\ t > 0s,\ w \in \mathbb{R}^n, \\ \dfrac{\partial w(t,x)}{\partial \eta} & = & 0,\ t > s,\ x \in \partial\Omega, \\ w(s,x) & = & \psi(x), x \in \Omega, \\ w_s(\tau,x) & = & \phi(\tau,x),\ \tau \in [-r,0),\ x \in \Omega, \end{array}\right. \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$ and D is a $n \times n$ diagonal matrix whose eigenvalues are semisimple with non negative real part.

Since we assume Neumann boundary conditions, then $\mathcal{P}$ is also periodic solution of the system (1.4). So, the open question is: **Under which condition this periodic orbit still asymptotically stable for the functional partial differential equation (1.4)**?.

Using some ideas from [8] and [5] we can try to solve this problem by considering the variational equation around this periodic orbit,i. e., by the change of variable $u = w - \mathcal{P}(t)$, study the stability of the zero solution of the system

$$\left\{\begin{array}{lcl} \dfrac{\partial u(t,x)}{\partial t} & = & D\Delta u(t,x) + \mathcal{B}(t)u_t(t,x) + H(t,u_t(t,x)),\ t > 0,\ u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} & = & 0,\ t > 0,\ x \in \partial\Omega, \end{array}\right. \tag{1.5}$$

where $\mathcal{B}(t) = h_y(\mathcal{P}_t)$ and $H(t,\phi) = \mathcal{O}(\|\phi\|_c), \quad \phi \in C([-r,0];I\!R^n)$.

The main objective of this work is to generalize Lemma 1.1 for a family of evolution operators and use it to derive a variation of constants formula for the solutions of system (1.2). We hope this result can be useful to solve the foregoing Legendary Open Problem.

2 Evolution Operators

In general, we are interested in the abstract Cauchy problem defined on a Hilbert space Z,

$$\begin{cases} \dfrac{dz(t)}{dt} &= A(t)z(t), \quad 0 \leq s \leq t < \infty, \\ z(s) &= z_0 \in Z, \end{cases} \tag{2.1}$$

where $z : [0, \infty) \longrightarrow Z$, $A(t)$ is a family of unbounded linear operators from $D(A(t)) = D$ in Z, independent on t, such that $A(\cdot)z \in C(\mathbb{R}^+, Z)$ for each $z \in D$.

This motives the study of evolution operators and the generator of these operators. We start this section with the definition of fundamental solution of (2.1).

Definition 2.1 *An operator-value function $U(t,s) \in \mathcal{L}(Z)$ which is strongly continuous jointly in t, s for $0 \leq s \leq t < \infty$, is called fundamental solution of (2.1) if*

1. *For all $z \in D$ the partial derivative $\dfrac{\partial}{\partial t}U(t,s)z$ exists in the strong topology of Z and it is strongly continuous in (t,s) for $0 \leq s \leq t < \infty$.*
2. *For all $z \in D$, $U(t,s)z \in D$.*
3. *For all $z \in D$, $\dfrac{\partial}{\partial t}U(t,s)z = A(t)U(t,s)z$, $0 \leq s \leq t < \infty$ and $U(s,s) = I$.*

Proposition 2.2 *The operator-value function $U(t,s)$ given by the foregoing definition satisfies the following properties:*

$$U(t,s)z_0 = z_0 + \int_s^t A(\tau)U(\tau,s)z_0 d\tau, \quad \forall z_0 \in D. \tag{2.2}$$

$$\lim_{r \longrightarrow t^+} \lim_{k \longrightarrow t^-} \frac{U(r,k)U(k,s)z_0 - U(k,s)z_0}{r-k} = \lim_{k \longrightarrow t^-} \lim_{r \longrightarrow t^+} \frac{U(r,k)U(k,s)z_0 - U(k,s)z_0}{r-k}. \tag{2.3}$$

Proof (2.2) is trivial. So, we only prove (2.3). In fact,

$$\begin{aligned}
\lim_{r\longrightarrow t^+}\lim_{k\longrightarrow t^-}\frac{U(r,k)U(k,s)z_0-U(k,s)z_0}{r-k} &= \lim_{r\longrightarrow t^+}\lim_{k\longrightarrow t^-}\frac{1}{r-k}\int_k^r A(\tau)U(\tau,k)U(k,s)z_0d\tau\\
&= \lim_{r\longrightarrow t^+}\lim_{k\longrightarrow t^-}\frac{1}{r-k}\int_k^r A(\tau)U(\tau,s)z_0d\tau\\
&= \lim_{r\longrightarrow t^+}\frac{1}{r-t}\int_t^r A(\tau)U(\tau,s)z_0d\tau\\
&= \lim_{h\longrightarrow 0^+}\frac{1}{h}\int_t^{t+h} A(\tau)U(\tau,s)z_0d\tau\\
&= A(t)U(t,s)z_0.
\end{aligned}$$

and

$$\begin{aligned}
\lim_{k\longrightarrow t^-}\lim_{r\longrightarrow t^+}\frac{U(r,k)U(k,s)z_0-U(k,s)z_0}{r-k} &= \lim_{k\longrightarrow t^-}\lim_{r\longrightarrow t^+}\frac{1}{r-k}\int_k^r A(\tau)U(\tau,k)U(k,s)z_0d\tau\\
&= \lim_{k\longrightarrow t^-}\lim_{r\longrightarrow t^+}\frac{1}{r-k}\int_k^r A(\tau)U(\tau,s)z_0d\tau\\
&= \lim_{k\longrightarrow t^-}\frac{1}{t-k}\int_k^t A(\tau)U(\tau,s)z_0d\tau\\
&= \lim_{h\longrightarrow 0^+}\frac{1}{h}\int_{t-h}^{t} A(\tau)U(\tau,s)z_0d\tau\\
&= A(t)U(t,s)z_0.
\end{aligned}$$

□

The above calculation motives the following definition.

