A Necessary and Sufficient Condition for the Controllability of Linear Systems in Hilbert Spaces and Applications *

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Abstract

As we have announced in the title of this work, we show that a broad class of linear evolution equations are exactly controllable. This class is represented by the following infinite dimensional linear control system:

$$\dot{z} = \mathcal{A}z + \mathcal{B}u(t), \quad t > 0, z \in Z, \quad u(t) \in U$$

where Z, U are Hilbert spaces, the control function u belong to $L^2(0, t_1; U), t_1 > 0$, $\mathcal{B} \in L(U, Z)$, $\mathcal{A}$ generates a strongly continuous semigroup operator $T(t)$ according to [5]. We give necessary and sufficient condition for the exact controllability of this system and apply this results to a linear controlled damped wave equation.

Key words. linear evolution equations,exact controllability,.

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1 Introduction

In this work we prove that a broad class of linear evolution equations are exactly controllable. This class is represented by the following linear infinite dimensional control system:

$$\dot{z} = \mathcal{A}z + \mathcal{B}u(t), \quad z(t) \in Z, \quad u(t) \in U, \quad t > 0, \tag{1.1}$$

where Z, U are infinite dimensional Hilbert spaces, the control function u belong to $L^2(0, t_1; U), t_1 > 0$, $\mathcal{B} \in L(U, Z)$, $\mathcal{A}$ generates a strongly continuous semigroup operator $T(t)$ according to [5].

As a motivation we shall consider the following finite dimensional linear control system

$$\dot{z} = Az + Bu(t), \quad z(t) \in I\!R^n, \quad u(t) \in I\!R^m, \quad t > 0, \tag{1.2}$$

where A and B are matrices of dimension $n \times n$ and $n \times m$ respectively, and the control function u belong to $L^2(0, t_1; I\!R^m)$. The following Lemma can be found in [3].

Lemma 1.1 *The following statements are equivalent:*

(a) System (1.2) is controllable on $[0, t_1]$.

(b) $B^* e^{A^* t} z = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow z = 0,$

(c) Rank $\left[B \vdots AB \vdots A^2 B \vdots \cdots A^{n-1} B\right] = n$

(d) The operator $\mathcal{W}(t_1) : I\!R^n \to I\!R^n$ *given by:*

$$\mathcal{W}(t_1) = \int_0^{t_1} e^{A(t_1 - s)} B B^* e^{A^*(t_1 - s)} ds, \tag{1.3}$$

is invertible.

Moreover, the control $u \in L^2(0, t_1; I\!R^m)$ that steers an initial state z_0 to a final state z_1 at time $t_1 > 0$ is given by the following formula:

$$u(t) = B^* e^{A^*(t_1 - t)} \mathcal{W}^{-1}(z_1 - e^{At_1} z_0). \tag{1.4}$$

In this work we generalize this result for the infinite dimensional linear system (1.1) in Hilbert spaces , in the following way: The system (1.1) is **exactly** controllable on $[0, t_1]$ iff the linear bounded operator $\mathcal{W}(t_1) : Z \to Z$ given by:

$$\mathcal{W}z = \int_0^{t_1} T(t_1 - s)\mathcal{B}\mathcal{B}^* T^*(t_1 - s) z ds, \tag{1.5}$$

is invertible. This result completes Theorem 4.1.7 from [2].

Moreover, the control $u \in L^2(0, t_1; U)$ that steers an initial state z_0 to a final state z_1 at time $t_1 > 0$ is given by the following formula:

$$u(t) = \mathcal{B}^* T^*(t_1 - t) \mathcal{W}^{-1}(z_1 - T(t_1) z_0). \tag{1.6}$$

Finally, we apply this result to the following controlled linear damped wave equation

$$\begin{cases} w_{tt} + c w_t - d w_{xx} = u(t, x), & 0 < x < 1 \\ w(t, 0) = w(t, 1) = 0, & t \in I\!R \end{cases} \tag{1.7}$$

where $u \in L^2(0, t_1; L^2[0, 1])$.

