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Exact Controllability for Semilinear Difference Equation and Application.

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Abstract

In this paper we study the exact controllability of the following semilinear difference equation

$$z(n+1) = A(n)z(n) + B(n)u(n) + f(z(n), u(n)), n \in I\!N$$

$z(n) \in Z$, $u(n) \in U$, where Z, U are Hilbert spaces, $A \in l^{\infty}(I\!N, L(Z))$, $B \in l^{\infty}(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$ and the nonlinear term $f : Z \times U \longrightarrow Z$ satisfies:

$$\|f(z_2, u_2) - f(z_1, u_1)\| \le L\{\|z_2 - z_1\| + \|u_2 - u_1\|\}.$$

We prove the following statement: If the linear equation is exactly controllable and $L << 1$, then the nonlinear equation is also exactly controllable. That it to say, the controllability of the linear equation is preserved under nonlinear perturbation $f(z,u)$. Finally, we apply this result to a discrete version of the semilinear wave equation.

Resumen

En este articulo estudiamos la controlabilidad exacta de la siguiente ecuación en diferencias semilineal

$$z(n+1) = A(n)z(n) + B(n)u(n) + f(z(n), u(n)), \quad n \in I\!N^*$$

$z(n) \in Z$, $u(n) \in U$, donde Z, U son espacios de Hilbert, $I\!N^* = I\!N \cup \{0\}$, $A \in l^{\infty}(I\!N, L(Z))$, $B \in l^{\infty}(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$ y el termino no lineal
$f : Z \times U \longrightarrow Z$ satisface:

$$\|f(z_2, u_2) - f(z_1, u_1)\| \le L\{\|z_2 - z_1\| + \|u_2 - u_1\|\}.$$

Probamos la siguiente afirmación: Si la ecuación lineal es exactamente controlable y $L << 1$, entonces ecuación no lineal es también exactamente controlable. Es decir, la controlabilidad de la ecuación lineal se preserva bajo la perturbación no lineal $f(z,u)$. Finalmente, aplicamos este resultado a una versión discreta de la ecuación del calor semilineal.

key words. difference equations, exact controllability, wave equation.

AMS(MOS) subject classifications. primary: 93B05; secondary: 93C25.

1 Introduction

In this paper we study the exact controllability of the following semilinear difference equation

$$z(n+1) = A(n)z(n) + B(n)u(n) + f(z(n), u(n)), \quad z(n) \in Z, u(n) \in U, n \in I\!N,$$

where Z, U are Hilbert spaces, $A \in l^{\infty}(I\!N, L(Z))$, $B \in l^{\infty}(I\!N, L(U,Z))$, $u \in l^2(I\!N, U)$, $L(U,Z)$ denotes the space of all bounded linear operators $L : U \longrightarrow Z$ and $L(Z,Z) = L(Z)$. The nonlinear term $f : Z \times U \longrightarrow Z$ is a continuous Lipschitzian function such that:

$$\|f(z_2, u_2) - f(z_1, u_1)\| \leq L\{\|z_2 - z_1\| + \|u_2 - u_1\|\}.$$

We prove the following statement: If the linear equation is exactly controllable and $L << 1$, then the nonlinear equation is also exactly controllable. That it to say, the controllability of the linear equation is preserved under nonlinear perturbation $f(z,u)$. Finally, we apply this result to a discrete version of the following semilinear wave equation:

$$\begin{cases} w_{tt} - w_{xx} = u(t,x) + f(w, w_t, u(t,x)), \quad 0 < x < 1 \\ w(t,0) = w(t,1) = 0, \quad t \in I\!R \end{cases} \tag{1.1}$$

where the distributed control $u \in L^2(0, t_1; L^2(0,1))$ and the nonlinear term $f(w,v,u)$ is a continuous function $f : I\!R^3 \to I\!R$ such that

$$|f(w_2, v_2, u_2) - f(w_1, v_1, u_1)| \leq L\left\{|w_2 - w_1| + |v_2 - v_1| + |u_2 - u_1|\right\}. \tag{1.2}$$

Finally, our technique can be used in a more general problem since it is based on the following Theorem use to characterize center manifolds in dynamical system theory.