Definition 2.3 *A two parameter family of bounded linear operators $U(t,s)\in\mathcal{L}(Z)$, $0\leq s\leq t<\infty$, is called an evolution operator if the following conditions are satisfied:*

1. $U(s,s)=I$ *and* $U(t,r)U(r,s)=U(t,s)$ *for* $0\leq s\leq t<\infty$.
2. $(t,s)\longrightarrow U(t,s)$ *is strongly continuous for* $0\leq s\leq t<\infty$.
3. *Exists a real valued continuous nonnegative function* $g(t,s)$ *with* $\| U(t,s)\|\leq g(t,s)$ *for all* $0\leq s\leq t<\infty$.

Now, we shall give a definition of generator of an evolution operator.

Definition 2.4 *The generator $A(t)$ of an evolution operator $U(t,s)$, $0 \leq s \leq t < \infty$ is defined as follows:*

$$A(t)z \quad = \quad \lim_{h \longrightarrow 0^+} \frac{U(t+h,t)z - z}{h}, \quad \forall z \in D(A(t)), \quad 0 \leq t < \infty.$$

where $D(A(t)) = D$ is given by

$$D = \left\{ z \in Z : \lim_{r \longrightarrow t^+} \lim_{k \longrightarrow t^-} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} = \lim_{k \longrightarrow t^-} \lim_{r \longrightarrow t^+} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} \right\}$$

and the limit exists for all $0 \leq s \leq t < \infty$.

Remark 2.1 *The foregoing definition is similar to the given in [7] pg 1902, and the generator of an evolution operator satisfies, for $z \in D$, the following property:*

$$A(t)z = \lim_{k \longrightarrow t^-} \frac{U(t,k)z - z}{t-k} = \lim_{r \longrightarrow t^+} \frac{U(r,t)z - z}{r-t}.$$

In fact,

$$\lim_{r \longrightarrow t^+} \lim_{k \longrightarrow t^-} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} = \lim_{k \longrightarrow t^-} \lim_{r \longrightarrow t^+} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k}, \ 0 \leq s \leq t < \infty.$$

If we put $s = t$, then

$$\begin{aligned} A(t)z \quad &= \quad \lim_{h \longrightarrow 0^+} \frac{U(t+h,t)z - z}{h} \\ &= \quad \lim_{k \longrightarrow t^-} \frac{U(t,k)z - z}{t-k} = \lim_{r \longrightarrow t^+} \frac{U(r,t)z - z}{r-t}. \end{aligned}$$

Lemma 2.5 *Let $U(t,s)$, $0 \leq s \leq t < \infty$ be an evolution operator on Z such that, $U(t,s)z \in D$ for all $z \in D$. Then for all $z \in D$ we have that*

$$\frac{\partial}{\partial t} U(t,s)z = A(t)U(t,s)z, \quad for \ \ 0 \leq s \leq t \leq T.$$

$$\frac{\partial}{\partial s} U(t,s)z = -U(t,s)A(s)z \ \ for \ \ 0 \leq s < t \leq T.$$

Proof If $z \in D$, then from the hypothesis we have that $U(t,s)z \in D$ for $0 \leq s \leq t < \infty$ and

$$\lim_{h \longrightarrow 0^+} \frac{U(t+h,s)z - U(t,s)z}{h} = \lim_{h \longrightarrow 0^+} \frac{U(t+h,t)U(t,s)z - U(t,s)z}{h}.$$

Since $U(t,s)z \in D$, we obtain that

$$\lim_{h \longrightarrow 0^+} \frac{U(t+h,s)z - U(t,s)z}{h} = A(t)U(t,s)z.$$

Now, suppose $t > s$ and $h \geq 0$ is small enough such that $t - h \geq s$. Then

$$\begin{aligned}
\lim_{h \longrightarrow 0^+} \frac{U(t-h,s)z - U(t,s)z}{-h} &= \lim_{h \longrightarrow 0^+} \frac{-U(t-h,s)z + U(t,s)z}{h} \\
&= \lim_{h \longrightarrow 0^+} \frac{U(t,t-h)U(t-h,s)z - U(t-h,s)z}{h} \\
&= \lim_{k \longrightarrow t^-} \frac{U(t,k)U(k,s)z - U(k,s)z}{t-k} \\
&= \lim_{k \longrightarrow t^-} \lim_{r \longrightarrow t^+} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} \\
&= \lim_{r \longrightarrow t^+} \lim_{k \longrightarrow t^-} \frac{U(r,k)U(k,s)z - U(k,s)z}{r-k} \\
&= \lim_{r \longrightarrow t^+} \frac{U(r,t)U(t,s)z - U(t,s)z}{r-t} \\
&= \lim_{h \longrightarrow 0^+} \frac{U(h+t,t)U(t,s)z - U(t,s)z}{h}, \quad (h = r-t) \\
&= A(t)U(t,s)z.
\end{aligned}$$

So,

$$\frac{\partial}{\partial t} U(t,s)z = A(t)U(t,s)z, \quad for \;\; 0 \leq s < t < \infty.$$

Finally,

$$\frac{\partial}{\partial t} U(t,s)z \mid_{t=s} = \lim_{h \longrightarrow 0^+} \frac{U(s+h,s)z - z}{h} = A(s)z.$$

Newly, suppose that $t > s$ and $h \geq 0$ is small enough such that $s + h < t$. Then

$$\begin{aligned}
\left\| \frac{U(t,s+h)z - U(t,s)z}{h} + U(t,s)A(s)z \right\| &= \left\| \frac{U(t,s+h)z - U(t,s+h)U(s+h,s)z}{h} \right. \\
&\quad \left. + U(t,s+h)U(s+h,s)A(s)z \right\| \\
&= \left\| -U(t,s+h) \left\{ \frac{U(s+h,s)z - z}{h} - U(s+h,s)A(s)z \right\} \right\| \\
&\leq g(t,s+h) \left\| \left\{ \frac{U(s+h,s)z - z}{h} - U(s+h,s)A(s)z \right\} \right\|.
\end{aligned}$$