2 Exact Controllability

Now, we shall give the definition of controllability for the linear system

$$\dot{z} = \mathcal{A}z + \mathcal{B}u(t) \quad z \in Z, \quad t \geq 0. \tag{2.1}$$

For all $z_0 \in Z$ the equation (2.1) has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t - s)\mathcal{B}u(s) ds, \quad 0 \leq t \leq t_1. \tag{2.2}$$

Definition 2.1 *(***Exact Controllability***) We say that system (2.1) is exactly controllable on* $[0, t_1]$*,* $t_1 > 0$*, if for all* $z_0, z_1 \in Z$ *there exists a control* $u \in L^2(0, t_1; U)$ *such that the solution* $z(t)$ *of (2.2) corresponding to* u*, verifies:* $z(t_1) = z_1$*.*

Consider the following bounded linear operators

$$G : L^2(0, t_1; U) \to Z, \quad Gu = \int_0^{t_1} T(t_1 - s)\mathcal{B}u(s)ds. \tag{2.3}$$

$$\mathcal{W} : Z \to Z, \quad \mathcal{W}z = \int_0^{t_1} T(t_1 - s)\mathcal{B}\mathcal{B}^* T^*(t_1 - s)zds. \tag{2.4}$$

Then, the following proposition is a characterization of the exact controllability of the system (2.1).

Proposition 2.1 *The system (2.1) is exactly controllable on* $[0, t_1]$ *if and only if, the operator* G *is surjective, that is to say*

$$G(L^2(0, t_1; U)) = Range(G) = Z.$$

The following Theorem is a version of Theorem 2.1 from [1], pg. 56 in Hilbert spaces.

Theorem 2.1 *If* $u \in L^2(0, t_1; U)$ *and* U*,* Z *are Hilbert spaces, then (2.1) is exactly controllable iff there exists* $\gamma > 0$ *such that*

$$\gamma \|\mathcal{B}^* T^*(t_1 - \cdot)z\|_{L^2(0,t;U)} \geq \|z\|_Z, \quad z \in Z. \tag{2.5}$$

Now, we are ready to formulate the main result on exact controllability of the linear system (2.1).

Theorem 2.2 *The system (2.1) is exactly controllable on* $[0, t_1]$ *if and only if the operator* $\mathcal{W}$ *is invertible. Moreover, the control* $u \in L^2(0, t_1; U)$ *steering an initial state* z_0 *to a final state* z_1 *at time* $t_1 > 0$ *is given by the following formula:*

$$u(t) = B^* T^*(t_1 - t)\mathcal{W}^{-1}(z_1 - T(t_1)z_0). \tag{2.6}$$

Proof Suppose the system (2.1) is exactly controllable on $[0, t_1]$. Then, from the foregoing Theorem we obtain

$$\gamma^2 \|\mathcal{B}^* T^*(t_1 - \cdot)z\|_{L^2}^2 \geq \|z\|_Z^2, \quad z \in Z.$$

i.e.,

$$\gamma^2 \int_0^{t_1} \|\mathcal{B}^* T^*(t_1 - s)z\|_U^2 \geq \|z\|_Z^2, \quad z \in Z.$$

i.e.,

$$\gamma^2 \int_0^{t_1} < \mathcal{B}^* T^*(t_1 - s)z, \mathcal{B}^* T^*(t_1 - s)z >_{U,U} \geq \|z\|_Z^2, \quad z \in Z.$$

i.e.,

$$\gamma^2 \int_0^{t_1} < T(t_1 - s)\mathcal{B}\mathcal{B}^* T^*(t_1 - s)z, z >_{U,U} \geq \|z\|_Z^2, \quad z \in Z.$$

Therefore,

$$< \mathcal{W}z, z > \geq \frac{1}{\gamma^2}\|z\|_Z^2, \quad z \in Z. \tag{2.7}$$

