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Variation Constant Formula for Functional Partial Parabolic Equations

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Abstract

In this paper we find a variational constant formula for the following system of functional partial parabolic equations

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = D\Delta u + Lu_t + f(t,x), \quad t > 0, \quad u \in \mathbb{R}^n \\ \dfrac{\partial u(t,x)}{\partial \eta} = 0, \quad t > 0, \quad x \in \partial\Omega \\ u(0,x) = \phi(x) \\ u(s,x) = \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases}$$

where Ω is a bounded domain in $I\!R^N$, D is a $n \times n$ non diagonal matrix whose eigenva- lues are semi-simple with non negative real part and $f : I\!R \times \Omega \to I\!R^n$ is a smooth function. The standard notation $u_t(x)$ defines a function from $[-\tau, 0]$ to $I\!R^n$ (with x fixed) by $u_t(x)(s) = u(t+s,x), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^2([-\tau, 0]; Z) \longrightarrow Z$ is a bounded linear(linear and continuous) with $Z = L^2(\Omega)$ and $\phi_0 \in Z$, $\phi \in L^2([-\tau, 0]; Z)$.

Resumen

En este artículo encontramos una fórmula de variación de parámetro para el siguiente sistema de ecuaciones parabólicas parciales funcionales:

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = D\Delta u + Lu_t + f(t,x), \quad t > 0, \quad u \in \mathbb{R}^n \\ \dfrac{\partial u(t,x)}{\partial \eta} = 0, \quad t > 0, \quad x \in \partial\Omega \\ u(0,x) = \phi(x) \\ u(s,x) = \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases}$$

donde Ω es un dominio acotado en $I\!R^N$, D es una matriz $n \times n$ no diagonal, cuyos autovalores son semisimples con parte real no negativa y $f : I\!R \times \Omega \to I\!R^n$ es una función suave. La notación estandar $u_t(x)$ define una función de $[-\tau, 0]$ en $I\!R^n$ (con x fijo) dada por $u_t(x)(s) = u(t+s,x), -\tau \leq s \leq 0$. Aquí $\tau \geq 0$ es el máximo retardo, el cual se supone finito. Se asume que el operador $L : L^2([-\tau,0];Z) \longrightarrow Z$ es lineal y acotado con $Z = L^2(\Omega)$ y $\phi_0 \in Z,\ \phi \in L^2([-\tau,0];Z)$.

key words. functional partial parabolic equations, variation constant formula, strongly continuous semigroups.

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Running Title: VARIATION CONSTANT FORMULA FOR FPD Eqs.

1 Introduction

In this paper we find a variational constant formula for the following system of functional partial parabolic equations

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} & = & D\Delta u + Lu_t + f(t,x), \quad t > 0, \quad u \in I\!R^n \\ \dfrac{\partial u(t,x)}{\partial \eta} & = & 0, \quad t > 0, \quad x \in \partial\Omega \\ u(0,x) & = & \phi(x) \\ u(s,x) & = & \phi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $I\!R^N$, D is a $n \times n$ matrix non diagonal whose eigenvalues are semi-simple with non negative real part and $f : I\!R \times \Omega \to I\!R^n$ is an smooth function. The standard notation $u_t(x)$ defines a function from $[-\tau, 0]$ to $I\!R^n$ (with x fixed) by $u_t(x)(s) = u(t+s,x)$, $-\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume the operator $L : L^2([-\tau,0];Z) \longrightarrow Z$ is linear and bounded with $Z = L^2(\Omega)$ and $\phi_0 \in Z,\ \phi \in L^2([-\tau,0];Z)$.

The variational constant formula plays an important role in the study of the stability, existence of bounded solutions and the asymptotic behavior of non linear ordinary differential equations. For the following finite dimensional semi-linear ordinary differential equations of the type:

$$\begin{cases} x'(t) & = & A(t) + f(t,x), \quad x \in I\!R^n \\ x(0) & = & x_0, \end{cases} \tag{1.2}$$

the variation constant formula is well known and is given by

$$x(t) = \Phi(t)x_0 + \int_0^t \Phi(t)\Phi^{-1}(s)f(s,x(s))ds$$

where $\Phi(\cdot)$ is the fundamental matrix of the system

$$x'(t) = A(t)x. \tag{1.3}$$

Due to the importance of this formula for semi linear ordinary differential equations, in 1961 the Russian mathematician Alekseev, V. M. [1], found a formula for the following non linear ordinary differential equation:

$$y'(t) = f(t, y) + g(t, y), \quad y(t_0) = y_0 \tag{1.4}$$

which is given by

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))g(s, y(s))ds,$$

where $x(t, t_0, y_0)$ is the solution of the initial value problem

$$x'(t) = f(t, x), \quad x(t_0) = y_0, \tag{1.5}$$

and

$$\Phi(t, s, \xi) = \frac{\partial x(t, t_0, y_0)}{\partial y_0}.$$

This formula is used to compare the solutions of (1.4) with solutions of (1.5). In fact, it was used in [9].

In infinite dimensional Banach spaces Z we have the following general situation. If A is the infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in Z and $f : [0, \beta] \to Z$ is a suitable function, then the solution of the initial value problem

$$\begin{cases} z'(t) &= Az(t) + f(t), \quad t > 0, \quad z \in Z \\ z(0) &= z_0, \end{cases} \tag{1.6}$$

is given by the variation constant formula

$$z(t) = T(t)z_0 + \int_0^t T(t-s)f(s)ds, \quad t \in [0, \infty). \tag{1.7}$$

So, any solution of the problem (1.6) is also solution of the integral equation (1.7), but not the conversely since a solution of (1.7) is not necessarily differentiable. We shall refer to a continuous solution of (1.7) as a mild solution of problem (1.6); a mild solution is thus a kind of generalized solution. However, if $\{T(t)\}_{t\geq 0}$ is an analytic semigroup and the function f satisfies the following Hölder condition

$$\|f(s) - f(t)\| \leq L|s - t|^{\theta}, \quad s, t \in [0, \beta],$$

with $L > 0$, $\theta \geq 1$, then the mild solution (1.7) is also solution of the initial value problem (1.6).

Our work and many others are motivated by the legendary paper du to Borisovic J.U.G and Turbabin A.S., see [3]; there they found a variational constant formula for the following system of nonhomogeneous differential equation with delay

$$\begin{cases} z'(t) &= Lz_t + f(t), \quad t > 0, \quad z \in \mathbb{R}^n \\ z(0) &= z_0, \\ z(s) &= \phi(s), \; s \in [-\tau, 0), \end{cases} \tag{1.8}$$

where $f : \mathbb{R}^+ \to \mathbb{R}^n$ is a suitable function. The standard notation z_t defines a function from $[-\tau, 0]$ to $\mathbb{R}^n$ by $z_t(s) = z(t+s), -\tau \leq s \leq 0$. Here $\tau \geq 0$ is the maximum delay, which is suppose to be finite. We assume that the operator $L : L^p([-\tau, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ is linear and bounded, and $z_0 \in \mathbb{R}^n, \; \phi \in L^p([-\tau, 0]; \mathbb{R}^n)$. Under some conditions they prove the existence and the uniqueness of solutions for this system and associate to it a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ in the Banach space $\mathbb{M}_p([-\tau, 0]; \mathbb{R}^n) = \mathbb{R}^n \oplus L_p([-\tau, 0]; \mathbb{R}^n)$.

Therefore, the systems (1.8) is equivalent to the following systems of ordinary differential equations in $\mathbb{M}_p$:

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0 \\ \\ W(0) \;\; = W_0 = (z_0, \phi(\cdot)) \end{cases} \tag{1.9}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$ and $\Phi(t) = (f(t), 0)$.