Since,

$$\lim_{h \longrightarrow 0^+} \frac{U(s+h,s)z - z}{h} = A(s)z,$$

we get that
$$\lim_{h\longrightarrow 0+}\frac{U(t,s+h)z-U(t,s)z}{h}=-U(t,s)A(s)z.$$
Analogously,
$$\begin{aligned}\lim_{h\longrightarrow 0^+}\frac{U(t,s-h)z-U(t,s)z}{-h} &= \lim_{h\longrightarrow 0^+}\frac{U(t,s)U(s,s-h)z-U(t,s)z}{-h}\\ &= -U(t,s)\lim_{h\longrightarrow 0^+}\frac{U(s,s-h)z-z}{h}\\ &= -U(t,s)A(s)z.\end{aligned}$$
Therefore,
$$\frac{\partial}{\partial s}U(t,s)z=-U(t,s)A(s)z,\quad for\ \ 0\le s\le t<\infty.$$

□

Theorem 2.6 *Let $U(t,s)$, $0\le s\le t<\infty$ be an evolution operator on Z satisfying the condition on Lemma 2.5, and $A(t)$ its generator with domain D. Then the Cauchy problem*
$$\begin{cases} z'(t) &= A(t)z(t),\ \ t\ge s,\\ z(s) &= z_0,\ \ z_0\in D,\end{cases}$$
has the unique solution
$$z(t)=U(t,s)z_0,\ \ t\ge s.$$

Proof From Lemma 2.5 we get that $z(t)=U(t,s)z_0$ is one solution of the Cauchy problem. Now, we shall prove the uniqueness; for this, we will suppose that $y(t)$ is another solution of the problem. Then the difference $w(t)=z(t)-y(t)$ satisfies the differential equation
$$\frac{dw}{dt}=A(t)w(t),\ t\ge s;\ \ w(s)=0$$
so we need to show that $w(t)\equiv 0$. For this, let us define $F(u)=U(t,u)w(u)$,
$0\le u\le s<t$. Then
$$\begin{aligned}F'(u) &= \frac{\partial}{\partial u}U(t,u)y(u)+U(t,u)\frac{d}{du}y(u)\\ &= -U(t,u)A(u)y(u)+U(t,u)A(u)y(u)\\ &= 0.\end{aligned}$$
Therefore, $F(u)=U(t,u)y(u)=c$ (constant). In particular, if we put $u=s$, we have that $F(s)=U(t,s)w(s)=0$.

Now, from the strongly continuity of $U(t,s)$ we get that

$$\begin{aligned} F(t) &= \lim_{s \longrightarrow t^-} F(s) \\ &= \lim_{s \longrightarrow t^-} U(t,s)w(s) \\ &= 0. \end{aligned}$$

Hence, $w(t) = z(t) - y(t) = 0$. □

Theorem 2.7 *Let $U(t,s)$, $0 \le s \le t < \infty$ be an evolution operator on Z satisfying the condition on Lemma 2.5, and $A(t)$ its generator with domain D. Consider the non-homogeneous Cauchy problem.*

$$\begin{cases} z'(t) = A(t)z(t) + f(t), & t \ge s, \\ z(s) = z_0, \quad z_0 \in Z, & 0 \le s \le t. \end{cases} \tag{2.4}$$

Suppose that either

(i) $z_0 \in Z$ and $f \in C(\mathbb{R}_+, Z)$ takes values on D and $(A(\cdot)f(\cdot))' \in C(\mathbb{R}_+, Z)$,
or
(ii) $z_0 \in D$ and $f \in C^1(\mathbb{R}_+, Z)$.
Then (2.4) has an unique solution $z \in C^1(\mathbb{R}_+, Z)$ with value on D. Moreover, this solution $z(t)$ is a solution of the following integral equation

$$z(t) = U(t,s)z_0 + \int_s^t U(t,\alpha)f(\alpha)d\alpha. \tag{2.5}$$

Definition 2.8 *A solution of (2.5) is called mild solution of (2.4).*

Remark 2.2 *A particular case of an evolution operator $U(t,s)$ which satisfies the condition on Lemma 2.5, is given by*

$$U(t,s)z = T(t-s)z + \int_s^t T(t-\alpha)B(\alpha)U(\alpha,s)zd\alpha, \tag{2.6}$$

where A is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\ge 0}$ in the Banach space Z and $B \in P_\infty([0,\infty); \mathcal{L}(Z))$, where

$$P_\infty([0,\infty); \mathcal{L}(Z)) := \left\{ B / \langle z_1, B(\cdot)z_2 \rangle \text{ is measurable for every } z_1, z_2 \in Z \text{ and } (ess \sup_{0 \le t < \infty}) \parallel B(t) \parallel_{\mathcal{L}(Z)} < \infty \right\}.$$

In this case, it is clear that the generator of this evolution operator is

$$A(t) = A + B(t),$$

with domain $D(A(t)) = D(A) = D$. *See [4].*

Proposition 2.9 *Consider the evolution operator* $U(t,s)$ *given by (2.6). Then*

1. $\dfrac{\partial}{\partial t}U(t,s)z = A(t)U(t,s)z, \;\; 0 \leq s \leq t < \infty.$
2. $\dfrac{\partial}{\partial s}U(t,s)z = -U(t,s)A(t)z, \;\; 0 \leq s < t < \infty.$
3. $A(t)$ *is closed.*

3 Mean Theorem

In this section we shall characterize a family of evolution operators that can be used to prove the existence and uniqueness of solutions for a general class of non-autonomous functional partial differential equations.

Lemma 3.1 *Let* Z *be a Hilbert Space,* $\{U_n(t,s)\}_{0\leq s\leq t<\infty}$ *a family of evolution operators and* $P_n(\cdot) : [0,\infty) \longrightarrow L(Z)$; $n = 1, 2, \dots,$ *a family of strongly continuous orthogonal projections on* Z*, which are complete and*

$$P_n(t)U_n(t,s) = U_n(t,s)P_n(s); \; n = 1, 2, \dots, \;\; 0 \leq s \leq t < \infty.$$

Let us define the following family of linear operators

$$U(t,s) = \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z, \;\; 0 \leq s \leq t < \infty.$$

Then, the following statements holds:

(i) $\{U(t,s)\}_{0\leq s\leq t<\infty}$ *is an evolution operator, if* $\|U_n(t,s)\| \leq g(t,s),\; n = 1, 2, \dots,$ *with* $g(t,s) \geq 0$*, continuous in* $0 \leq s \leq t < \infty$.