This implies that $\mathcal{W}$ is one to one. Now, we shall prove that $\mathcal{W}$ is surjective. That is to say

$$\mathcal{R}(\mathcal{W}) = \text{Range}(\mathcal{W}) = Z.$$

For the purpose of contradiction, let us assume that $\mathcal{R}(\mathcal{W})$ is estrictly contained in Z. Using Cauchy Schwarz‘s inequality and (2.7)we get

$$\|\mathcal{W}z\| \geq \frac{1}{\gamma^2}\|z\|_Z, \quad z \in Z,$$

which implies that $\mathcal{R}(\mathcal{W})$ is closed. Then, from Hahn Banachs Theorem there exists $z_0 \in Z$ with $z_0 \neq 0$ such that

$$< \mathcal{W}z, z_0 >= 0, \quad \forall z \in Z.$$

In particular, putting $z = z_0$ we get from (2.7) that

$$0 =< \mathcal{W} z_0, z_0 > \geq \frac{1}{\gamma^2} \|z_0\|_Z^2.$$

Then $z_0 = 0$, which is a contradiction. Hence, $\mathcal{W}$ is a bijection and from the open mapping Theorem $\mathcal{W}^{-1}$ is a bounded linear operator.

Now, suppose $\mathcal{W}$ is invertible. Then, given $z \in Z$ we shall prove the existence of a control $u \in L^2$ such that $Gu = z$. This control u can be taking as follows

$$u(t) = B^* T^*(t_1 - t) \mathcal{W}^{-1} z.$$

In fact,

$$Gu = \int_0^{t_1} T(t_1 - s) \mathcal{B} u(s) ds = \int_0^{t_1} T(t_1 - s) \mathcal{B} \mathcal{B}^* T^*(t_1 - s) \mathcal{W}^{-1} z ds = \mathcal{W} \mathcal{W}^{-1} z = z.$$

In the same way we can prove that the control u given by (2.6) steers the initial state z_0 to the final state z_1 in time t_1.

□

Lemma 2.1 *Suppose system(2.1) is exactly controllable. Consider $z \in Z$, the control*

$$u_0(t) = B^* T^*(t_1 - t) \mathcal{W}^{-1} z$$

and the set

$$S_z = \{u \in L^2(0, t_1; U) : Gu = z\}.$$

Then

$$\|u_0\| = \inf\{\|u\| : u \in S_z\}$$

Proof Consider the following equalities

$$\|u\|^2 = \|u_0 + (u - u_0)\|^2 = \|u_0\|^2 + 2\mathrm{Re} < u_0, u - u_0 > + \|u - u_0\|^2, \quad u \in S_z.$$

on the other hand,

$$\begin{aligned} < u_0, u - u_0 > &= < \int_0^{t_1} \mathcal{B}^* T^*(t_1 - s)\mathcal{W}^{-1} z, u(s) - u_0(s) > ds \\ &= < \int_0^{t_1} \mathcal{W}^{-1} z, T(t_1 - s)\mathcal{B}u(s) - T(t_1 - s)\mathcal{B}u_0(s) > ds \\ &= < \mathcal{W}^{-1} z, Gu - Gu_0 > = < \mathcal{W}^{-1} z, z - z > = 0. \end{aligned}$$

Hence,

$$\|u\|^2 - \|u_0\|^2 = \|u - u_0\|^2 \geq 0, \quad u \in S_z.$$

Therefore, $\|u_0\| \leq \|u\|, \quad u \in S_z$ and $\|u_0\| = \|u\|$ iff $u_0 = u$.

□

3 Applications

As we have announced in the introduction of this work we apply this result to the following controlled linear damped wave equation

$$\begin{cases} w_{tt} + cw_t - dw_{xx} = u(t, x), & 0 < x < 1 \\ w(t, 0) = w(t, 1) = 0, & t \in I\!R \end{cases} \tag{3.1}$$

where $u \in L^2(0, t_1; L^2[0, 1])$.