Hence, the solution of system (1.8) is given by the variational constant formula o mild solution:

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\Phi(s)ds. \tag{1.10}$$

Finally, the formula we found here is valid for those system of PDEs that can be rewritten in the form $\frac{\partial}{\partial t}u = D\Delta u$, like damped nonlinear vibration of a string or a beam, thermoplastic plate equation, etc; for information about this, one can see the paper by Luiz de Oliveira ([12]).

To the best of our knowledge, there are variational constant formulas for reaction diffusion equations, functional equations and neutral equations [6], but for functional partial parabolic equations we are not aware of results similar to the one presented here. At the same time, if we change the Neumann boundary condition by Dirichlet boundary condition, the result follows trivially.

2 Abstract Formulation of the Problem

In this section we choose a Hilbert Space where system (1.1) can be written as an abstract functional differential equation, to this end, we consider the following hypothesis:

H1). The matrix D is semi simple (block diagonal) and the eigenvalues $d_i \in \boldsymbol{C}$ of D satisfy $\mathrm{Re}(d_i) \geq 0$. Consequently, if $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ are the eigenvalues of $-\Delta$ with homogeneous Neumann boundary conditions, then there exists a constant $M \geq 1$ such that :
$\|e^{-\lambda_n D t}\| \leq M, \quad t \geq 0, \quad n = 1, 2, 3, ...$
H2). For all $I > 0$ and $z \in L^2_{loc}([-\tau, 0); Z)$ we have the following inequality

$$\int_0^t \mid Lz_s \mid ds \leq M_0(t) \mid z \mid_{L^2([-\tau,t),Z)}, \ \ \forall t \in [0, I],$$

where $M_0(\cdot)$ is a positive continuous function on $[0, \infty)$.

Consider $H = L^2(\Omega, I\!R)$ and $0 = \lambda_1 < \lambda_2 < ... < \lambda_n \longrightarrow \infty$ the eigenvalues of $-\Delta$, each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Then :

(i) There exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvectors of $-\Delta$.

(ii) For all $\xi \in D(-\Delta)$ we have

$$-\Delta\xi = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n \xi, \tag{2.1}$$

where $< \cdot, \cdot >$ is the inner product in H and

$$E_n x = \sum_{k=1}^{\gamma_n} < \xi, \phi_{n,k} > \phi_{n,k}. \tag{2.2}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in H and
$\xi = \sum_{n=1}^{\infty} E_n \xi, \quad \xi \in H.$
(iii) Δ generates an analytic semigroup $\{T_\Delta(t)\}$ given by

$$T_\Delta(t)\xi = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n \xi. \tag{2.3}$$

Now, we denote by Z the Hilbert space $L^2(\Omega, I\!R^n)$ and define the following operator

$$A : D(A) \subset Z \longrightarrow Z, \quad A\psi = -D\Delta\psi$$

with $D(A) = H^2(\Omega, I\!R^n) \cap H^1_0(\Omega, I\!R^n)$.
Therefore, for all $z \in D(A)$ we obtain,

$$Az = \sum_{n=1}^{\infty} \lambda_n D P_n z$$

and

$$z = \sum_{n=1}^{\infty} P_n z, \quad \| z \|^2 = \sum_{n=1}^{\infty} \| P_n z \|^2, \quad z \in Z$$

where

$$P_n = diag(E_n, E_n, ..., E_n)$$

is a family of complete orthogonal proyections in Z.
Consequently, system (1.1) can be written as an abstract functional differential equation in Z:

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + Lz_t + f^e(t), \quad t > 0 \\ z(0) = \phi_0 \\ z(s) = \phi(s), \ \ s \in [-\tau, 0) \end{cases} \tag{2.4}$$

Here $f^e : (0, \infty) \longrightarrow Z$ is a function defined as follows:

$$f^e(t)(x) = f(t, x), \ \ t > 0, \ \ x \in \Omega.$$

3 Preliminaries Results

From now on, we will use the following generalization of lemma 2.1 from [8].

Lemma 3.1 *Let Z be a separable Hilbert space, $\{S_n(t)\}_{n\geq 1}$ a family of strongly continuous semigroups and $\{P_n\}_{n\geq 1}$ a family of complete orthogonal projection in Z such that:*

$$\Lambda_n P_n = P_n \Lambda_n, \ \ n \geq 1, 2, \ldots$$

where Λ_n is the infinitesimal generator of S_n.

Define the following family of linear operators

$$S(t)z = \sum_{n=1}^{\infty} S_n(t) P_n z, \qquad t \geq 0.$$

Then:
(a) $S(t)$ is a linear and bounded operator if $\| S_n(t) \| \leq g(t), n = 1, 2, ...,$ with $g(t) \geq 0$, continuous for $t \geq 0$.
(b) $\{S(t)\}_{t\geq 0}$ is an strongly continuous semigroup in the Hilbert space Z whose infinitesimal ge-nerator Λ is given by

$$\Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z, \qquad z \in D(\Lambda)$$

with

$$D(\Lambda) = \left\{ z \in Z \,/\, \sum_{n=1}^{\infty} \parallel \Lambda_n P_n z \parallel^2 < \infty \right\}$$

(c) the spectrum $\sigma(\Lambda)$ *of* Λ *is given by*

$$\sigma(\Lambda) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}, \tag{3.1}$$

where $\bar{\Lambda}_n = \Lambda_n P_n : \mathcal{R}(P_n) \to \mathcal{R}(P_n)$.

Proof First, from Hille-Yosida Theorem we obtain

$$S_n(t)P_n = P_n S_n(t) \quad \text{since} \quad \Lambda_n P_n = P_n \Lambda_n.$$

So, $\{S_n(t)P_n z\}_{n\geq 1}$ is a family of orthogonal vectors in Z. Then

$$\begin{aligned}
\parallel S(t)z \parallel^2 &= \langle S(t)z, S(t)z \rangle \\
&= \left\langle \sum_{n=1}^{\infty} S_n(t)P_n z, \sum_{m=1}^{\infty} S_m(t)P_m z \right\rangle \\
&= \sum_{n=1}^{\infty} \parallel S_n(t)P_n z \parallel^2 \\
&\leq (g(t))^2 \sum_{n=1}^{\infty} \parallel P_n z \parallel^2 \\
&= (g(t) \parallel z \parallel)^2
\end{aligned}$$

Therefore, $S(t)$ is a bounded linear operator.

Second, we have the following relations:

(i)

$$\begin{aligned}
S(t)S(s)z &= \sum_{n=1}^{\infty} S_n(t)P_n S(s)z \\
&= \sum_{n=1}^{\infty} S_n(t)P_n \left(\sum_{m=1}^{\infty} S_m(s)P_m z \right) \\
&= \sum_{n=1}^{\infty} S_n(t+s)P_n z \\
&= S(t+s)z
\end{aligned}$$

(ii)

$$S(0)z = \sum_{n=1}^{\infty} S_n(0)P_n z = \sum_{n=1}^{\infty} P_n z = z$$

(iii)

$$\begin{aligned}
\| S(t)z - z \|^2 &= \| \sum_{n=1}^{\infty} S_n(t)P_n z - \sum_{n=1}^{\infty} P_n z \|^2 \\
&= \sum_{n=1}^{\infty} \| (S_n(t) - I)P_n z \|^2 \\
&= \sum_{n=1}^{N} \| (S_n(t) - I)P_n z) \|^2 + \sum_{n=N+1}^{\infty} \| (S_n(t) - I)P_n z \|^2 \\
&\leq \sup_{1\leq n\leq N} \| (S_n(t) - I)P_n z \|^2 \sum_{n=1}^{N} + K \sum_{n=N+1}^{\infty} \| P_n z \|^2
\end{aligned}$$

where $K = \sup_{0\leq t\leq 1;\, n\geq 1} \| (S_n(t) - I) \|^2 \leq (g(t) + 1)^2$.