(ii) The generator $A(t) : D \longrightarrow Z$ *of* $\{U(t,s)\}_{0\leq s\leq t<\infty}$ *is given by*

$$A(t)z = \sum_{n=1}^{\infty} A_n(t)P_n(t)z, \;\; z \in D,$$

where

$$D \subset \mathcal{W} = \left\{ z \in Z : \sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 < \infty, \quad \forall t \in [0, \infty) \right\},$$

and if $A(t)$ is a closed operator, then $D = \mathcal{W}$.

(iii) Suppose $A(t)$ is a closed operator. If $z \in D$, then $U(t,s)z \in D$.

Proof We show first that $U(t,s)$ is a bounded linear operator for fixed $s \leq t$. In fact, let $z \in Z$. Then

$$\begin{aligned}
\| U(t,s)z \|^2 &= \left\langle \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z, \sum_{m=1}^{\infty} U_m(t,s)P_m(s)z \right\rangle \\
&= \sum_{n,m=1}^{\infty} \langle U_n(t,s)P_n(s)z, U_m(t,s)P_m(s)z \rangle \\
&= \sum_{n,m=1}^{\infty} \langle P_n(t)U_n(t,s)P_n(s)z, P_m(t)U_m(t,s)P_m(s)z \rangle \\
&= \sum_{n=1}^{\infty} \| U_n(t,s)P_n(s)z \|^2 \\
&\leq (g(t,s))^2 \| z \|^2 .
\end{aligned}$$

This proves that $U(t,s)$ is bounded.

Now, we will show that $U(t,r)U(r,s) = U(t,s)$ for $0 \leq r \leq s \leq t < \infty$

$$\begin{aligned}
U(t,r)U(r,s)z &= \sum_{n=1}^{\infty} U_n(t,r)P_n(r) \left(\sum_{i=1}^{\infty} U_i(r,s)P_i(s)z \right) \\
&= \sum_{n=1}^{\infty} U_n(t,r)U_n(r,s)P_n(s)z \\
&= \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z \\
&= U(t,s)z.
\end{aligned}$$

Next, we show that $U(t,s)$ is strongly continuous in $[0,\infty)$. In fact:

$$\begin{aligned}
\| U(t,s)z - z \|^2 &= \left\| \sum_{n=1}^{\infty} U_n(t,s)P_n(s)z - \sum_{n=1}^{\infty} P_n(s)z \right\|^2 \\
&= \sum_{n=1}^{\infty} \| (U_n(t,s) - I)P_n(s)z \|^2 \\
&= \sum_{n=1}^{N} \| (U_n(t,s) - I)P_n(s)z \|^2 + \sum_{n=N+1}^{\infty} \| (U_n(t,s) - I)P_n(s)z \|^2 \\
&\leq \sup_{1\leq n\leq N} \| (U_n(t,s) - I)P_n(s)z \|^2 \, N + K \sum_{n=N+1}^{\infty} \| P_n(s)z \|^2,
\end{aligned}$$

where $K = \sup_{0\leq s\leq t\leq 1;\, n\geq 1} \| U_n(t,s) - I \|^2 \leq (g(t,s)+1)^2$.

Since $\{U_n(t,s)\}_{0\leq s\leq t\leq T}$ $(n = 1, 2, \dots)$ is a strongly continuous evolution operator and $\{P_n(s)\}_{n\geq 1}$ is a complete orthogonal projections, given an arbitrary $\epsilon > 0$ we have, for some natural number N and $0 < s < t < 1$, the following estimates:

$$\sum_{n=N+1}^{\infty} \| P_n(s)z \|^2 < \frac{\epsilon}{2K}, \quad \sup_{1\leq n\leq N} \| (U_n(t,s) - I)P_n(s)z \|^2 < \frac{\epsilon}{2N}$$

and hence

$$\begin{aligned}
\| U(t,s)z - z \|^2 &< \frac{\epsilon}{2N} N + K \frac{\epsilon}{2K}, \\
&= \epsilon.
\end{aligned}$$

Therefore, $U(t,s)$ is a strongly continuous evolution operator in $[0,\infty)$.

Let $A(t)$ be the generator of this evolution operator. Then, from definition 2.2, we have for all $z \in D$,

$$A(t)z = \lim_{h\longrightarrow 0^+} \frac{U(t+h,t)z - z}{h} = \lim_{h\longrightarrow 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t+h,t) - I)}{h} P_n(t)z.$$

Therefore,

$$\begin{aligned}
P_m(t)A(t)z &= P_m(t) \left(\lim_{h\longrightarrow 0^+} \sum_{n=1}^{\infty} \frac{(U_n(t+h,t) - I)}{h} P_n(t)z \right) \\
&= \lim_{h\longrightarrow 0^+} \frac{(U_m(t+h,t) - I)}{h} P_m(t)z \\
&= A_m(t)P_m(t)z.
\end{aligned}$$

Hence,

$$A(t)z = \sum_{n=1}^{\infty} P_n(t)A(t)z = \sum_{n=1}^{\infty} A_n(t)P_n(t)z,$$

and

$$D \subset \mathcal{W} = \left\{ z \in Z : \sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 < \infty, \ \ \forall t \in [0,\infty) \right\}.$$

Now, suppose $A(t)$ is closed and $z \in \left\{ z \in Z : \sum_{k=1}^{\infty} \| A_k(t)P_k(t)z \|^2 < \infty, \ \ \forall t \in [0,\infty) \right\}$. Then

$$\sum_{k=1}^{\infty} \| A_k(t)P_k(t)z \|^2 < \infty, \ \ t \in [0,\infty) \ \text{ and } \ y = \sum_{k=1}^{\infty} A_k(t)P_k(t)z \in Z.$$

Therefore, if we consider $z_n = \sum_{k=1}^{n} P_k(t)z$, then $z_n \in D$ and $A(t)z_n = \sum_{k=1}^{n} A_k(t)P_k(t)z$.
Hence, $\lim_{n \longrightarrow \infty} z_n = z$ and $\lim_{n \longrightarrow \infty} A(t)z_n = y$ and since $A(t)$ is a closed linear operator we get that $z \in D$ and $A(t)z = y$.

