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BY

T.V. PANCHAPAGESAN

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ABSTRACT

Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. For $f : T \to \mathbb{K}$ the concept of $\mathbf{m}$-measurability is defined in such a way as to coincide with that in Definition 2.8 when X is a normed space. For $\mathbf{m}$-measurable functions on T, the concepts of (KL) $\mathbf{m}$-integrability and (BDS) $\mathbf{m}$-inegrability are generalized and shown to coincide when X is quasicomplete. The results and Definitions in Part II are generalized here when X is quasicomplte. Moreover, when X is sequentially complete and f is $\sigma(\mathcal{P})$-measurable, the above mentioned results are further generalized. A section is devoted to the study of the separability of $\mathcal{L}_P$-spcaes for $1 \le p < \infty$ when X is quasicomplete (resp. sequentially complete).

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In the sequel, Definitions, Propositions, Theorems, Remarks, etc., of Parts I ([P3]) and II ([P4]) such as Definition 2.3, Propositions 2.10, Theorem 6.3, etc., will be referred to without any explcit reference to Part I or II. Moreover, the enumeration of sections will be continued from Part II. We adopt the same notation and terminology in Parts I and II.

10. ADDITIONAL NOTATION, TERMINOLOGY AND BASIC RESULTS

In this part (Part III), X denotes an lcHs with topology τ and $\tilde{X}$ denotes its completion with the lcHs topology $\tilde{\tau}$. Γ is the family of continuous seminorms on X so that Γ generates the topology τ on X. X^* denotes the tolpological dual of X.

For $q \in \Gamma$, let $\Pi_q : X \to X_q = X/q^{-1}(0)$ be the canonical quotient map. If we define $|x + q^{-1}(0)|_q = q(x)$, $x \in X$, then $|\cdot|_q$ ia a well defined norm on X_q and the Banach space completion of X_q with respect to $|\cdot|_q$ is denoted by $\widetilde{X_q}$.

In the sequel, $\mathbf{m} : \mathcal{P} \to X$, $\mathcal{P}$ a δ-ring of subsets of T, is σ-additive. For $q \in \Gamma$, let $\mathbf{m}_q : \mathcal{P} \to X_q \subset \widetilde{X_q}$ be defined by $\mathbf{m}_q(A) = \Pi_q \circ \mathbf{m}(A)$, $A \in \mathcal{P}$. Then clearly $\mathbf{m}_q$ is σ-additive on $\mathcal{P}$ and $||\mathbf{m}_q||$ as well as $||\mathbf{m}||_q$ denote the semivariation of $\mathbf{m}_q$ on $\sigma(\mathcal{P})$; $v(\mathbf{m}_q)$ is the variation of $\mathbf{m}_q$ on $\sigma(\mathcal{P})$. Note that for defining $||\mathbf{m}||_q$ and $v(\mathbf{m}_q)$ it suffices that $\mathbf{m}$ is just additive on $\mathcal{P}$.

Definition 10.1. Let $q \in \Gamma$ be fixed. $\widetilde{\sigma(\mathcal{P})_q}$ is the generalized Lebesgue-completion of $\sigma(\mathcal{P})$ with respect to $||\mathbf{m}||_q$ so that $\widetilde{\sigma(\mathcal{P})_q} = \{A = B_q \cup N_q : B_q \in \sigma(\mathcal{P}), N_q \subset M_q \in \sigma(\mathcal{P}) \text{ with } ||\mathbf{m}||_q(M_q) = 0\}$. Then $\sigma(\mathcal{P}) \subset \widetilde{\sigma(\mathcal{P})_q}$ and $\widetilde{\sigma(\mathcal{P})_q}$ is a σ-ring. We define $\overline{||\mathbf{m}||_q}(A) = ||\mathbf{m}||_q(B_q)$ if A, B_q are as in the above and clearly $\overline{||\mathbf{m}||_q}$ is well defined, extends $||\mathbf{m}||_q$ to $\widetilde{\sigma(\mathcal{P})_q}$ and is a σ-subadditive submeasure on $\widetilde{\sigma(\mathcal{P})_q}$. We shall use the symbol $||\mathbf{m}||_q$ to denote $\overline{||\mathbf{m}||_q}$ also. Sets $N \in \widetilde{\sigma(\mathcal{P})_q}$ with $||\mathbf{m}||_q(N) = 0$

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are called $\mathbf{m}_q$-null. (See Section 2).