Since $\{S_n(t)\}_{t\geq 0}$ $(n = 1, 2, ...)$ is an strongly continuous semigroup and $\{P_n\}_{n\geq 1}$ is a complete orthogonal projections, given an arbitrary $\epsilon > 0$ we have, for some natural number N and $0 < t < 1$, the following estimates:

$$\sum_{n=N+1}^{\infty} \| P_n z \|^2 < \frac{\epsilon}{2K}, \quad \sup_{1\leq n\leq N} \| (S_n(t) - I)P_n z \|^2 \leq \frac{\epsilon}{2N} \text{ and } \| S(t)z - z \|^2 < \frac{\epsilon}{2N} \sum_{n=1}^{N} + K\frac{\epsilon}{2K} < \epsilon$$

Hence, $S(t)$ is an strongly continuous semigroup.

Let Λ be the infinitesimal generator of this semigroup. By definition, we have for all $z \in D(\Lambda)$

$$\Lambda z = \lim_{t\longrightarrow 0^+} \frac{S(t)z - z}{t} = \lim_{t\longrightarrow 0^+} \sum_{n=1}^{\infty} \frac{(S_n(t) - I)}{t} P_n z.$$

Next,

$$P_m \Lambda z = P_m \left(\lim_{t\longrightarrow 0^+} \sum_{n=1}^{\infty} \frac{(S_n(t) - I)}{t} P_n z \right) = \lim_{t\longrightarrow 0^+} \frac{S_m(t) - I}{t} P_m z = \Lambda_m P_m z$$

So,

$$\begin{aligned}
\Lambda z &= \sum_{n=1}^{\infty} P_n \Lambda z \\
&= \sum_{n=1}^{\infty} \Lambda_n P_n z
\end{aligned}$$

and,

$$D(\Lambda) \subset \left\{ z \in Z / \sum_{n=1}^{\infty} \| \Lambda_n P_n z \|^2 < \infty \right\}$$

On the other hand, if we suppose that $z \in \left\{ z \in Z / \sum_{n=1}^{\infty} \| \Lambda_n P_n z \|^2 < \infty \right\}$, then

$$\sum_{n=1}^{\infty} \Lambda_n P_n z = y \in Z$$

Next, making $z_n = \sum_{k=1}^{n} P_k z$ we obtain that

$$\lim_{t \longrightarrow 0^+} \frac{S(t)z_n - z_n}{t} = \sum_{k=1}^{n} P_k \Lambda_k z < \infty.$$

Therefore, $z_n \in D(\Lambda)$ and $\Lambda z_n = \sum_{k=1}^{n} P_k \Lambda_k z$.

Finally, if $z_n \longrightarrow z$ when $n \longrightarrow \infty$ and $\lim_{t \longrightarrow 0^+} \Lambda z_n = y$, then, since Λ is closed, we obtain that $z \in D(\Lambda)$ and $\Lambda z = y$.

To complete the proof of the lemma, we shall prove part (c). It is equivalent to prove the following:

$$\bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n) \subset \sigma(\Lambda) \quad \text{and} \quad \sigma(\Lambda) \subset \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}.$$

To prove the first part, We shall show that $\rho(\Lambda) \subset \bigcap_{n=1}^{\infty} \rho(\bar{\Lambda}_n)$. In fact, let λ be in $\rho(\Lambda)$. Then $(\lambda - \Lambda)^{-1} : Z \to D(\Lambda)$ is a bounded linear operator. We need to prove that

$$(\lambda - \bar{\Lambda}_m)^{-1} : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$$

exists and is bounded for $m \geq 1$. Suppose that $(\lambda - \bar{\Lambda}_m)^{-1} P_m z = 0$. Then

$$\begin{aligned} (\lambda - \Lambda)P_m z &= \sum_{n=1}^{\infty} (\lambda - \Lambda_n) P_n P_m z \\ &= (\lambda - \Lambda_m) P_m z = (\lambda - \bar{\Lambda}_m) P_m z = 0. \end{aligned}$$

Which implies that, $P_m z = 0$. So, $(\lambda - \bar{\Lambda}_m)$ is one to one.

Now, given y in $\mathcal{R}(P_m)$ we want to solve the equation $(\lambda - \bar{\Lambda}_m)w = y$. In fact, since $\lambda \in \rho(\Lambda)$ there exists $z \in Z$ such that

$$(\lambda - \Lambda)z = \sum_{n=1}^{\infty} (\lambda - \Lambda_n) P_n z = y.$$

Then, applying P_m to the both side of this equation we obtain

$$P_m(\lambda - \Lambda)z = (\lambda - \Lambda_m)P_m z = (\lambda - \bar{\Lambda}_m)P_m z = P_m y = y.$$

Therefore, $(\lambda - \bar{\Lambda}_m) : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$ is a bijection. Since $\bar{\Lambda}_m$ is close, then, by the closed-graph theorem, we get that

$$\lambda \in \rho(\bar{\Lambda}_m) = \{\lambda \in \boldsymbol{C} : (\bar{\Lambda}_m - \lambda I) \text{ is bijective}\} = \{\lambda \in \boldsymbol{C} : (\bar{\Lambda}_m - \lambda I)^{-1} \text{ is bounded}\}$$

for all $m \geq 1$. We have proved that

$$\rho(\Lambda) \subset \bigcap_{n=1}^{\infty} \rho(\bar{\Lambda}_n) \iff \bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n) \subset \sigma(\Lambda).$$

Now, we shall prove the other part of (c), that is to say:

$$\sigma(\Lambda) \subset \overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}.$$

In fact, if $\lambda \in \sigma(\Lambda)$, then

(1) $\lambda \in \sigma_p(\Lambda) = \{\lambda \in \boldsymbol{C} : (\Lambda - \lambda I) \text{ is not injective}\}$

(2) $\lambda \in \sigma_r(V) = \{\lambda \in \boldsymbol{C} : (\Lambda - \lambda I) \text{ is injective , but } \overline{R(\Lambda - \lambda I)} \neq Z\}$

(3) $\lambda \in \sigma_c(\Lambda) = \{\lambda \in \boldsymbol{C} : (\Lambda - \lambda I) \text{ is injective , } \overline{R(\Lambda - \lambda I)} = Z, \text{but } R(\Lambda - \lambda I) \neq Z\}$.

(1) If $(A\Lambda - \lambda I)$ is not injective, then there exists $z \in Z$ non zero such that: $(\Lambda - \lambda I)z = 0$. This implies that for some n_0 we have:

$$(\overline{\Lambda_{n_0}} - \lambda I)P_{n_0}z = 0, \quad P_{n_0}z \neq 0.$$

From here we obtain that $\lambda \in \sigma(\overline{\Lambda_{n_0}})$, and therefore $\lambda \in \overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}$.

(2) If $\overline{R(\Lambda - \lambda I)} \neq Z$, then there exists $z_0 \in Z$ non zero such that:

$$\langle z_0, (A\Lambda - \lambda I)z\rangle = 0, \quad \forall z \in D(A).$$

But, $z = \sum_{n=1}^{\infty} P_n z$, so:

$$\langle z_0, \sum_{n=1}^{\infty}(\overline{\Lambda_n} - \lambda I)P_n z\rangle = 0.$$

Now, if $z_0 \neq 0$, then there is $n_0 \in \mathbf{N}$ such that $P_{n_0}z_0 \neq 0$. Hence,

$$\begin{aligned} 0 &= \langle z_0, \sum_{n=1}^{\infty}(\overline{\Lambda_n} - \lambda I)P_n z\rangle \\ &= \langle z_0, (\overline{\Lambda_{n_0}} - \lambda I)P_{n_0} z\rangle \\ &= \langle P_{n_0} z_0, (\overline{\Lambda_{n_0}} - \lambda I)P_{n_0} z\rangle \end{aligned}$$

So, $R(\overline{\Lambda}_{n_0} - \lambda I) \neq P_{n_0}Z$. Therefore, $\lambda \in \sigma(\overline{\Lambda}_{n_0}) \subset \overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}$.