(iii) If $A(t)$ is a closed operator, then $D = \mathcal{W}$. Now, let $z \in D$ and consider
$\sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 < \infty$, for all $0 \leq t < \infty$. Then,

$$\begin{aligned}
\left\| \sum_{n=1}^{\infty} A_n(t)P_n(t)U(t,s)z \right\|^2 &= \left\| \sum_{n=1}^{\infty} A_n(t)P_n(t) \left(\sum_{k=1}^{\infty} U_k(t,s)P_k(s)z \right) \right\|^2 \\
&= \left\| \sum_{n=1}^{\infty} A_n(t)U_n(t,s)P_n(s)z \right\|^2 \\
&\leq \sum_{n=1}^{\infty} \| A_n(t)U_n(t,s)P_n(s)z \|^2 \\
&\leq (g(t,s))^2 \sum_{n=1}^{\infty} \| A_n(t)P_n(t)z \|^2 \\
&< \infty,
\end{aligned}$$

for all $0 \leq s \leq t < \infty$. Hence, $U(t,s)z \in D$ for all $z \in D$ and $0 \leq s \leq t < \infty$. $\square$

4 Applications

In this section we shall use the foregoing result to find a variation of constants formula for the following system of functional partial parabolic equations:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} &= D\Delta u(t,x) + B(t)u_t(t,x) + f(t,x),\ t > s,\ u \in \mathbb{R}^n, \\ \dfrac{\partial u(t,x)}{\partial \eta} &= 0,\ t > s,\ x \in \partial\Omega, \\ u(s,x) &= \psi(x), x \in \Omega, \\ u_s(\tau,x) &= \phi(\tau,x),\ \tau \in [-r,s),\ x \in \Omega, \end{cases} \tag{4.1}$$

where $s \geq 0$, Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$, D is a $n \times n$ diagonal matrix whose eigenvalues are semisimple with non negative real part and $f : \mathbb{R} \times \Omega \longrightarrow \mathbb{R}^n$ is a smooth function. The standard notation $u_s(x)$ define a function from $[-r, 0]$ to $\mathbb{R}^n$(with x fixed) by $u_s(x)(\tau) = u(\tau + s, x),\ -r \leq \tau \leq 0$. Here $r \geq 0$ is the maximum delay, which is supposed to be finite. We assume that, for each $T \geq 0$, the operator $B \in L^\infty([0,\infty); \mathcal{L}(Z_1))$ with $Z_1 = L^2([-r, 0], Z)$, $Z = L^2(\Omega)$.

4.1 Abstract Formulation of the Problem

In this section we choose a Hilbert space where system (4.1) can be written as an abstract functional differential equation; for this, we consider the following hypothesis:

H1). The matrix D is semi simple (block diagonal) and the eigenvalues $d_i \in \boldsymbol{C}$ of D satisfy $\mathrm{Re}(d_i) \geq 0$. Consequently, if $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ are the eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, then there exists a constant $M \geq 1$ such that :
$\|e^{-\lambda_n D t}\| \leq M, \quad t \geq 0, \quad n = 1, 2, 3, ...$
H2). For all $I > 0$ and $z \in L^2_{loc}([-r,\infty); Z)$ we have the following inequality

$$\int_0^t \mid B(\alpha)z_\alpha \mid d\alpha \leq M_0(t) \mid z \mid_{L^2([-r,t),Z)},\ \ \forall t \in [0, I],$$

where $M_0(\cdot)$ is a positive continuous function on $[0,\infty)$.

Consider $H = L^2(\Omega, I\!R)$ and $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Then :

(i) There exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$ we have

$$-\Delta\xi = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi, \tag{4.2}$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_n x = \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k}. \tag{4.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in H and
$\xi = \sum_{n=1}^{\infty} E_n \xi, \quad \xi \in H.$

(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)\xi = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n \xi. \tag{4.4}$$

Now, we denote by Z the Hilbert space $L^2(\Omega, I\!R^n)$ and define the following operator

$$A : D(A) \subset Z \longrightarrow Z, \quad A\psi = -D\Delta\psi$$

with $D(A) = H^2(\Omega, I\!R^n) \cap H_0^1(\Omega, I\!R^n)$.

Therefore, for all $z \in D(A)$ we obtain,

$$Az = \sum_{n=1}^{\infty} \lambda_n D P_n z$$

and

$$z = \sum_{n=1}^{\infty} P_n z, \quad \| z \|^2 = \sum_{n=1}^{\infty} \| P_n z \|^2, \quad z \in Z,$$

where

$$P_n = diag(E_n, E_n, ..., E_n)$$

is a family of complete orthogonal proyections in Z.

Consequently, system (4.1) can be written as an abstract functional differential equation in Z:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + B(t)z_t + f^e(t), \quad t > 0, \\ \\ z(s) = \psi_0, \\ \\ z_s(\tau) = \phi(\tau), \quad \tau \in [-r, s). \end{cases} \tag{4.5}$$

Here $f^e : (0, \infty) \longrightarrow Z$ is a function defined as follows:

$$f^e(t)(x) = f(t, x), \quad t > 0, \quad x \in \Omega.$$

4.2 Existence and Uniqueness of Solutions

In case that $f^e \equiv 0$ the system (4.5)is given by:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + B(t)z_t, \quad t > 0, \\ \\ z(s) = \psi_0 = z_0, \\ \\ z_s(\tau) = \phi(\tau), \ \ \tau \in [-r, s). \end{cases} \tag{4.6}$$

So, the system (4.6) admits only one solution.