By Proposition 2.2 we have the following

Proposition 10.2. Let $q \in \Gamma$. Then $||\mathbf{m}||_q$ is a continuous submeasure on $\mathcal{P}$. If $\mathcal{P}$ is a σ-ring $\mathcal{S}$, then $||\mathbf{m}||_q(T) < \infty$.

Definition 10.3. Let $\widetilde{\sigma(\mathcal{P})} = \{A = B \cup N : B \in \sigma(\mathcal{P}), N \subset M \in \sigma(\mathcal{P}) \text{ with } ||\mathbf{m}||_q(M) = 0 \text{ for all } q \in \Gamma\}$. Then $\widetilde{\sigma(\mathcal{P})}$ is called the generalized Lebesgue-completion of $\sigma(\mathcal{P})$ with respect to $\mathbf{m}$. A set N in T is said to be $\mathbf{m}$-null if $||\mathbf{m}||_q(N) = 0$ for all $q \in \Gamma$.

Clearly, $\sigma(\mathcal{P}) \subset \widetilde{\sigma(\mathcal{P})} \subset \widetilde{\sigma(\mathcal{P})_q}$ for $q \in \Gamma$ and $\widetilde{\sigma(\mathcal{P})}$ is a σ-ring.

Definition 10.4. Let $q \in \Gamma$. A property P is said to be true $\mathbf{m}$-a.e. (resp. $\mathbf{m}_q$-a.e.) in T if there exists an $\mathbf{m}$-null (resp. $\mathbf{m}_q$-null) set $N \in \sigma(\mathcal{P})$ such that P holds for all $t \in T \backslash N$.

Remark 10.5. All the definitions and results in the sequel hold good if Γ is replaced by any subfamily of seminorms which generates the toplogy τ on X.

Definition 10.6. A function $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ is said to be $\mathbf{m}$-measurable if, for each $q \in \Gamma$, f is $\mathbf{m}_q$-measurable (i.e., $\widetilde{\sigma(\mathcal{P})_q}$-measurable). In that case, there exists $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$ such that $f\chi_{T\backslash N_q}$ is $\sigma(\mathcal{P})$-measurable. (See Proposition 2.10.)

Definition 10.7. An $\mathbf{m}$-measurable function $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ is said to be $\mathbf{m}$-essentially bounded in T if there exists an $\mathbf{m}$-null set $N \in \sigma(\mathcal{P})$ such that f is bounded in $T \backslash N$. Then we define $\text{ess sup}_{t\in T}|f(t)| = \inf\{\alpha > 0 : |f(t)| \le \alpha \text{ for } t \in T\backslash N_\alpha, N_\alpha \in \sigma(\mathcal{P}),\ N_\alpha\ \mathbf{m}\text{-null}\}$.

Since $||\mathbf{m}||_q, q \in \Gamma$, are σ-subadditive on $\sigma(\mathcal{P})$, it follows that there exists an $\mathbf{m}$-null set $N \in \sigma(\mathcal{P})$ such that

$$\text{ess sup}_{t\in T}|f(t)| = \sup_{t\in T\backslash N} |f(t)|. \qquad (10.7.1)$$

Notation 10.8. $\mathcal{I}_{\widetilde{\sigma(\mathcal{P})_q}}$ denotes the family of all $\widetilde{\sigma(\mathcal{P})_q}$-simple functions on T for $q \in \Gamma$. However, as in Notation 2.9, $\mathcal{I}_s$ is the family of all $\mathcal{P}$-simple funtions.