(3) Assume that $(\Lambda - \lambda I)$ is injective, $\overline{R(\Lambda - \lambda I)} = Z$ and $R(\Lambda - \lambda I) \subseteq Z$.

For the purpose of get a contradiction, we suppose that $\lambda \in \left(\overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}\right)^C$.

But,

$$\begin{aligned} \left(\overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})}\right)^C &\subset \left(\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})\right)^C \\ &= \bigcap_{n\geq 1} \left(\sigma(\overline{\Lambda_n})\right)^C \\ &= \bigcap_{n\geq 1} \rho(\overline{\Lambda_n}), \end{aligned}$$

which implies that, $\lambda \in \rho(\overline{\Lambda_n})$, for all $n \geq 1$. Then we get that:

$$(\overline{\Lambda_n} - \lambda I) : R(P_n) \longrightarrow R(P_n)$$

is invertible, with $(\overline{\Lambda_n} - \lambda I)^{-1}$ bounded.

Hence, for all $z \in D(\Lambda)$ we obtain that

$$P_j(\Lambda - \lambda I)z = (\overline{\Lambda_j} - \lambda I)P_j z, \quad j = 1, 2, \dots$$

i.e.,

$$(\overline{\Lambda_j} - \lambda I)^{-1} P_j(\Lambda - \lambda I)z = P_j z, \quad j = 1, 2, \dots$$

Now, since $D(A)$ is dense in Z, we may extend the operator $(\overline{\Lambda_j} - \lambda I)^{-1} P_j(\Lambda - \lambda I)$ to a bounded operator T_j defined on Z. Therefore, it follows that

$$T_j z = P_j z, \quad \forall z \in Z, \quad j = 1, 2, \dots,$$

and

$$\|T_j\| = \|P_j\| \leq 1, \quad j = 1, 2, \dots.$$

Since $\overline{R(\Lambda - \lambda I)} = Z$, we get that

$$\|(\overline{\Lambda_j} - \lambda I)^{-1}\| \leq 1, \quad j = 1, 2, \dots. \tag{3.2}$$

Now we shall see that $R(\Lambda - \lambda I) = Z$. In fact, given $z \in Z$ we define y as follows

$$y = \sum_{j=1}^{\infty} (\overline{\Lambda_j} - \lambda I)^{-1} P_j z.$$

From (3.2) we get that y is well defined. We shall see now that $y \in D(\Lambda)$ and $(\Lambda - \lambda I)y = z$. In fact, we know that:

$$y \in D(\Lambda) \iff \sum_{j=1}^{\infty} \|\Lambda_j P_j y\|^2 < \infty.$$

On the other hand, we have that

$$\sum_{j=1}^{\infty} \|\overline{\Lambda}_j P_j y\|^2 = \sum_{j=1}^{\infty} \|\Lambda_j (\overline{\Lambda_j} - \lambda I)^{-1} P_j z\|^2 = \sum_{j=1}^{\infty} \|\{I + \lambda(\overline{\Lambda_j} - \lambda I)^{-1}\} P_j z\|^2.$$

So,

$$\sum_{j=1}^{\infty} \|\Lambda_j P_j y\|^2 \leq \sum_{j=1}^{\infty} \|(1 + |\lambda|)^2 \|P_j z\|^2 = (1 + |\lambda|)^2 \|z\|^2 < \infty.$$

Then, $y \in D(\Lambda)$ and $(\Lambda - \lambda I) = z$.

Therefore $R(\Lambda - \lambda I) = Z$, which is a contradiction that came from the assumption: $\lambda \in \left(\overline{\bigcup_{n=1}^{\infty} \sigma(\overline{\Lambda_n})} \right)^C$.

□

Lemma 3.2 *Let Z be a separable Hilbert space, $\{S_n(t)\}_{t \geq 0}$ a family of strongly continuous semigroups with generators Λ_n and $\{P_n\}_{n \geq 1}$ a family of complete orthogonal projections such that*

$$\Lambda_n P_m = P_m \Lambda_n, \quad n, m = 1, 2, \ldots \tag{3.3}$$

If the operator

$$\Lambda z = \sum_{n=1}^{\infty} \Lambda_n P_n z, \quad z \in D(\Lambda)$$

with

$$D(\Lambda) = \{z \in Z : \sum_{n=1}^{\infty} \| \Lambda_n P_n z \|^2 < \infty\}$$

generates a strongly continuous semigroup $\{S(t)\}_{t \geq 0}$, then

$$S(t)z = \sum_{n=1}^{\infty} S_n(t) P_n z, \quad z \in Z.$$

Proof If $z_0 \in Z$, then $P_n z_0 \in D(\Lambda)$ and the mild solution of the problem

$$\begin{cases} z'(t) &= \Lambda z(t) \\ z(0) &= P_n z_0 \end{cases} \tag{3.4}$$

is given by $z_n(t) = S(t)P_n z_0$ and it is a classic solution.
Using (3.3) and the Hille-Yosida Theorem, we get that $P_n S(t) = S(t)P_n$, which implies that:

$$S(t)z_0 = \sum_{n=1}^{\infty} P_n S(t) z_0 = \sum_{n=1}^{\infty} S(t) P_n z_0. \tag{3.5}$$

On the other hand, since $z_n(t)$ is a classic solution of (3.4), we obtain that

$$\begin{aligned} z_n'(t) &= \Lambda z_n(t) \\ &= \Lambda S(t) P_n z_0 \\ &= \sum_{m=1}^{\infty} \Lambda_m P_m S(t) P_n z_0 \\ &= \Lambda_n P_n S(t) P_n z_0 \\ &= \Lambda_n S(t) P_n z_0 = \Lambda_n z_n(t) \end{aligned}$$

So, $z_n(t) = S_n(t)P_n z_0 = S(t)P_n z_0$ and from (3.5) we get that

$$S_n(t) z_0 = \sum_{n=1}^{\infty} S_n(t) P_n z_0.$$

□

Now, applying Lemma 3.1 we can prove the following result.

Theorem 3.3 *The operator $-A$ is the infinitesimal generator of a strongly continous semigroup $\{T_A(t)\}_{t\geq 0}$ in the space Z, given by*

$$T_A(t) z = \sum_{n=1}^{\infty} e^{-\lambda_n D t} P_n z, \quad z \in Z, \quad t \geq 0. \tag{3.6}$$

3.1 Existence and Uniqueness of Solutions

In this part we study the existence and the uniqueness of the solutions for system (2.4) in case that $f^e \equiv 0$. That is, we analyze the following homogeneous system

$$\begin{cases} \dfrac{dz(t)}{dt} = -Az(t) + Lz_t, \quad t > 0 \\ z(0) = \phi_0 = z_0 \\ z(s) = \phi(s), \;\; s \in [-\tau, 0) \end{cases} \tag{3.7}$$

Definition 3.4 *A function $z(\cdot)$ define on $[-\tau,\alpha)$ is called a Mild Solution of (3.7) if*

$$z(t) = \begin{cases} \phi(t); \quad -\tau \le t < 0 \\ \\ T_A(t)z_0 + \int_0^t T_A(t-s)Lz_s ds, \quad t \in [0,\alpha) \end{cases}$$

Theorem 3.5 *The problem (3.7) admits only one mild solution defined on $[-\tau,\infty)$.*

Proof Consider the following initial function

$$\varphi(s) = \begin{cases} \phi(s); \quad -\tau \le s < 0 \\ \\ T_A(s)z_0; \quad s \ge 0 \end{cases}$$

which belongs to $L^2_{loc}([-\tau,\infty), Z)$. For a moment we shall set the problem on $[-\tau, I]$, $I > 0$ and denote by G the set:

$$G = \{\psi : \psi \in L^2[[-\tau,\alpha], Z] \quad and \quad \mid \psi - \varphi \mid_{L^2} \le \rho, \quad \rho > 0\},$$

where $\alpha > 0$ is a number to be determine. It is clear that G endowed with the norm of $L^2([-\tau,\alpha]; Z)$ is a complete metric space.