Theorem 1.1 *Let Z be a Banach space and $K : Z \to Z$ a Lipschitz function with a Lipschitz constant $k < 1$ and consider $G(z) = z + Kz$. Then G is an homemorphis whose inverse is a Lipschitz function with a Lipschitz constant* $(1+k)^{-1}$.

2 Exact Controllability of the Linear Equation

In this section we shall study the exact controllability of the linear difference equation

$$z(n+1) = A(n)z(n) + B(n)u(n), n \in I\!N, z(0) = z_0 \tag{2.3}$$

To this end, we shall give a variation constant formula for the solution of (2.3), the definition of exact controllability and prove some characterization of exact controllability needed to prove our main theorem.

Consider the set $\Delta = \{(m,n) \in I\!N \times I\!N : m \geq n\}$ and let $\Phi = \{\Phi(m,n)\}_{(m,n)\in\Delta}$ be the evolution operator associated to A, i.e., $\Phi(m,n) = A(m-1)\cdots A(n)$ and $\Phi(m,n) = I$, for $m = n$.

Then the solution of (2.3) is given by the discrete variation constant formula:

$$z(n) = \Phi(n,0)z(0) + \sum_{k=1}^{n} \Phi(n,k)B(k-1)u(k-1), n \in I\!N \tag{2.4}$$

Definition 2.1 *(**Exact Controllability**) The system (2.3) is said to be exactly controlable if there is $n_0 \in I\!N$ such that for every z_0, $z_1 \in X$ there exists $u \in l^\infty(I\!N, U)$ such that $z(0) = z_0$ and $z(n_0) = z_1$.*

Definition 2.2 *For the system (2.3) we define the following concepts:*

a) The controllability map (for $n \in I\!N$) is define as follows $B^n : l^2(I\!N, U) \longrightarrow Z$ by

$$B^n u = \sum_{k=1}^{n} \Phi(n,k)B(k-1)u(k-1) \tag{2.5}$$

b) The grammian map (for $n \in I\!N$) is define by $L_{B^n} = B^n B^{n*}$

The following theorem is a discrete version of theorem 4.1.7 from [2].

Theorem 2.1 *The equation (2.3) is exactly controllable for some $n_0 \in I\!N$ if, and only if, one of the following statements holds:*

a) $Ran(B^{n_0}) = Z$

b) There exists $\gamma > 0$ such that

$$\langle L_{B^{n_0}} z, z\rangle \geq \gamma \|z\|_Z^2$$

c) There exists $\gamma > 0$ such that

$$\|B^{n_0 *} z\|_{l^2(I\!N,U)} \geq \gamma \|z\|_Z, z \in Z$$

The following proposition is easy to prove.

Proposition 2.1 *The adjoint $B^{n_0 *}$ of the operator B^{n_0} is given by*
$B^{n_0 *} : Z \longrightarrow l^2(I\!N, U)$

$$(B^{n_0 *} z)(k-1) = \begin{cases} B^*(k-1)\Phi^*(n_0,k)z, & k \leq n_0 \\ 0 \qquad , & k > n_0 \end{cases} \tag{2.6}$$

and

$$L_{B^{n_0}} z = \sum_{k=1}^{n_0} \Phi(n_0, k) B(k-1) B^*(k-1) \Phi^*(n_0, k) z, z \in Z \tag{2.7}$$

Lemma 2.1 *The equation (2.3) is exactly controllable for $n_0 \in I\!N$ if, and only if, $L_{B^{n_0}}$ is invertible. Moreover, in this case $S = B^{n_0 *} L^{-1}_{B_{n_0}}$ is right inverse of B^{n_0} and the control $u \in l^2(I\!N, U)$ steering an initial state z_0 to a final state z_1 is given by:*

$$u = B^{n_0 *} L^{-1}_{B^{n_0}} (z_1 - \Phi(n_0, 0) z_0). \tag{2.8}$$

Proof Suppose the system (2.3) is exactly controlable. Then from theorem (2.1) part c) there is $\gamma > 0$ such that $\|B^{n_0 *} z\| \geq \gamma \|z\|$, for all $z \in Z$, i.e.,