The proof of Proposition 2.10 can be adapted to prove the following

Proposition 10.9. Let $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$. For $q \in \Gamma$, f is $\mathbf{m}_q$-measurable if and only if there exists a sequence $(s_n^{(q)})_{n=1}^\infty \subset \mathcal{I}_{\widetilde{\sigma(\mathcal{P})_q}}$ (resp. $(s_n^{(q)})_1^\infty \subset \mathcal{I}_s$) such that $s_n^{(q)} \to f$ and $|s_n^{(q)}| \nearrow |f|$ pointwise in T (resp. $s_n \to f$ and $|s_n| \nearrow |f|$ pointwise in $T\backslash N_q$ where $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$-so that $f\chi_{T\backslash N_\alpha}$ is $\sigma(\mathcal{P})$-measurable.). f is $\sigma(\mathcal{P})$-measurable if and only if there exists a sequence $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $s_n \to f$ and $|s_n| \nearrow |f|$ pointwise in T.

Notation 10.10. $\mathcal{E}$ denotes the family of equicontinuous subsets of X^* and for each $E \in \mathcal{E}$, $q_E(x) = \sup_{x^*\in E} |x^*(x)|$, $x \in X$. $\Gamma_{\mathcal{E}} = \{q_E : E \in \mathcal{E}\}$.

By Proposition 7, §4, Ch. 3 of [Ho], we have the following

Proposition 10.11. $\Gamma_{\mathcal{E}} \subset \Gamma$ and $\Gamma_{\mathcal{E}}$ generates the topology τ on X.

Proposition 10.12. Let $\eta : \mathcal{P} \to X$ be additive and let $E \in \mathcal{E}$. Then:

(i) For $x^* \in E$, $\Psi_{x^*} : X_{q_E} \to \boldsymbol{K}$ given by $\Psi_{x^*}(x + q_E^{-1}(0)) = x^*(x)$, $x \in X$, is well defined, linear and continuous. Thus $\Psi_{x^*}(\Pi_{q_E}(x)) = x^*(x)$ for $x^* \in E$.

(ii) $\{\Psi_{x^*} : x^* \in E\}$ is a norm determining subset of the closed unit ball of $(X_{q_E})^*$.

(iii) $||\Pi_{q_E} \circ \eta||(A) = \sup_{x^* \in E} v(x^*\eta)(A)$, $A \in \sigma(\mathcal{P})$.

(iv) If $E = \{x^*\}$, $x^* \in X^*$, then $||\Pi_{q_E} \circ \eta||(A) = v(x^*\eta)(A) = ||x^*\eta||(A)$, $A \in \sigma(\mathcal{P})$.

Proof. (i) Clearly, Ψ_{x^*} is well defined and linear for $x^* \in E$. Moreover, $|\Psi_{x^*}(x + q_E^{-1}(0))| = |x^*(x)| \le q_E(x)$, $x \in X$ and hence $\Psi_{x^*} \in (X_{q_E})^*$ with
$|\Psi_{x^*}| \le 1$. (10.12.1)

(ii) $|x + q_E^{-1}(0)|_{q_E} = q_E(x) = \sup_{x^* \in E} |x^*(x)| = \sup_{x^* \in E} |\Psi_{x^*}(x + q_E^{-1}(0))|$ and hence by (10.12.1), (ii) holds.

(iii) Note that the proof of Proposition I.1.11 of [DU] holds for any additive set function γ on $\mathcal{P}$ with values in a normed space Y and for any norm determining subset of the closed unit ball of Y^*. Moreover, by replacing π on p.5 of [DU] by $\pi = \{(A_i)_1^r \subset \mathcal{P},\ A_i \cap A_j = \emptyset,\ \bigcup_1^r A_i \subset A\}$ for $A \in \sigma(\mathcal{P})$, continuing with the proof of I.1.11 and using (i) and (ii) we have

$$||\Pi_{q_E} \circ \eta||(A) = \sup_{x^* \in E} v(\Psi_{x^*}(\Pi_{q_E} \circ \eta))(A) = \sup_{x^* \in E} v(x^*\eta)(A),\ A \in \sigma(\mathcal{P}).$$

(iv) This is immediate from (iii) and the fact that $||\mu||(A) = v(\mu)(A)$, $A \in \sigma(\mathcal{P})$ for a scalar valued additive set functiuon μ on $\mathcal{P}$.