Now, we consider the application $S : G \to Z$ given by

$$(Sz)(t) = Sz(t) = \begin{cases} \phi(t), \quad -\tau \le t < 0 \\ \\ T_A(t)z_0 + \int_0^t T_A(t-s)Lz_s ds, \quad t \in [0,\alpha] \end{cases} \qquad \forall z \in G.$$

Claim 1. There exists $\alpha > 0$ such that:

(i) $Sz \in G, \quad \forall z \in G.$

(ii) S is a contraction mapping.

In fact, we prove (i) in the following way:

$$\begin{aligned} \mid Sz(t) - \varphi(t) \mid &\le \int_0^t \mid T_A(t-s)Lz_s \mid ds \\ &\le \int_0^\alpha M \mid Lz_s \mid ds \\ &\le MM_0(\alpha) \mid z \mid_{L^2([-\tau,\alpha),Z)} . \end{aligned}$$

Integrating we have:

$$\mid Sz - \varphi \mid_{L^2} \quad \le \quad K\alpha^{\frac{1}{2}} \mid z \mid_{L^2}$$

where $K = max\{MM_0(\alpha)/ \ \alpha \in [0, I]\}$.

From here we get :

$$| Sz - \varphi |_{L^2} \quad \leq \quad K\alpha^{\frac{1}{2}}(| \varphi |_{L^2} + \rho), \quad z \in G.$$

Taking

$$\alpha \quad < \quad \left(\frac{\rho}{K(| \varphi |_{L^2} + \rho)} \right) 2$$

we obtain that $Sz \in G, \quad \forall z \in G$.

In order to prove (ii), we use the linearity of L to obtain:

$$| Sz - Sw |_{L^2} \quad \leq \quad K\alpha^{\frac{1}{2}} \; | z - w |_{L^2}, \quad \forall z, w \in G.$$

Next, in order to prove that S it is a contraction and $S(G) \subset G$ it is enough to choose α as follows :

$$\alpha \quad < \quad min \left\{ \left(\frac{1}{K} \right)^2, \left(\frac{\rho}{K(| \varphi |_{L^2} + \rho)} \right)^2 \right\}$$

Therefore, S is a contraction mapping.

So, if we apply the contraction mapping Theorem, there exists a unique point $z \in G$ such that $Sz = z$. i.e.,

$$z(t) = Sz(t) = \begin{cases} \phi(t); \quad -\tau \leq t < 0 \\ \\ T_A(t)z_0 + \displaystyle\int_0^t T_A(t-s)Lz_s ds, \quad t \in [0, \alpha], \end{cases}$$

which proves the existence and the uniqueness of the mild solution of the initial value problem (3.7) on $[-\tau, \alpha]$.

Claim 2. α could be equal to ∞. In fact, let z be the unique mild solution define in a maximal interval $[-\tau, \delta)(\delta \geq \alpha)$.

By contradiction, let us suppose that $\delta < \infty$. Since z is a mild solution of (3.7), we have that

$$z(t) = T_A(t)z_0 + \int_0^t T_A(t-s)Lz_s ds, \quad t \in [0, \delta).$$

Consider the sequence $\{t_n\}$ such that $t_n \longrightarrow \delta^-$. Let us prove that $\{z(t_n)\}$ is a Cauchy sequence. In fact,

$$\begin{aligned} | z(t_n) - z(t_m) | \quad &= \quad | T_A(t_n)z_0 - T_A(t_m)z_0 + \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds | \\ &\leq \quad | (T_A(t_n) - T_A(t_m))z_0 | + | \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds | \end{aligned}$$

But,

$$
\begin{aligned}
\left| \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t_m} T_A(t_m - s)Lz_s ds \right| &\leq \left| \int_0^{t_m} (T_A(t_n - s) - T_A(t_m - s))Lz_s ds \right| \\
&+ \left| \int_{t_n}^{t_m} T_A(t_n - s)Lz_s ds \right|
\end{aligned}
$$

Now, for $z \in L^2([-\tau, \delta])$ we obtain that

$$
\int_0^{t_m} | (T_A(t_n - s) - T_A(t_m - s))Lz_s | \, ds \leq \int_0^{\delta} | (T_A(t_n - s) - T_A(t_m - s))Lz_s | \, ds
$$

We know that:

$$
\lim_{n,m \longrightarrow \infty} | (T_A(t_n - s) - T_A(t_m - s))Lz_s | = 0
$$

and

$$
| (T_A(t_n - s) - T_A(t_m - s))Lz_s | \leq 2M \, | Lz_s |
$$

But, from the hypothesis H1), we obtain that:

$$
\int_0^{\delta} 2M \, | Lz_s | \, ds \leq 2MM_0(\delta) \, | z |_{L^2([-\tau,\delta);Z)}
$$

Therefore, applying the Lebesgue Dominated Convergence Theorem we obtain

$$
\lim_{n,m \longrightarrow \infty} \int_0^{\delta} | (T_A(t_n - s) - T_A(t_m - s))Lz_s | \, ds = 0
$$

Then, since the family $\{T_A(t)\}_{t \geq 0}$ is strongly continuous and $t_n, t_m \longrightarrow \delta^-$ when $n, m \longrightarrow \infty$, the sequence $\{z(t_n)\}$ is a Cauchy sequence and therefore there exists $B \in Z$ such that:

$$
\lim_{n \longrightarrow \infty} z(t_n) = B.
$$

Now, for $t \in [0, \delta)$ we obtain that

$$
\begin{aligned}
| z(t) - B | &\leq | z(t) - z(t_n) | + | z(t_n) - B | \\
&\leq | (T_A(t) - T_A(t_n))z_0 | + | z(t_n) - B | \\
&+ \left| \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t} T_A(t - s)Lz_s ds \right|
\end{aligned}
$$

But,

$$
\begin{aligned}
\left| \int_0^{t_n} T_A(t_n - s)Lz_s ds - \int_0^{t} T_A(t - s)Lz_s ds \right| &\leq \int_0^{t_n} | (T_A(t - s) - T_A(t_n - s))Lz_s | \, ds \\
&+ \int_t^{t_n} | T_A(t - s)Lz_s | \, ds.
\end{aligned}
$$

On the other hand, for $z \in L^2([-\tau, \delta])$ we get the following estimate:

$$\int_0^{t_n} \mid (T_A(t-s) - T_A(t_n - s))Lz_s \mid ds \;\leq\; \int_0^{\delta} \mid (T_A(t-s) - T_A(t_n - s))Lz_s \mid ds$$

Therefore, applying the Lebesgue Dominated Convergence Theorem, we obtain

$$\lim_{n \longrightarrow \infty} \int_0^{\delta} \mid (T_A(t-s) - T_A(t_n - s))Lz_s \mid = \; 0$$

Then, since the family $\{T_A(t)\}_{t\geq 0}$ is strongly continuous and $t_n \longrightarrow \delta^-$ when $n \longrightarrow \infty$, it follows that $z(t) \longrightarrow B$ as $t \longrightarrow \delta^-$.