$$\|B^{n_0 *} z\|^2 \geq \gamma^2 \|z\|^2, z \in Z$$

i.e.,

$$\langle B^{n_0} B^{n_0 *} z, z \rangle \geq \gamma^2 \|z\|^2, z \in Z$$

i.e.,

$$\langle L_{B^{n_0}} z, z \rangle \geq \gamma^2 \|z\|^2, z \in Z \tag{2.9}$$

This implies that $L_{B^{n_0}}$ is one to one. Now, we shall prove that $L_{B^{n_0}}$ is surjective. That is to say

$$R(L_{B^{n_0}}) = Range(L_{B^{n_0}}) = Z.$$

For the purpose of contradiction, let us assume that $R(L_{B^{n_0}})$ is strictly contained in Z. On the other hand, using Cauchy Schwarz's inequality and (2.9) we get

$$\|L_{B^{n_0}} z\|_{l^2} \geq \gamma^2 \|z\|^2, z \in Z$$

which implies that $R(L_{B^{n_0}})$ is closed. Then, from Hahn Banach's Theorem there exist $z_0 \neq 0$ such that

$$\langle L_{B^{n_0}} z, z_0 \rangle = 0, \forall z \in Z$$

In particular, putting $z = z_0$ we get from (2.9) that

$$0 = \langle L_{B^{n_0}} z_0, z_0 \rangle \geq \gamma^2 \|z_0\|^2$$

Then $z_0 = 0$, which is a contradiction. Hence, $L_{B^{n_0}}$ is a bijection and from the Open Mapping Theorem, $L_{B^{n_0}}^{-1}$ is a bounded linear operator.

Now suppose $L_{B^{n_0}}$ is invertible. Then, from Theorem (2.1) it is enough to prove that $R(B^{n_0}) = Z$. For $z \in Z$ we define the control $u_z \in l^2(I\!N, U)$ as follows

$$u_z = Sz = B^{n_0*} L_{B^{n_0}}^{-1} z.$$

Then $B^{n_0} u_z = z$. The rest of the proof follows from here. ♠

3 Exact Controllability of the Nonlinear Equation.

Through this section we shall assume that the linear system is exactly controllable for some $n_0 \in I\!N$. Now, we shall give the definition of controllability in terms of the non-linear systems In this section we shall study the exact controllability of the nonlinear difference equation

$$z(n+1) = A(n)z(n) + B(n)u(n) + f(z(n), u(n)), z(0) = z_0, \quad n \in I\!N \tag{3.10}$$

where the nonlinear term $f : Z \times U \longrightarrow Z$ satisfies:

$$\|f(z_2, u_2) - f(z_1, u_1)\| \leq L\{\|z_2 - z_1\| + \|u_2 - u_1\|\}.$$

To this end, we shall assume that the equation (2.3) is controllable for some n_0, i.e., $R(B^{n_0}) = Z$.

For $z_0 \in Z$ the equation (3.10) has a unique solution given by

$$z(n) = \Phi(n, 0)z(0) + \sum_{k=1}^{n} \Phi(n, k)[B(k-1)u(k-1) + f(z(k-1), u(k-1))], n \in I\!N \tag{3.11}$$

Definition 3.1 *(***Exact Controllability***) The system (3.10) is said to be exactly controlable if there is $n_0 \in I\!N$ such that for every z_0, $z_1 \in X$ there exists $u \in l^\infty(I\!N, U)$ such that $z(0) = z_0$ and $z(n_0) = z_1$.*

Consider the following non-linear operator $B_f^{n_0} : l^2(I\!N, U) \longrightarrow Z$ define by

$$\begin{aligned} B_f^{n_0} u &= \sum_{k=1}^{n_0} \Phi(n_0, k)B(k-1)u(k-1) + \sum_{k=0}^{n_0} \Phi(n_0, k) f(z(k-1), u(k-1)) \\ &= B^{n_0} u + \sum_{k=0}^{n_0} \Phi(n_0, k) f(z(k-1), u(k-1)). \end{aligned} \tag{3.12}$$

Then, following proposition is a characterization of the exact controllability of the nonlinear system (3.10).