Definition 4.1 *A function $z(\cdot)$ define on $[s-r, \alpha)$ is called a Mild Solution of (4.6) if*

$$z(t) = \begin{cases} \phi(t-s); \quad s-r \leq t < s, \\ \\ T_A(t-s)z_0 + \displaystyle\int_s^t T_A(t-\gamma)B(\gamma)z_\gamma d\gamma, \quad t \in [s, \infty). \end{cases}$$

Theorem 4.2 *The problem (4.6) admits only one mild solution defined on $[s-r, \infty)$.*

5 The Variation of Constants Formula

Now we are ready to find the formula announced in the title of this paper for the system (4.5), but first we need to write this system as an abstract ordinary differential equation in an appropriate Hilbert space. In fact, we consider the Hilbert space $\mathbb{M}_2([-r, 0]; Z) = Z \oplus L^2([-r, 0]; Z)$ with the usual inner product given by:

$$\left\langle \begin{pmatrix} \phi_{01} \\ \phi_1 \end{pmatrix}, \begin{pmatrix} \phi_{02} \\ \phi_2 \end{pmatrix} \right\rangle = \langle \phi_{01}, \phi_{02} \rangle_Z + \langle \phi_1, \phi_2 \rangle_{L^2}.$$

Define the following operator in the space $\mathbb{M}_2$ for $t \geq s \geq 0$ by

$$U(t,s) \begin{pmatrix} \psi_0 \\ \phi(\cdot) \end{pmatrix} = \begin{pmatrix} z(t) \\ z_t(\cdot) \end{pmatrix} = \begin{pmatrix} z(t, s, \phi(\cdot), \psi_0) \\ z_t(\cdot, s, \phi(\cdot), \psi_0) \end{pmatrix} \tag{5.1}$$

where $z(\cdot)$ is the unique mild solution of the system (4.6).

Theorem 5.1 *The family of operators $\{U(t,s)\}_{t \geq s \geq 0}$ defined by (5.1) is a strongly continuous evolution operator on $\mathbb{M}_2$ such that*

$$U(t,s)W = \sum_{n=1}^{\infty} U_n(t,s)Q_nW, \ \ W \in \mathbb{M}_2, \ \ t \geq s \geq 0, \tag{5.2}$$

where,

$$Q_n = \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix},$$

with $(\widetilde{P}_n\phi)(\alpha) = P_n\phi(\alpha), \phi \in L^2([-r,0];Z),\ \alpha \in [-r,0],$ *and* $\{\{U_n(t,s)\}_{t\geq s\geq 0}, n = 1,2,3,..\}$ *is a family of strongly continuous evolution operators on* $\mathbb{M}_2^n = Q_n\mathbb{M}_2$ *is given defined as follows*

$$U_n(t,s)\begin{pmatrix} \psi_0^n \\ \phi^n(\cdot) \end{pmatrix} = \begin{pmatrix} z^n(t) \\ z^n(t+\cdot) \end{pmatrix}, \ \begin{pmatrix} \psi_0^n \\ \phi^n(\cdot) \end{pmatrix} \in \mathbb{M}_2^n,$$

where $z^n(\cdot)$ *is the unique solution of the initial value problem*

$$\begin{cases} \dfrac{dz(t)}{dt} = -\lambda_n Dz(t) + B_n(t)z_t, \quad t > s, \\ z(s) = \psi_0^n, \\ z_s(\tau) = \phi^n(\tau), \quad \tau \in [-r,s) \end{cases} \tag{5.3}$$

and $B_n(t) = B(t)\widetilde{P}_n = P_nB(t)$*.*

Proof We prove first that

$$U(t,s)W = \sum_{n=1}^{\infty} U_n(t,s)Q_nW, \ \ W \in \mathbb{M}_2, \ \ t \geq s.$$

In fact, let $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{M}_2$.

$$
\begin{aligned}
\sum_{n=1}^{\infty} U_n(t,s)Q_nW &= \sum_{n=1}^{\infty} U_n(t,s)\begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} U_n(t,s)\begin{pmatrix} P_n w_1 \\ \widetilde{P}_n w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} z^n(t) \\ z^n(t+\cdot) \end{pmatrix}; \quad z^n(\cdot) \text{ the unique mild solution of (5.3)} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} e^{-\lambda_n D(t-s)}P_n w_1 + \int_s^t e^{-\lambda_n D(t-\gamma)}B_n(\gamma)(\widetilde{P}_n z^n(\gamma+\cdot))d\gamma \\ \\ (\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=1}^{\infty} e^{-\lambda_n D(t-s)}P_n w_1 + \int_s^t \sum_{n=1}^{\infty} e^{-\lambda_n D(t-\gamma)}P_n\left(B(\gamma)\sum_{m=1}^{\infty}(\widetilde{P}_m z(\gamma+\cdot))\right)d\gamma \\ \\ \sum_{n=1}^{\infty}(\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} T_{\mathcal{A}}(t-s)w_1 + \int_s^t T_{\mathcal{A}}(t-\gamma)B(\gamma)z(\gamma+\cdot)d\gamma \\ \\ z(t+\cdot) \end{pmatrix} \\
&= \begin{pmatrix} z(t) \\ z_t(\cdot) \end{pmatrix}; \quad z(\cdot) \text{ the unique mild solution of (4.6)} \\
&= U(t,s)W.
\end{aligned}
$$

In the same way as in [1], we can prove that the infinitesimal generator of $\{U_n(t,s)\}_{t\geq s\geq 0}$ is given by:

$$
\Lambda_n(t)\begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\lambda_n D w_n^0 + B_n(t)w_n(\alpha) \\ \\ \dfrac{\partial w_n(\cdot)}{\partial \alpha} \end{pmatrix}, \; -r \leq \alpha \leq 0
$$

with

$$
D(\Lambda_n(t)) = \left\{ \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} \in \mathbb{M}_2^n : w_n \; is \text{ a.c.}, \; \frac{\partial w_n(\cdot)}{\partial \alpha} \in L^2([-\tau,0];Q_nZ) \; and \; w_n(0) = w_n^0 \right\}.
$$

$\{Q_n\}_{n\geq 1}$ is a family of complete orthogonal projection on $\mathbb{M}_2$ such that

$$
Q_nU_n(t,s) = U_n(t,s)Q_n, \quad n = 1,2,3,\dots,
$$

and $\|U_n(t,s)\| \le g(t,s),\ n = 1, 2, \dots,$ for some continuous function $g(t,s) \ge 0$.
Therefore, applying Lemma 3.1, we obtain that $U(t,s)$ is bounded and $\{U(t,s)\}_{t\ge s}$ is a strongly continuous evolution operator on the Hilbert space $\mathbb{M}_2$, whose generator Λ is given by

$$\Lambda(t)W = \sum_{n=1}^{\infty} \Lambda_n(t)Q_nW, \qquad W \in D(\Lambda),$$

with

$$D(\Lambda(t)) \subset \left\{ W \in \mathbb{M}_2 / \sum_{n=1}^{\infty} \| \Lambda_n(t)Q_nW \|^2 < \infty \right\}.$$