The function

$$\varphi(s) = \begin{cases} z(s); & \delta - \tau \leq s < \delta \\ T_A(s)B, & s \geq \delta \end{cases}$$

belong to $L^2_{loc}([\delta - \tau, \infty), Z)$. So, if we apply again the contraction mapping Theorem to the Cauchy problem

$$\begin{cases} \dfrac{dy(t)}{dt} = -Ay(t) + Ly_t, \quad t > \delta \\ y(\delta) = B \\ y(s) = z(s), \;\; s \in [\delta - \tau, \delta) \end{cases} \tag{3.8}$$

where $z(\cdot)$ is the unique solution of the system (3.7), then we get that (3.8) admits only one solution $y(\cdot)$ on the interval $[\delta - \tau, \delta + \epsilon]$ with $\epsilon > 0$. Therefore, the function

$$\widetilde{z}(s) = \begin{cases} z(s); & -\tau \leq s < \delta \\ y(s), & \delta \leq s < \delta + \epsilon \end{cases}$$

is also mild solution of (3.7) which is a contradiction. So, $\delta = \infty$. □

4 The Variation Constant Formula

Now we are ready to find the formula announced in the title of this paper for the system (2.4), but first we need to write this system as an abstract ordinary differential equation in an appropriate Hilbert space. In fact, we consider the Hilbert space $\mathbb{M}_2([-\tau, 0]; Z) = Z \oplus L_2([-\tau, 0]; Z)$ with the usual inerproduct given by:

$$\left\langle \begin{pmatrix} \phi_{01} \\ \phi_1 \end{pmatrix}, \begin{pmatrix} \phi_{02} \\ \phi_2 \end{pmatrix} \right\rangle = \langle \phi_{01}, \phi_{02} \rangle_Z + \langle \phi_1, \phi_2 \rangle_{L_2}.$$

Define the following operator in the space $\mathbb{M}_2$ for $t \geq 0$ by

$$T(t)\begin{pmatrix} \phi_0 \\ \phi(.) \end{pmatrix} = \begin{pmatrix} z(t) \\ z_t \end{pmatrix} \tag{4.1}$$

where $z(\cdot)$ is the only mild solution of the system (3.7).

Theorem 4.1 *The family of operators $\{T(t)\}_{t\geq 0}$ defined by (4.1) is an strongly continuous semigroup on $\mathbb{M}_2$ such that*

$$T(t)W = \sum_{n=1}^{\infty} T_n(t)Q_nW, \quad W \in \mathbb{M}_2, \quad t \geq 0, \tag{4.2}$$

where,

$$Q_n = \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix},$$

with $(\widetilde{P}_n\phi)(s) = P_n\phi(s), \phi \in L^2([-\tau, 0]; Z),\ s \in [-\tau, 0]$, and $\{\{T_n(t)\}_{t\geq 0}, n = 1, 2, 3, ..\}$ is a family of strongly continuous semigroups on $\mathbb{M}_2^n = Q_n\mathbb{M}_2$ given in the same way as in Theorem 2.4.4 from [5] and defined as follows

$$T_n(t)\begin{pmatrix} w_n^0 \\ w_n \end{pmatrix} = \begin{pmatrix} W^n(t) \\ W^n(t+\cdot) \end{pmatrix}, \quad \begin{pmatrix} w_n^0 \\ w_n \end{pmatrix} \in \mathbb{M}_2^n,$$

where $W^n(\cdot)$ is the unique solution of the initial value problem

$$\begin{cases} \dfrac{dw(t)}{dt} = -\lambda_n Dw(t) + L_n w_t, & t > 0 \\ w(0) = w_n^0 \\ w(s) = w_n(s), & s \in [-\tau, 0) \end{cases} \tag{4.3}$$

and $L_n = L\widetilde{P}_n = P_nL$, as it is in most the case practical problems.

Proof of Theorem 4.1 First, we shall prove that

$$T(t)W = \sum_{n=1}^{\infty} T_n(t)Q_nW, \quad W \in \mathbb{M}_2, \quad t \geq 0.$$

In fact, let $W = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \in \mathbb{M}_2$.

$$
\begin{aligned}
\sum_{n=1}^{\infty} T_n(t) Q_n W &= \sum_{n=1}^{\infty} T_n(t) \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} T_n(t) \begin{pmatrix} P_n w_1 \\ \widetilde{P}_n w_2 \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} z^n(t) \\ z^n(t+\cdot) \end{pmatrix}; \quad z^n(\cdot) \text{ the only mild solution of } (\mathbf{??}) \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} e^{\mathcal{A}_n t} P_n w_1 + \int_0^t e^{\mathcal{A}_n (t-s)} L_n (\widetilde{P}_n z^n(s+\cdot)) ds \\ \\ (\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} \sum_{n=1}^{\infty} e^{\mathcal{A}_n t} P_n w_1 + \int_0^t \sum_{n=1}^{\infty} e^{\mathcal{A}_n (t-s)} P_n \left(L \sum_{m=1}^{\infty} (\widetilde{P}_m z(s+\cdot)) \right) ds \\ \\ \sum_{n=1}^{\infty} (\widetilde{P}_n z(t+\cdot)) \end{pmatrix} \\
&= \begin{pmatrix} T_{\mathcal{A}}(t) w_1 + \int_0^t T_{\mathcal{A}}(t-s) L z(s+\cdot) ds \\ \\ z(t+\cdot) \end{pmatrix} \\
&= \begin{pmatrix} z(t) \\ z_t(\cdot) \end{pmatrix}; \quad z(\cdot) \text{ the only mild solution of } (3.7) \\
&= T(t) W.
\end{aligned}
$$

In the same way as in Theorem 2.4.4 of [5] we can prove that the infinitesimal generator of $\{T_n(t)\}_{t \geq 0}$ is given by:

$$
\Lambda_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\Lambda_n D w_n^0 + L_n w_n(\cdot) \\ \\ \dfrac{\partial w_n(\cdot)}{\partial s} \end{pmatrix}
$$

with

$$
D(\Lambda_n) = \left\{ \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} \in \mathbb{M}_2^n : w_n \; is \; a.c., \; \frac{\partial w_n(\cdot)}{\partial s} \in L_2([-\tau, 0]; Q_n Z) \; and \; w_n(0) = w_n^0 \right\}.
$$

Furthermore, the spectrum of Λ_n is discrete and given by

$$\sigma(\Lambda_n) = \sigma_p(\Lambda_n) = \{\lambda \in \boldsymbol{C} : \det(A_n(\lambda)) = 0\}, \tag{4.4}$$

where $A_n(\lambda)$ is given by

$$\Lambda_n(\lambda)z = \lambda z + \lambda_n Dz - L_n e^{\lambda(\cdot)} z, \quad z \in Z_n = P_n Z,$$

which can be considered as a matrix since $\dim(Z_n) < \infty$.