Proposition 3.1 *The system (3.10) is exactly controllable for* n_0 *if, and only if,* $R(B_f^{n_0}) = Z$.

Lemma 3.1 *Let* u_1, $u_2 \in l^2(\mathbb{N}, U)$ *and* z_1, z_2 *the corresponding solutions of (3.10). Then, the following estimate holds:*

$$\|z_1(j) - z_2(j)\|_X \leq M[\|B^{n_0}\| + L]\sqrt{n_0}e^{MLn_0}\|u_1 - u_2\|_{l^2(\mathbb{N},U)} \tag{3.13}$$

where $j \leq n_0$ *and* $M = \sup_{1 \leq j,k \leq n_0} \{\|\Phi(j,k)\|\}$.

Proof Let z_1, z_2 be solutions of (3.10) corresponding to u_1, u_2 respectively. Then

$$\begin{aligned} \|z_1(j) - z_2(j)\| &= \sum_{k=0}^{j} \|\Phi(j,k)\|\|B^{n_0}\|\|u_1(k-1) - u_2(k-1)\| \\ &+ \sum_{k=0}^{j} \|\Phi(j,k)\|\|f(z_1(k-1), u_1(k-1)) - f(z_2(k-1), u_2(k-1))\| \\ &\leq M[\|B^{n_0}\| + L]\sum_{k=1}^{j-1} \|u_1(k) - u_2(k)\| + ML\sum_{k=1}^{j-1} \|z_1(k) - z_2(k)\| \\ &\leq M[\|B^{n_0}\| + L]\sqrt{n_0}\|u_1 - u_2\| + ML\sum_{k=1}^{j-1} \|z_1(k) - z_2(k)\|. \end{aligned} \tag{3.14}$$

Using Discrete Gronwall inequality [see Laksmikanham and Trigiante Cor. 1.6.2.] we obtain

$$\|z_1(j) - z_2(j)\|_X \leq M[\|B^{n_0}\| + L]\sqrt{n_0}e^{MLn_0}\|u_1 - u_2\|_{l^2(\mathbb{N},U)}, j \leq n_0.$$

♠

Now, we are ready to formulate and prove the main result of this work.

Theorem 3.1 *If the following estimate holds*

$$L_H = ML(\Gamma + 1)\|B^{n_0 *}\|\|L_{B^{n_0}}^{-1}\|n_0 < 1, \tag{3.15}$$

where $\Gamma = M[\|B^{n_0}\| + L]\sqrt{n_0}e^{MLn_0}$, *then the nonlinear system (3.10) is exactly controllable in* n_0.

Proof want to prove that

$$B_f^{n_0} l^2(\mathbb{N}; U) = \text{Range}(B_f^{n_0}) = Z.$$

But, from the exact controllability of the linear system (2.3) we know due lemma (2.1) that the operator $S = B^{n_0 *}L_{B_{n_0}}^{-1}$ is a right inverse of B^{n_0}. Then, it is enough to prove that the operator

$\widetilde{B}_f^{n_0} = B_f^{n_0} \circ S$ is surjective. From the equation (3.12) we obtain the following expression for this oparator

$$\widetilde{B}_f^{n_0}\xi = \xi + \sum_{k=1}^{n_0} \Phi(n_0,k) f(z(k-1), S(\xi)(k-1)). \tag{3.16}$$

Now, if we define the operator $K : Z \longrightarrow Z$ by

$$K\xi = \sum_{k=1}^{n_0} \Phi(n_0,k) f(z(k-1), S(\xi)(k-1)), \tag{3.17}$$

then the equation (3.16) takes the nice form

$$\widetilde{B}_f^{n_0} = I + K. \tag{3.18}$$

The function H is globally Lipschitz. In fact, let z_1, z_2 be solutions of (3.10) corresponding to the controls $S\xi_1$, $S\xi_2$ respectively. Then