In the space $X = L^2[0, 1]$ this system can be written as an abstract second order ordinary differential equation. To this end, we consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\phi_{xx}$, where

$$D(A) = \{\phi \in X : \phi, \phi_x, \text{ are a.c}, \quad \phi_{xx} \in X; \quad \phi(0) = \phi(1) = 0\}. \tag{3.2}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty,$$

each one with multiplicity one. Therefore,

a) There exists a complete orthonormal set $\{\phi_n\}$ of eigenvectors of A.

b) For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n < x, \phi_n > \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{3.3}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n x =< x, \phi_n > \phi_n. \tag{3.4}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and $x = \sum_{n=1}^{\infty} E_n x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{3.5}$$

d) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{3.6}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with norm given by:

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2.$$

Using the change of variables $w' = v$, the second order equation (3.1) can be written as a first order system of ordinary differential equations in the Hilbert space $Z_{1/2} = D(A^{1/2}) \times X = X^{1/2} \times X$ as:

$$z' = \mathcal{A}z + Bu, \ z \in Z_{1/2}, \quad t \geq 0, \tag{3.7}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -dA & -cI_X \end{bmatrix}. \tag{3.8}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$.

We shall use the following Lemma from [4] to prove the next Theorem:

Lemma 3.1 *Let Z be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in Z with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_n P_n = P_n A_n, \quad n = 1, 2, 3, \ldots \tag{3.9}$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \qquad t \geq 0. \tag{3.10}$$

Then:

(a) $T(t)$ is a linear bounded operator if

$$\|e^{A_n t}\| \leq g(t), \quad n = 1, 2, 3, \ldots \tag{3.11}$$

for some continuous real-valued function $g(t)$.

(b) under the condition (3.11) $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup in the Hilbert space Z whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}) \tag{3.12}$$

with

$$D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\} \tag{3.13}$$

(c) the spectrum $\sigma(\mathcal{A})$ *of* $\mathcal{A}$ *is given by*

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{3.14}$$

where $\bar{A}_n = A_n P_n$.

Theorem 3.1 *The operator* $\mathcal{A}$ *given by (3.8), is the infinitesimal generator of a strongly continuous group* $\{T(t)\}_{t\in I\!R}$ *given by*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z_{1/2}, \quad t \geq 0 \tag{3.15}$$

where $\{P_n\}_{n\geq 0}$ *is a complete family of orthogonal projections in the Hilbert space* $Z_{1/2}$*:*

$$P_n = diag\left[E_n, E_n\right] \ , n \geq 1 \ , \tag{3.16}$$

and

$$A_n = B_n P_n, \quad B_n = \begin{bmatrix} 0 & 1 \\ -d\lambda_n & -c \end{bmatrix}, n \geq 1. \tag{3.17}$$

This group decays exponentially to zero. In fact, we have the following estimate

$$\|T(t)\| \leq M(c,d) e^{-\frac{c}{2}t}, \quad t \geq 0, \tag{3.18}$$

where

$$\frac{M(c,d)}{2\sqrt{2}} = \sup_{n\geq 1}\left\{ 2\left| \frac{c \pm \sqrt{4d\lambda_n - c^2}}{\sqrt{c^2 - 4d\lambda_n}} \right|, \left| (2+d)\sqrt{\frac{\lambda_n}{4d\lambda_n - c^2}} \right| \right\}.$$

It is known that the linear damped wave equation

$$z' = \mathcal{A}z + Bu \quad z \in Z_{1/2}, \quad t \geq 0, \tag{3.19}$$

is controllable on $[0, t_1]$ for $t_1 > 0$(see [1] and [2]). Nevertheless, we will give here a different and nicer proof of it, for better understanding of the reader and self-contained work. To this end, we project the system (3.19) on the range $\mathcal{R}(P_j)$ of P_j to obtain the following family of finite dimensional systems

$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \ldots, \infty. \tag{3.20}$$

Then, the following proposition can be shown the same way as Lemma 1 from [3].