Notation 10.13. For $q \in \Gamma$, U_q denotes the set $\{x \in X : q(x) \le 1\}$ and U_q^o is the polar of U_q.

Proposition 10.14. For $q \in \Gamma$ the following hold:

(i) $q(x) = \sup_{x^* \in U_q^o} |x^*(x)|$, $x \in X$.

(ii) If $\eta : \mathcal{P} \to X$ is additive, then the following hold:

(a) For $x^* \in U_q^o$, $\Psi_{x^*} : X_q \to \boldsymbol{K}$ given by $\Psi_{x^*}(x + q^{-1}(0)) = x^*(x)$, $x \in X$, is well defined, linear and continuous. Thus $(\Psi_{x^*} \circ \Pi_q)(x) = x^*(x)$, $x^* \in U_q^o$.

(b) $\{\Psi_{x^*} : x^* \in U_q^o\}$ is a norm determining subset of the closed unit ball of $(X_q)^*$.

(c) $||\Pi_q \circ \eta||(A) = \sup_{x^* \in U_q^o} v(x^*\eta)(A)$, $A \in \sigma(\mathcal{P})$.

Proof. (i) If $q(x) = 0$, then $q(nx) = 0$ for $n \in \boldsymbol{N}$ so that $nx \in U_q$. Then, for $x^* \in U_q^o$, we have $|x^*(nx)| = n|x^*(x)| \le 1$ so that $|x^*(x)| \le \frac{1}{n}$ for all n. Hence $\sup_{x^* \in U_q^o} |x^*(x)| = 0$. Hence (i) holds.

If $q(x) \ne 0$, then $\frac{x}{q(x)} \in U_q$ and hence $\sup_{x^* \in U_q^o} |x^*(x)| \le q(x)$. We claim that $\sup_{x^* \in U_q^o} |x^*(x)| = q(x)$. Otherwise, $\alpha = \sup_{x^* \in U_q^o} |x^*(x)| < q(x)$. Then $q(\frac{x}{\alpha}) > 1$ and hence $\frac{x}{\alpha} \notin U_q$. As U_q is τ-closed, it is weakly closed. Moreover, U_q is absolutely convex. Hence by the bipolar theorem (see Theorem 1, §3, Ch. 3 of [Ho]), $U_q = U_q^{oo}$. Consequently, there exists $x^* \in U_q^o$ such that $|x^*(\frac{x}{\alpha})| > 1$ and hence $|x^*(x)| > \alpha$, a contradiction. Hence (i) holds.

(ii) By Proposition 6, §4, Ch. 3 of [Ho], $E = U_q^0 \in \mathcal{E}$ and hence, by (i), $q = q_E$. Then (ii) holds by Proposition 10.12.

Notation and Convention 10.15. The lcHs completion $(\tilde{X},\tilde{\tau})$ of the lcHs (X,τ) is unique upto a toplogical isomorphism, X is $\tilde{\tau}$ dense in $\tilde{X}$ and $\tilde{\tau}|_X = \tau$. Each $q \in \Gamma$ has a unique continuous extension $\tilde{q}$ to $\tilde{X}$ and $\tilde{q}$ is thus a $\tilde{\tau}$-continuous seminorm on $\tilde{X}$. Moreover, $\{\tilde{q} : q \in \Gamma\}$ generates the topology $\tilde{\tau}$ on $\tilde{X}$. By an abuse of notation, we dentoe by q itself to dentote its continuous extension to $\tilde{X}$ and by an abuse of language we also say that Γ generates the topology of $\tilde{X}$. See Theorem 1, §9, Ch. 2 of [Ho] and pp. 134-135 of [Ho].