$$\begin{aligned}
\|K\xi_1 - K\xi_2\| &\leq \sum_{k=1}^{n_0} \|\Phi(n_0,k)\| \|f(z_1(k-1), S(\xi_1)(k-1)) - f(z_2(k-1), S(\xi_2)(k-1))\| \\
&\leq \sum_{k=1}^{n_0} ML\{\|z_1(k-1) - z_2(k-1)\| + \|(S\xi_1)(k-1) - (S\xi_2)(k-1)\|\} \\
&\leq \sum_{k=1}^{n_0} ML(\Gamma+1)\|(S\xi_1)(k-1) - (S\xi_2)(k-1)\| \\
&\leq ML(\Gamma+1)\|B^{n_0*}\|\|L_{B^{n_0}}^{-1}\| \sum_{k=1}^{n_0} \|\xi_1 - \xi_2\| \\
&= ML(\Gamma+1)\|B^{n_0*}\|\|L_{B^{n_0}}^{-1}\| n_0 \|\xi_1 - \xi_2\|.
\end{aligned}$$

Therefore, K is Lipschitzian with lipschitz constant $L_K = ML(\Gamma+1)\|B^{n_0*}\|\|L_{B^{n_0}}^{-1}\|n_0$, and the assumption (3.15) implies that $L_K < 1$. Hence, from Theorem 1.1 we get that $\widetilde{B}_{n_0}^f = I + K$ is an homeomorphism and consequently the operator $B_{n_0}^f$ is surjective, that is to say

$$B_f^{n_0} l^2(I\!N; U) = \text{Range}(B_f^{n_0}) = Z.$$

♠

Corollary 3.1 *The control steering an initial state z_0 to a final state z_1 is given by*

$$u = B^{n_0*} L_{B^{n_0}}^{-1} (I+K)^{-1} (z_1 - \Phi(n_0,0) z_0).$$

Corollary 3.2 *The The operator $\Gamma : Z \to Z$ define by $\Gamma = S \circ (I + K_{\alpha_0})^{-1}$ is a right inverse of the non linear operator $B_f^{n_0}$. That is to say, $B_f^{n_0} \circ \Gamma = I$.*

4 Applications

As an application of the main results of this paper we shall consider a discrete version of the controlled nonlinear wave equation in 1 dimension.

$$\begin{cases} y_{tt} = y_{xx} + u(t,x) + g(y, u(t,x)) \\ y(t,0) = y(t,1) = 0 \\ y(0,x) = y_0, y_t(0,x) = y_1(x) \end{cases} \tag{4.19}$$

The system (4.19) can be written as an abstract second order equation in the Hilbert space $X = L^2[0,1]$ as follows:

$$\begin{cases} y'' = -Ay + u(t) + g(y, u(t)) \\ y(0) = y_0, y'(0) = y_1 \end{cases} \tag{4.20}$$

where the operator A is given by $A\phi = -\phi_{xx}$ with domain $D(A) = H^2 \cap H_0^1$, and has the following spectral decomposition.

For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \langle x, \phi_n \rangle \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x$$

where $\lambda_n = n^2\pi^2$, $\phi_n(x) = \sin n\pi x$, $\langle \cdot, \cdot \rangle$ is the inner product in X and $E_n x = \langle x, \phi_n \rangle \phi_n$.

So, $\{E_n\}$ is a family of complete orthogonal projections in X and $x = \sum_{n=1}^{\infty} E_n x$, $x \in X$.

Using the change of variables $y' = v$. the second order equation (4.20) can be written as a first order system of ordinary differential equations in the Hilbert space $Z = X^{1/2} \times X$ as

$$\begin{cases} z' = \mathcal{A}z + Bu(t) + f(z, u(t)), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.21}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \;\; B = \begin{bmatrix} 0 \\ I \end{bmatrix}, \;\; \mathcal{A} = \begin{bmatrix} 0 & I \\ -A & 0 \end{bmatrix}, \tag{4.22}$$

is an unbounded linear operator with domain $D() = D(A) \times X$ and

$$f(z,u) = \begin{bmatrix} 0 \\ g(y,u) \end{bmatrix}, u \in L^2(0, \tau, X) = U.$$

The proof of the following theorem follows from Theorem 3.1 (see, [4]) by putting $c = 0$ and $d = 1$.