Proposition 3.1 *The following statements are equivalent:*

(a) System (3.20) is controllable on $[0, t_1]$.

(b) $B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0,$

(c) Rank $\left[P_j B \vdots A_j P_j B \right] = 2$

(d) The operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ *given by:*

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} B B^* e^{-A_j^* s} ds, \tag{3.21}$$

is invertible.

Now, we are ready to prove the exact controllability of the linear system (3.19).

Theorem 3.2 *The system (3.19) is exactly controllable on* $[0, t_1]$ *and the control* $u \in L^2(0, t_1; X)$ *that steers an initial state* z_0 *to a final state* z_1 *at time* $t_1 > 0$ *is given by the following formula:*

$$u(t) = B^* T^*(-t) \sum_{j=1}^{\infty} \mathcal{W}_j^{-1}(t_1) P_j (T(-t_1) z_1 - z_0). \tag{3.22}$$

Moreover,

$$\mathcal{W}(t_1) z = \int_0^{t_1} T(-s) B B^* T^*(-s) z ds = \sum_{j=1}^{\infty} W_j(t_1) P_j z,$$

and

$$\mathcal{W}^{-1}(t_1)z = \sum_{j=1}^{\infty} \mathcal{W}_j^{-1}(t_1)P_j z.$$

Proof . First, we shall prove that each of the following finite dimensional systems is controllable on $[0, t_1]$

$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \ldots, \infty. \tag{3.23}$$

In fact, we can check the condition for controllability of the systems

$$B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0.$$

In this case the operators $A_j = B_j P_j$ and $\mathcal{A}$ are given by

$$B_j = \begin{bmatrix} 0 & 1 \\ -d\lambda_j & -c \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -dA & -cI \end{bmatrix},$$

and the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$ of the matrix B_j are given by

$$\sigma_1(j) = -\mu + i l_j, \quad \sigma_2(j) = -\mu - i l_j,$$

where,

$$\mu = \frac{c}{2} \quad \text{and} \quad l_j = \frac{1}{2}\sqrt{4d\lambda_j - c^2}.$$

Therefore, $A_j^* = B_j^* P_j$ with

$$B_j^* = \begin{bmatrix} 0 & -1 \\ d\lambda_j & -c \end{bmatrix},$$

and

$$\begin{aligned} e^{B_j t} &= e^{-\mu t}\left\{\cos l_j t I + \frac{1}{l_j}\sin l_j t\,(B_j + cI)\right\} \\ &= e^{-\mu t}\begin{bmatrix} \cos l_j t + \frac{c}{2l_j}\sin l_j t & \frac{\sin l_j t}{l_j} \\ -dS(j)\lambda_j^{1/2}\sin l_j t & \cos l_j t - \frac{c}{2l_j}\sin l_j t \end{bmatrix}, \end{aligned}$$

$$
\begin{aligned}
e^{B_j^* t} &= e^{-\mu t}\left\{\cos l_j t I + \frac{1}{l_j}\sin l_j t\left(B_j^* + \mu I\right)\right\} \\
&= e^{-\mu t}\begin{bmatrix} \cos l_j t + \frac{c}{2l_j}\sin l_j t & -\frac{\sin l_j t}{l_j} \\ dS(j)\lambda_j^{1/2}\sin l_j t & \cos l_j t - \frac{c}{2l_j}\sin l_j t \end{bmatrix},
\end{aligned}
$$

$$
B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad B^* = [0, I_X] \ \text{ and } \ BB^* = \begin{bmatrix} 0 & 0 \\ 0 & I_X \end{bmatrix}.
$$

Now, let $y = (y_1, y_2)^T \in \mathcal{R}(P_j)$ such that

$$
B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1].
$$

Then,

$$
e^{-\mu t}\left[dS(j)\lambda_j^{1/2}\sin l_j t y_1 + \left(\cos l_j t - \frac{c}{2l_j}\sin l_j t\right) y_2\right] = 0, \quad \forall t \in [0, t_1],
$$

which implies that $y = 0$.