On the other hand, $\{Q_n\}_{n\geq 1}$ is a family of complete orthogonal projection on $\mathbb{M}_2$ and

$$\Lambda_n Q_n = Q_n \Lambda_n, \quad n = 1, 2, 3, ...$$

In fact,

$$\begin{aligned}
\Lambda_n Q_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix} &= \Lambda_n \begin{pmatrix} Pnw_n^0 \\ \widetilde{P_n} w_n(\cdot) \end{pmatrix} = \begin{pmatrix} -\Lambda_n D P_n w_n^0 + L_n \widetilde{P_n} w_n(\cdot) \\ \\ \dfrac{\partial \widetilde{P_n} w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= \begin{pmatrix} -\Lambda_n D P_n w_n^0 + L \widetilde{P_n}\widetilde{P_n} w_n(\cdot) \\ \\ \widetilde{P_n} \dfrac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} = \begin{pmatrix} -\Lambda_n D P_n w_n^0 + P_n L_n w_n(\cdot) \\ \\ \widetilde{P_n} \dfrac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} \\
&= \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} -\Lambda_n D w_n^0 + L_n w_n(\cdot) \\ \\ \dfrac{\partial w_n(\cdot)}{\partial s} \end{pmatrix} = Q_n \Lambda_n \begin{pmatrix} w_n^0 \\ w_n(\cdot) \end{pmatrix}
\end{aligned}$$

Now, we shall check condition (a) of Lemma 3.1, to this end we need to prove the following claim:

Claim. If $W^n(t)$ is the solution of (4.3), then the following inequalities hold:

$$\| W^n(t) \|_Z \leq \; c_2 e^{c_1 t} \|w_n^0\|, \quad t \geq 0, \tag{4.5}$$

$$\int_0^t \| W^n(u) \|_Z \, du \; \leq \; k e^{c_2 t} \|w_n^0\|, \quad t \geq 0. \tag{4.6}$$

In fact, if we put $M_1 = \text{máx}\{M, \| L \|\}$, then we get:

$$\| W^n(t+\theta) \|_Z \leq \; M_1 \|w_n^0\| + M_1^2 \int_0^t \| W_s^n \|_{L^2} \, ds; \quad \theta \in [-\tau, 0],$$

this implies that

$$\| W^n(t+\theta) \|_Z^2 \leq \; \left(M_1 \|w_n^0\| + M_1^2 \int_0^t \| W_s^n \|_{L^2} \, ds \right)^2 .$$

Next,

$$\begin{aligned}
\int_{-\tau}^{0} \| W^n(t+\theta) \|_Z^2 \, d\theta &\leq \int_{-\tau}^{0} \left(M_1 \|w_n^0\| + M_1^2 \int_0^t \| W_s^n \|_{L^2} \, ds \right)^2 d\theta \\
&\leq \int_{-\tau}^{0} 2^2 \left(M_1^2 \|w_n^0\|^2 + M_1^4 \left(\int_0^t \| W_s^n \|_{L^2} \, ds \right)^2 \right) d\theta \\
&= 2^2 \tau M_1^2 \|w_n^0\|^2 + M_1^4 \left(\int_0^t \| W_s^n \|_{L^2} \, ds \right)^2 \int_{-\tau}^{0} d\theta \\
&= c_2^2 \|w_n^0\|^2 + c_1^2 \left(\int_0^t \| W_s^n \|_{L^2} \, ds \right)^2 \\
&\leq \left(c_2 \|w_n^0\| + c_1 \left(\int_0^t \| W_s^n \|_{L^2} \, ds \right) \right)^2
\end{aligned}$$

So,

$$\| W_t^n \|_{L^2} \leq \; c_2 \|w_n^0\| + c_1 \left(\int_0^t \| W_s^n \|_{L^2} \, ds \right)$$

Therefore, applying Gronwall's lemma we obtain that

$$\| W_t^n \|_{L^2} \leq \; c_2 e^{c_1 t} \|w_n^0\|, \;\; t \geq 0.$$

On the other hand, we obtain the following estimate

$$\begin{aligned}
\| W^n(t) \|_Z &\leq \| T_{A_n}(t) w_n^0 \| + \| \int_0^t T_{A_n}(t-s) L_n W^n(s+\cdot) ds \| \\
&\leq M_1 \|w_n^0\| + M_1^2 \int_0^t \| W^n(s+\cdot) ds \| \\
&\leq M_1 \|w_n^0\| + M_1^2 \int_0^t c_1 e^{c_2 t} \|w_n^0\| ds \\
&= \left(M_1 + \frac{M_1^2 c_1}{c_2} e^{c_2 t} \right) \|w_n^0\| \\
&\leq c e^{c_2 t} \|w_n^0\|, \qquad c = M_1 + \frac{M_1^2 c_1}{c_2}, \; t \geq 0.
\end{aligned}$$

Finally, we get

$$\int_0^t \| W^n(u) \|_Z \, du \; \leq \; k e^{c_2 t} \|w_n^0\|, \qquad k = \frac{c}{c_2}, \; t \geq 0.$$

This completes the proof of the claim.

Now, we will use the above inequalities:

$$\begin{aligned}\| T_n(t)\begin{pmatrix} w_n^0 \\ w_n \end{pmatrix} \|^2 &= \| W^n(t) \|_Z^2 + \int_{-\tau}^{0} \| W^n(t+\tau) \|_Z^2 \, d\tau \\ &= \| W^n(t) \|_Z^2 + \int_{t-\tau}^{t} \| W^n(u) \|_Z^2 \, du \\ &\leq \| W^n(t) \|_Z^2 + \int_{0}^{t} \| W^n(u) \|_Z^2 \, du + \| w_n \|_{L^2}^2 \\ &\leq \left(c_2^2 e^{2c_2 t} + k^2 e^{2c_2 t}\right) \|w_n^0\|^2 + \| w_n \|_{L^2}^2 \\ &\leq g(t)2\left(\|w_n^0\|^2 + \| w_n \|_{L^2}^2\right), \qquad n \geq 1, 2, \dots .\end{aligned}$$

Hence,

$$\| T_n(t) \| \leq g(t), \qquad n \geq 1, 2, \dots .$$

Therefore, applying Lemma 3.1, we obtain that $T(t)$ is bounded and $\{T(t)\}_{t\geq 0}$ is a strongly continuous semigroup on the Hilbert space $\mathbb{M}_2$, whose generator Λ is given by

$$\Lambda W = \sum_{n=1}^{\infty} \Lambda_n Q_n W, \qquad W \in D(\Lambda),$$

with

$$D(\Lambda) = \left\{ W \in \mathbb{M}_2 / \sum_{n=1}^{\infty} \| \Lambda_n Q_n W \|^2 < \infty \right\}$$

and the spectrum $\sigma(\Lambda)$ of Λ is given by

$$\sigma(\Lambda) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{\Lambda}_n)}, \tag{4.7}$$

where $\bar{\Lambda}_n = \Lambda_n Q_n : \mathcal{R}(Q_n) \to \mathcal{R}(Q_n)$. □

Lemma 4.2 *Let Λ be the infinitesimal generator of the semi-group $\{T(t)\}_{t\geq 0}$. Then*

$$\Lambda \tilde{\varphi}(s) = \begin{pmatrix} -A\varphi(0) + L\phi(s) \\ \\ \dfrac{\partial \phi(s)}{\partial s} \end{pmatrix} ; \ -\tau \leq s \leq 0,$$

$$D(\Lambda) = \{ \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} \in \mathbb{M}_2 : \phi_0 \in D(A), \phi \ \ is \ \ a.c., \frac{\partial \phi(s)}{\partial s} \in L^2([-\tau, 0]; Z) \ \ and \ \ \phi(0) = \phi_0 \},$$

and

$$\sigma(\Lambda) = \overline{\bigcup_{n=1}^{\infty} \{\lambda \in \boldsymbol{C} : det(\Lambda_n(\lambda)) = 0\}}$$