Lemma 5.2 *Let $\Lambda(t)$ be the infinitesimal generator of the evolution operator $\{U(t,s)\}_{t\ge s}$. Then*

$$\Lambda(t)\tilde{\varphi}(\alpha) = \begin{pmatrix} -A\psi(0) + B(t)\phi(\alpha) \\ \\ \dfrac{\partial \phi(\cdot)}{\partial \alpha} \end{pmatrix} ; \ -r \le \alpha \le 0,$$

$$D(\Lambda(t)) = \left\{ \begin{pmatrix} \psi_0 \\ \phi(\cdot) \end{pmatrix} \in \mathbb{M}_2 : \psi_0 \in D(A), \phi \ \ \textit{is} \ \ a.c., \frac{\partial \phi(\cdot)}{\partial \alpha} \in L^2([-r,0];Z) \ \ \textit{and} \ \ \phi(0) = \psi_0 \right\}.$$

Proof Consider $\begin{pmatrix} \psi_0 \\ \phi(\cdot) \end{pmatrix}$ in $\mathbb{M}_2$. Then

$$\begin{aligned} \Lambda(t)W = \Lambda(t)\begin{pmatrix} \psi_0 \\ \phi(\cdot) \end{pmatrix} &= \sum_{n=1}^{\infty} \Lambda_n(t)Q_nW \\ &= \sum_{n=1}^{\infty} \Lambda_n(t)\begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix}\begin{pmatrix} \psi_0 \\ \phi(\cdot) \end{pmatrix} = \sum_{n=1}^{\infty} \Lambda_n(t)\begin{pmatrix} P_n\psi_0 \\ \widetilde{P_n}\phi(\cdot) \end{pmatrix} \\ &= \sum_{n=1}^{\infty} \begin{pmatrix} -\lambda_n D\widetilde{P_n}\psi(0) + B_n(t)\widetilde{P_n}\phi(\alpha) \\ \\ \dfrac{\partial \widetilde{P}_n\phi(\cdot)}{\partial \alpha} \end{pmatrix} \\ &= \begin{pmatrix} -\displaystyle\sum_{n=1}^{\infty} \lambda_n DP_n\psi(0) + B(t)\sum_{n=1}^{\infty} \widetilde{P}_n\phi(\alpha) \\ \\ \dfrac{\partial}{\partial \alpha}\left(\displaystyle\sum_{n=1}^{\infty} \widetilde{P}_n\phi(\cdot)\right) \end{pmatrix} \\ &= \begin{pmatrix} -A\psi(0) + B(t)\phi(s) \\ \\ \dfrac{\partial \phi(\cdot)}{\partial \alpha} \end{pmatrix}. \end{aligned}$$

In consequent, for each t fixed, $\Lambda(t)$ is the infinitesimal generator of a strongly continuous semi-group.

Hence, $\Lambda(t)$ is closed and the result follows from part ii) of Lemma 3.1.

$\square$

Therefore, the systems (4.6) and (4.5) are equivalent to the following two systems of ordinary di-fferential equations in $\mathbb{M}_2$ respectively:

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda(t)W(t), \quad t > s, \\ \\ W(s) \ = W_0 = (\psi_0, \phi(\cdot)), \end{cases} \tag{5.4}$$

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda(t)W(t) + \Phi(t), \quad t > s, \\ \\ W(s) \ = W_0 = (\psi_0, \phi(\cdot)), \end{cases} \tag{5.5}$$

where $\Lambda(t)$ is the infinitesimal generator of the evolution operator $\{U(t,s)\}_{t\geq s}$ and $\Phi(t) = (f^e(t), 0)$.

The steps we have to arrive here allow us find The Variation of Constants Formula for Parabolic-Delay Equations. This result is presented in the final Theorem of this work.

Theorem 5.3 *The abstract Cauchy problem in the Hilbert space* $\mathbb{M}_2$

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda(t)W(t) + \Phi(t), \quad t > s, \\ \\ W(s) \ = W_0, \end{cases}$$

where $\Lambda(t)$ *is the infinitesimal generator of the evolution operator* $\{U(t,s)\}_{t\geq s}$ *and* $\Phi(t) = (f^e(t), 0)$ *is a function taking values in* $\mathbb{M}_2$*, admits one and only one mild solution given by:*

$$W(t) = U(t,s)W_0 + \int_s^t U(t,\gamma)\Phi(\gamma)d\gamma, \quad t \geq s. \tag{5.6}$$

Corollary 5.4 *If* $z(t)$ *is a solution of (4.5), then the function* $W(t) := (z(t), z_t)$ *is solution of the equation (5.5)*

6 Conclusion

As we can see, this work can be applied to a broad class of time-dependent functional reaction diffusion equation in a Hilbert space Z of the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \mathcal{A}z(t) + B(t)z_t + F(t), \quad t > s, \\ \\ z(s) = \phi_0, \\ \\ z_s(\alpha) = \phi(\alpha), \ \ \alpha \in [-r, 0), \end{cases} \tag{6.1}$$

where $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}), \tag{6.2}$$

$B \in L^{\infty}([0,\infty); \mathcal{L}(Z_1, Z))$ with $Z_1 = L^2([-r,0], Z)$ and $F : [-r, \infty) \longrightarrow Z$ is a suitable function. Some examples of this class are the following well known systems of partial differential equations with delay:

Example 6.1 *The equation modeling the damped flexible beam:*

$$\begin{cases} \dfrac{\partial 2z}{\partial 2t} &= -\dfrac{\partial 3z}{\partial 3x} + 2\alpha \dfrac{\partial 3z}{\partial t \partial 2x} + b(t)z(t-\tau, x) + f(t,x) \ \ t > s, \ \ 0 \leq x \leq 1, \\ \\ z(t,1) &= z(t,0) = \dfrac{\partial 2z}{\partial 2x}(0,t) = \dfrac{\partial 2z}{\partial 2x}(1,t) = 0, \\ \\ z(s,x) &= \phi_0(x), \ \ \dfrac{\partial z}{\partial t}(s,x) = \psi_0(x), \ \ 0 \leq x \leq 1, \\ \\ z(\alpha, x) &= \phi(\alpha, x), \ \ \dfrac{\partial z}{\partial t}(\alpha, x) = \psi(\alpha, x), \ \ \alpha \in [s-r, s), \ \ 0 \leq x \leq 1, \end{cases} \tag{6.3}$$

where $\alpha > 0$, $b(t)$ is a bounded continuous function, $f : I\!R \times [0,1] \to I\!R$ is a smooth function, $\phi_0, \psi_0 \in L^2[0,1]$ and $\phi, \psi \in L^2([-r,0]; L^2[0,1])$.