From Proposition 3.1 the operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ given by:

$$
W_j(t_1) = \int_0^{t_1} e^{-A_j s} BB^* e^{-A_j^* s} ds = P_j \int_0^{t_1} e^{-B_j s} BB^* e^{-B_j^* s} ds P_j = P_j \overline{W}_j(t_1) P_j
$$

is invertible.

Since

$$
\|e^{-A_j t}\| \leq M(c,d)e^{\mu t}, \quad \|e^{-A_j^* t}\| \leq M(c,d)e^{\mu t},
$$

$$
\|e^{-A_j t} BB^* e^{-A_j^* t}\| \leq M^2(c,d)\|BB^*\|e^{2\mu t},
$$

we have

$$
\|W_j(t_1)\| \leq M^2(c,d)\|BB^*\|e^{2\mu t_1} \leq L(c,d), \quad j = 1, 2, \ldots.
$$

Now, we shall prove that the family of linear operators,

$$
\mathcal{W}_j^{-1}(t_1) = \overline{W}_j^{-1}(t_1) P_j : Z_{1/2} \to Z_{1/2}
$$

is bounded and $\|\mathcal{W}_j^{-1}(t_1)\|$ is uniformly bounded. To this end, we shall compute explicitly the matrix $\overline{W}_j^{-1}(t_1)$. From the above formulas we obtain that

$$e^{B_j t} = e^{-\mu t}\begin{bmatrix} a(j) & b(j) \\ -a(j) & c(j) \end{bmatrix}, \quad e^{B_j^* t} = e^{-\mu t}\begin{bmatrix} a(j) & -b(j) \\ d(j) & c(j) \end{bmatrix},$$

where

$$a(j) = \cos l_j t + \frac{c}{2l_j}\sin l_j t, \quad b(j) = \frac{\sin l_j t}{l_j},$$
$$c(j) = dS(j)\lambda_j^{1/2}\sin l_j t, \quad d(j) = \cos l_j t - \frac{c}{2l_j}\sin l_j t,$$

and

$$S(j) = \sqrt{\frac{\lambda_j}{4d\lambda_j - c^2}}.$$

Then

$$e^{-B_j s}BB^* e^{-B_j^* s} = \begin{bmatrix} b(j)c(j)\lambda_j^{1/2} I & -b(j)d(j)I \\ -d(j)c(j)\lambda_j^{1/2} I & d^2(j)I \end{bmatrix}.$$

Therefore,

$$\overline{W}_j(t_1) = \begin{bmatrix} \frac{dS(j)\lambda_j^{1/2}}{l_j}k_{11}(j) & \frac{1}{l_j}k_{12}(j) \\ -dS(j)\lambda_j^{1/2}k_{21}(j) & k_{22}(j) \end{bmatrix},$$

where

$$\begin{aligned} k_{11}(j) &= \int_0^{t_1} e^{2cs}\sin^2 l_j s ds \\ k_{12}(j) &= -\int_0^{t_1} e^{2cs}\left[\sin l_j s\cos l_j s - \frac{c\sin^2 l_j s}{2l_j}\right]ds \\ k_{21}(j) &= \int_0^{t_1} e^{2cs}\left[\sin l_j s\cos l_j s - \frac{c\sin^2 l_j s}{2l_j}\right]ds \\ k_{22}(j) &= \int_0^{t_1} e^{2cs}\left[\cos l_j s - \frac{c\sin l_j s}{2l_j}\right]^2 ds. \end{aligned}$$