Theorem 4.1 *The operator given by (4.22), is the infinitesimal generator of a strongly continuous semigroup* $\{T(t)\}_{t\in I\!R}$ *given by*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, z \in Z, t \geq 0 \tag{4.23}$$

where $\{P_n\}_{n\geq 1}$ *is a complete family of orthogonal projections in the Hilbert space* Z *given by*

$$P_n = diag[E_n, E_n], n \geq 1 \tag{4.24}$$

and

$$A_n = B_n P_n, \quad B_n = \begin{bmatrix} 0 & 1 \\ -\lambda_n & 0 \end{bmatrix}, n \geq 1. \tag{4.25}$$

Now, the discretization of (4.21) on flow is given by

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n) + f(z(n), u(n)), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.26}$$

where

$$f : Z \times U \longrightarrow Z, \quad u \in l^2(I\!N, U), \quad B : U \longrightarrow Z, \quad Bu = \begin{bmatrix} 0 \\ u \end{bmatrix}.$$

In this case, the evolution operator associated to $T(\cdot)$, is given by $\Phi(m,n) = T(m-1)T(m-2)\ldots T(n)$, $n < m$ and $\Phi(m,m) = I$.

Note that $\phi(m,n) = T(\Theta(m,n))$ where $\Theta(m,n) = \dfrac{m^2 - n^2 + n - m}{2} \in I\!N$, $m > n$.

We considere the linear difference equation

$$\begin{cases} z(n+1) = T(n)z(n) + B(n)u(n), z \in Z \\ z(0) = z_0 \end{cases} \tag{4.27}$$

We want to show that (4.27) is exactly controllable. In this case, we have

$$B^n : l^2(I\!N, U) \longrightarrow Z, \quad B^n u = \sum_{k=1}^{n} T(\Theta(n,k))Bu(k-1)$$

and

$$L_{B^n} : Z \longrightarrow Z, \quad L_{B^n} = B^n B^{n*}$$

Now we considere the following family of finite dimensional systems:

$$\begin{cases} P_j z(n+1) = e^{A_j n} P_j z(n) + P_j B(n)u(n), z \in Z \\ P_j z(0) = P_j z_0 \end{cases} \tag{4.28}$$

or

$$\begin{cases} y(n+1) = e^{A_j n} y(n) + B_j u(n), y \in R(P_j) \\ y(0) = y_0 \in R(P_j) \end{cases} \tag{4.29}$$

where $R(P_j) = Range(P_j)$, $B_j = P_j B$.

For (4.29) we have:

$$B_j^n u = \sum_{k=1}^{n} e^{A_j \Theta(n,k)} B_j u(k-1) = P_j B^n u$$

and $L_{B_j^n} = B_j^n B_j^{n*}$.

The verification that $P_n BB^* = BB^* P_n$ and $T^*(t) = T(-t)$ is trivial.

Then

$$\begin{aligned} L_{B^n} z &= \sum_{k=1}^{n} T(\Theta(n,k)) BB^* T^*(\Theta(n,k)) z \\ &= \sum_{k=1}^{n} \sum_{j=1}^{\infty} e^{A_j \Theta(n,k)} P_j BB^* \sum_{i=1}^{\infty} e^{-A_j \Theta(n,k)} P_j z \\ &= \sum_{j=1}^{\infty} \sum_{k=1}^{n} e^{A_j \Theta(n,k)} BB^* e^{-A_j \Theta(n,k)} P_j z \\ &= \sum_{j=1}^{\infty} L_{B_j^n} P_j z. \end{aligned}$$

Hence, $L_{B^n} = \sum_{j=1}^{\infty} L_{B_j^n}$.

In consequence, to show that (4.27) is exactly controllable we shall prove that (4.29) is exactly controllable, i.e., we shall prove that $L_{B_j^n}$ satisface Theorem 2.1 (b), i.e., the exist $\gamma > 0$ such that $\langle L_{B_j^n} y, y \rangle \geq \gamma \|y\|^2$. . . . . .

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