Example 6.2 *The strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} \dfrac{\partial 2w}{\partial 2t} + \eta(-\Delta)^{1/2} \dfrac{\partial w}{\partial t} + \gamma(-\Delta)w = B(t)w_t + f(t,x), \quad t > s, \quad x \in \Omega, \\ \\ w(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ \\ w(s,x) = \phi_0(x), \quad \dfrac{\partial z}{\partial t}(s,x) = \psi_0(x), \quad x \in \Omega, \\ \\ w(\alpha, x) = \phi(s,x), \quad \dfrac{\partial z}{\partial t}(\alpha, x) = \psi(\alpha, x), \quad \alpha \in [s-r, s), \quad x \in \Omega, \end{cases} \tag{6.4}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^N$, $B \in L^\infty([0,\infty);\mathcal{L}(Z_1,Z))$ with $Z_1 = L^2([-r,0],Z)$, $Z = L^2(\Omega)$, $f : I\!R \times \Omega \to I\!R$ is a smooth function, $\phi_0,\psi_0 \in Z$ and $\phi,\psi \in L^2([-\tau,0];Z)$ and $r \geq 0$ is the maximum delay, which is supposed to be finite.

Example 6.3 *The thermoelastic plate equation with Dirichlet boundary conditions*

$$\begin{cases} \dfrac{\partial 2w}{\partial 2t} + \Delta^2 w + \alpha\Delta\theta = B_1(t)w_t + f_1(t,x) \quad t > s, \quad x \in \Omega, \\ \dfrac{\partial \theta}{\partial t} - \beta\Delta\theta - \alpha\Delta\dfrac{\partial w}{\partial t} = B_2(t)\theta_t + f_2(t,x) \quad t > s, \quad x \in \Omega, \\ \theta = w = \Delta w = 0, \quad t \geq 0, \quad x \in \partial\Omega, \\ w(s,x) = \phi_0(x), \quad \dfrac{\partial w}{\partial t}(s,x) = \psi_0(x), \quad \theta(s,x) = \xi_0(x), \quad x \in \Omega, \\ w(\alpha,x) = \phi(\alpha,x), \quad \dfrac{\partial w}{\partial t}(\alpha,x) = \psi(\alpha,x), \quad \theta(\alpha,x) = \xi(\alpha,x), \quad \alpha \in [s-r,s), \quad x \in \Omega, \end{cases} \tag{6.5}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^N$, $B_1, B_2 \in L^\infty([0,\infty);\mathcal{L}(Z_1,Z))$ with $Z_1 = L^2([-r,0],Z)$, $Z = L^2(\Omega)$, $f_1, f_2 : I\!R \times \Omega \to I\!R$ are smooth functions, $\phi_0,\psi_0,\xi_0 \in Z$ and $\phi,\psi,\xi \in L^2([-r,0];Z)$ and $r \geq 0$ is the maximum delay, which is supposed to be finite.

Referencias

[1] A. CARRASCO AND H. LEIVA, "Variation of Constants Formula for Functional Partial Parabolic Equations", Electronic Journal of Differential Equations, Vol.2007(2007), No. 130, pp. 1-20.

[2] A. CARRASCO AND H. LEIVA, "A Negative Result on Controllability of the Damped Wave Equation with Delay", International Mathematical Forum, 3, 2008, No. 11, 525-537.

[3] A. CARRASCO AND H. LEIVA, "Approximate Controllability of a System of Parabolic Equations with Delay ", J. Math. Anal. Appl. doi:10.1016/j.jmaa.2008.04.068.

[4] R.F. CURTAIN AND H.J. ZWART, "An Introduction to Infinite Dimensional Linear Systems Theory", Tex in Applied Mathematics, Vol. 21. Springer Verlag, New York (1995).

[5] L.A. F. DE OLIVEIRA " Instability of Homogeneous Periodic Solutions of Parabolic-Delay Equations", J. Differential Eqs. 109, 42-76 (1994).

[6] E. ITURRIAGA and H. LEIVA, "A Necessry and Sufficient Condition for the Controllability of Linear Systems in Hilbert Spaces and Applications", IMA J. Math. Control Inf., Advance Accsess Published on May 15, 2007; doi:10.1093/imamci/dnm017

[7] M. V. KARTASHOV, "General Time-Dependent Bounded Perturbation of a Strongly Continuous Semigroup", Ukrainian Mathematical Journal, Vol. 57, No. 12,(2005).

[8] H. LEIVA, "Stability of a Periodic Solution for a System of Parabolic Equations", Applicable Analyis, Vol. 60, 277-300, 19996.

[9] H. LEIVA, "A Lemma on C_0-Semigroups and Applications PDEs Systems" Quaestions Mathematicae, Vol. 26, pp. 247-265 (2003).

[10] H. LEIVA, "Exact controllability of the suspension bridge model proposed by Lazer and McKenna", J. Math. Anal. Appl. 309 (2005), 404-419.

[11] H. LEIVA, "Exact controllability of semilinear evolution equation and applications", Int. J. Systems and communications Vol.1, No. 1, 2008.

[12] H. LEIVA and J. UZCATEGUI, "Exact controllability of semilinear difference equation and application", J. of Difference Equations and Applications, Vol. 14, No. 7, July 2008, 671-679.

ALEXANDER CARRASCO
Universidad Centroccidental Lisandro Alvarado
Decanato de Ciencias, Departamento de Matemática.
Barquisimeto 3001 - Venezuela
e-mail: acarrasco@ucla.edu.ve

HUGO LEIVA
Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: hleiva@ula.ve