Proof Consider $\begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix}$ in $\mathbb{M}_2$. Then

$$\begin{aligned}
\Lambda W = \Lambda \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} &= \sum_{n=1}^{\infty} \Lambda_n Q_n W \\
&= \sum_{n=1}^{\infty} \Lambda_n \begin{pmatrix} P_n & 0 \\ 0 & \widetilde{P}_n \end{pmatrix} \begin{pmatrix} \phi_0 \\ \phi(\cdot) \end{pmatrix} = \sum_{n=1}^{\infty} \Lambda_n \begin{pmatrix} P_n \phi_0 \\ \widetilde{P_n} \phi(\cdot) \end{pmatrix} \\
&= \sum_{n=1}^{\infty} \begin{pmatrix} -\Lambda_n D \widetilde{P_n} \phi(0) + L_n \widetilde{P_n} \phi \\ \\ \dfrac{\partial \widetilde{P}_n \phi(\cdot)}{\partial(s)} \end{pmatrix} \\
&= \begin{pmatrix} -\displaystyle\sum_{n=1}^{\infty} \Lambda_n D P_n \phi(0) + L \sum_{n=1}^{\infty} \widetilde{P}_n \phi \\ \\ \dfrac{\partial}{\partial s} \left(\displaystyle\sum_{n=1}^{\infty} \widetilde{P}_n \phi(\cdot) \right) \end{pmatrix} \\
&= \begin{pmatrix} -A\phi(0) + L\phi(\cdot) \\ \\ \dfrac{\partial \phi(\cdot)}{\partial s} \end{pmatrix}.
\end{aligned}$$

The other part of the lemma follows from (4.7) □

Therefore, the systems (3.7) and (2.4) are equivalent to the following two systems of ordinary di-fferential equations in $\mathbb{M}_2$ respectively:

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t), \quad t > 0 \\ \\ W(0) \ = W_0 = (\phi_0, \phi(\cdot)) \end{cases} \tag{4.8}$$

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0 \\ \\ W(0) \ = W_0 = (\phi_0, \phi(\cdot)) \end{cases} \tag{4.9}$$

where Λ is the infinitesimal generator of the semigroup $\{T(t)\}_{t \geq 0}$ and $\Phi(t) = (f^e(t), 0)$.

The steps we have to arrive here allow us to conclude the proof of the main result of this work: The Variation Constant Formula for Functional Partial Parabolic Equations. This result is presented in the final Theorem of the this work.

Theorem 4.3 *The abstract Cauchy problem in the Hilbert space* $\mathbb{M}_2$

$$\begin{cases} \dfrac{dW(t)}{dt} = \Lambda W(t) + \Phi(t), \quad t > 0 \\ W(0) \ = W_0 \end{cases}$$

where Λ *is the infinitesimal generator of the semigroup* $\{T(t)\}_{t\geq 0}$ *and* $\Phi(t) = (f^e(t), 0)$ *is a function taking values in* $\mathbb{M}_2$*, admits one and only one mild solution given by:*

$$W(t) = T(t)W_0 + \int_0^t T(t-s)\Phi(s)ds \tag{4.10}$$

Corollary 4.4 *If* $z(t)$ *is a solution of (2.4), then the function* $W(t) := (z(t), z_t)$ *is solution of the equation (4.9)*

5 Conclusion

As one can see, this work can be generalized to a broad class of functional reaction diffusion equation in a Hilbert space Z of the form:

$$\begin{cases} \dfrac{dz(t)}{dt} = \mathcal{A}z(t) + Lz_t + F(t), \quad t > 0 \\ z(0) = \phi_0 \\ z(s) = \phi(s), \ \ s \in [-\tau, 0), \end{cases} \tag{5.1}$$

where $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}), \tag{5.2}$$

where $L : L^2([-\tau, 0]; Z) \longrightarrow Z$ is linear and bounded $F : [-\tau, \infty) \longrightarrow Z$ is a suitable function. Some examples of this class are the following well known systems of partial differential equations with delay:

Example 5.1 *The equation modeling the damped flexible beam:*

$$\begin{cases} \dfrac{\partial 2z}{\partial 2t} &= -\dfrac{\partial 3z}{\partial 3x} + 2\alpha \dfrac{\partial 3z}{\partial t \partial 2x} + z(t-\tau, x) + f(t,x) \quad t \geq 0, \quad 0 \leq x \leq 1 \\ z(t,1) &= z(t,0) = \dfrac{\partial 2z}{\partial 2x}(0,t) = \dfrac{\partial 2z}{\partial 2x}(1,t) = 0, \\ z(0,x) &= \phi_0(x), \quad \dfrac{\partial z}{\partial t}(0,x) = \psi_0(x), \quad 0 \leq x \leq 1 \\ z(s,x) &= \phi(s,x), \quad \dfrac{\partial z}{\partial t}(s,x) = \psi(s,x), \quad s \in [-\tau, 0), \quad 0 \leq x \leq 1 \end{cases} \tag{5.3}$$

where $\alpha > 0$*,* $f : I\!R \times [0,1] \to I\!R$ *is a smooth function,* $\phi_0, \psi_0 \in L^2[0,1]$ *and* $\phi, \psi \in L^2([-\tau, 0]; L^2[0,1])$.

Example 5.2 *The strongly damped wave equation with Dirichlet boundary conditions*

$$\begin{cases} \dfrac{\partial 2w}{\partial 2t} + \eta(-\Delta)^{1/2}\dfrac{\partial w}{\partial t} + \gamma(-\Delta)w = Lw_t + f(t,x), \qquad t \geq 0, \quad x \in \Omega, \\ w(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega. \\ w(0,x) = \phi_0(x), \quad \dfrac{\partial z}{\partial t}(0,x) = \psi_0(x), \quad x \in \Omega, \\ w(s,x) = \phi(s,x), \quad \dfrac{\partial z}{\partial t}(s,x) = \psi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega, \end{cases} \tag{5.4}$$

where Ω *is a sufficiently smooth bounded domain in* $I\!R^N$*,* $f : I\!R \times \Omega \to I\!R$ *is a smooth function,* $\phi_0, \psi_0 \in L^2(\Omega)$ *and* $\phi, \psi \in L^2([-\tau, 0]; L^2(\Omega))$ *and* $\tau \geq 0$ *is the maximum delay, which is supposed to be finite. We assume that the operators* $L : L^2([-\tau, 0]; Z) \longrightarrow Z$ *is linear and bounded and* $Z = L^2(\Omega)$.

Example 5.3 *The thermoelastic plate equation with Dirichlet boundary conditions*

$$\begin{cases} \dfrac{\partial 2w}{\partial 2t} + \Delta^2 w + \alpha\Delta\theta = L_1 w_t + f_1(t,x) \quad t \geq 0, \quad x \in \Omega, \\ \dfrac{\partial \theta}{\partial t} - \beta\Delta\theta - \alpha\Delta\dfrac{\partial w}{\partial t} = L_2\theta_t + f_2(t,x) \quad t \geq 0, \quad x \in \Omega, \\ \theta = w = \Delta w = 0, \quad t \geq 0, \quad x \in \partial\Omega. \\ w(0,x) = \phi_0(x), \quad \dfrac{\partial w}{\partial t}(0,x) = \psi_0(x), \quad \theta(0,x) = \xi_0(x) \quad x \in \Omega, \\ w(s,x) = \phi(s,x), \quad \dfrac{\partial w}{\partial t}(s,x) = \psi(s,x), \quad \theta(0,x) = \xi(s,x), \quad s \in [-\tau, 0), \quad x \in \Omega, \end{cases} \tag{5.5}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^N$, $f_1, f_2 : I\!R \times \Omega \to I\!R$ are smooth functions, $\phi_0, \psi_0, \xi_0 \in L^2(\Omega)$ and $\phi, \psi, \xi \in L^2([-\tau, 0]; L^2(\Omega))$ and $\tau \geq 0$ is the maximum delay, which is supposed to be finite. We assume that the operators $L_1, L_2 : L^2([-\tau, 0]; Z) \longrightarrow Z$ are linear and bounded and $Z = L^2(\Omega)$.

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