The determinant $\Delta(j)$ of the matrix $\overline{W}_j(t_1)$ is given by

$$\Delta(j) = \frac{dS(j)\lambda_j^{1/2}}{l_j}\left[k_{11}(j)k_{22}(j) - k_{12}(j)k_{21}(j)\right]$$

$$
\begin{aligned}
= & \quad \frac{dS(j)\lambda_j^{1/2}}{l_j}\{\left(\int_0^{t_1} e^{2\mu s}\sin^2 l_j s ds\right)\left(\int_0^{t_1} e^{2\mu s}\left[\cos l_j s - \frac{c\sin l_j s}{2l_j}\right]^2 ds\right) \\
- & \quad \left(\int_0^{t_1} e^{2\mu s}\left[\sin l_j s\cos l_j s - \frac{c\sin^2 l_j s}{2l_j}\right] ds\right)^2\}.
\end{aligned}
$$

Passing to the limit as j goes to ∞, we obtain,

$$\lim_{j\to\infty}\Delta(j) = \frac{(e^{2\mu t_1}-1)(1-2e^{\mu t_1}+e^{2\mu t_1})}{2^4\mu^3}.$$

Therefore, there exist constants $R_1, R_2 > 0$ such that

$$0 < R_1 < |\Delta(j)| < R_2, \quad j = 1, 2, 3, \ldots$$

Hence,

$$
\begin{aligned}
\overline{W}^{-1}(j) &= \frac{1}{\Delta(j)}\begin{bmatrix} k_{22}(j) & -\frac{1}{l_j}k_{12}(j) \\ dS(j)\lambda_j^{1/2}k_{21}(j) & \frac{dS(j)\lambda_j^{1/2}}{l_j}k_{11}(j)\end{bmatrix} \\
&= \begin{bmatrix} b_{11}(j) & b_{12}(j) \\ b_{21}(j)\lambda_j^{1/2} & b_{22}(j)\end{bmatrix},
\end{aligned}
$$

where $b_{n,m}(j), \quad n = 1,2; m = 1,2; j = 1,2,\ldots$ are bounded. We can prove the existence of constant $L_2(c,d)$ such that

$$\|\mathcal{W}_j^{-1}(t_1)\|_{Z_{1/2}} \le L_2(c,d), \quad j = 1, 2, \ldots.$$

Now, we define the following linear bounded operators

$$\mathcal{W}(t_1): Z_{1/2} \to Z_{1/2}, \quad \mathcal{W}^{-1}(t_1): Z_{1/2} \to Z_{1/2},$$

by

$$\mathcal{W}(t_1)z = \sum_{j=1}^{\infty} W_j(t_1)P_j z, \quad \mathcal{W}^{-1}(t_1)z = \sum_{j=1}^{\infty}\mathcal{W}_j^{-1}(t_1)P_j z.$$

Using the definition we see that, $\mathcal{W}(t_1)\mathcal{W}^{-1}(t_1)z = z$ and

$$\mathcal{W}(t_1)z = \int_0^{t_1} T(-s)BB^*T^*(-s)z ds.$$

Next, we will show that given $z \in Z_{1/2}$ there exists a control $u \in L^2(0, t_1; X)$ such that $Gu = z$. In fact, let u be the following control

$$u(t) = B^*T^*(-t)\mathcal{W}^{-1}(t_1)z, \quad t \in [0, t_1].$$

Then,

$$\begin{aligned} Gu &= \int_0^{t_1} T(-s)Bu(s)ds \\ &= \int_0^{t_1} T(-s)BB^*T^*(-s)\mathcal{W}^{-1}(t_1)zds \\ &= \left(\int_0^{t_1} T(-s)BB^*T^*(-s)ds\right)\mathcal{W}^{-1}(t_1)z \\ &= \mathcal{W}(t_1)\mathcal{W}^{-1}(t_1)z = z. \end{aligned}$$

Then, the control steering an initial state z_0 to a final state z_1 in time $t_1 > 0$ is given by

$$\begin{aligned} u(t) &= B^*T^*(-t)\mathcal{W}^{-1}(t_1)(T(-t_1)z_1 - z_0) \\ &= B^*T^*(-t)\sum_{j=1}^{\infty}\mathcal{W}_j^{-1}(t_1)P_j(T(-t_1)z_1 - z_0). \end{aligned}$$

□

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