Notation 10.16. Let Γ and $\Gamma_{\mathcal{E}}$ be directed by the partial order $q_1 \le q_2$ if $q_1(x) \le q_2(x)$ for $x \in X$ and $q_{E_1} \le q_{E_2}$ if $E_1 \subset E_2$, for $q_1, q_2 \in \Gamma$ and $E_1, E_2 \in \mathcal{E}$. For $q_1 \le q_2$, $A_{q_1q_2} : X_{q_2} \to X_{q_1}$ defined by $A_{q_1q_2}(x + q_2^{-1}(0)) = x + q_1^{-1}(0)$ is a continuous onto linear mapping. Similarly, $A_{q_{E_1}q_{E_2}}$ is defined if $E_1 \subset E_2$ (so that $q_{E_1} \le q_{E_2}$). Then $\Pi_{q_1} = Aq_1q_2\Pi_{q_2}$ (resp., $\Pi_{q_{E_1}} = A_{q_{E_1}q_{E_2}}\Pi_{q_{E_2}}$). Let $Y = \tilde{X}$, the completion of X. Let $\tilde{\Gamma} = \{\tilde{q} : q \in \Gamma\}$ as in 10.15. For $q \in \Gamma$, $\Pi_{\tilde{q}} : Y \to Y_{\tilde{q}}$ and $\Pi_{\tilde{q}}|_X = \Pi_q$ so that $X_q = \Pi_{\tilde{q}}(X) \subset \widetilde{X_q}$. As X is dense in Y, it follows that $X_q \subset \Pi_{\tilde{q}}(Y) = Y_{\tilde{q}} \subset \widetilde{X_q}$ and hence the completion of $\Pi_{\tilde{q}}(Y)$ is equal to $\widetilde{Y_{\tilde{q}}} = \widetilde{X_q}$ for $q \in \Gamma$. If $q_1, q_2 \in \Gamma$ with $q_1 \le q_2$, then clearly, $\tilde{q_1} \le \tilde{q_2}$ and hence by 5.4, Ch. II of [Scha], Y is topologically isomorphic to the projective limit $\varprojlim A_{\tilde{q_1}\tilde{q_2}} Y_{\tilde{q_2}}$. If $\widetilde{A_{\tilde{q_1}\tilde{q_2}}}$ is the continuous extension of $A_{\tilde{q_1}\tilde{q_2}}$ to $\widetilde{Y_{\tilde{q_2}}}\ (= \widetilde{X_{q_2}})$ with values in $\widetilde{Y_{\tilde{q_1}}}$, then again by 5.4, Ch. II of [Scha], Y is topologically isomorphic to $\varprojlim \widetilde{A_{\tilde{q_1}\tilde{q_2}}}\widetilde{Y_{\tilde{q_2}}} = \varprojlim \widetilde{A_{\tilde{q_1}\tilde{q_2}}}\widetilde{X_{q_2}}$. Thus each $y \in Y$ is written as $y = \varprojlim x_{\tilde{q}}$ with $x_{\tilde{q}} = \Pi_{\tilde{q}}(y) \in Y_{\tilde{q}}$, $q \in \Gamma$ and is also written as $y = \varprojlim x_{\tilde{q}}$, $q \in \Gamma$ with $x_{\tilde{q}} \in \widetilde{X_q}$ (without mentioning the partial order in Γ and the transformations $A_{\tilde{q_1}\tilde{q_2}}$ and $\widetilde{A_{\tilde{q_1}\tilde{q_2}}}$). Similar description holds in terms of $(q_E : E \in \mathcal{E})$. For details, see pp.53-54 of [Scha].

11. (KL) m-INTEGRABILITY (m LCHS-VALUED)

Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. The concept of $\mathbf{m}$-measurable functions and (KL) $\mathbf{m}$-integrable functions given in Sections 2 and 3 are suitabley generalized here. Theorem 11.4 below plays a key role in the subsequent theory of (KL) $\mathbf{m}$-integrability. While (i)-(iv) and (viii) of Theorem 3.5 are generalized to an arbitrary lcHs-valued σ-additive measure $\mathbf{m}$ on $\mathcal{P}$, the remaining parts of Theorem 3.5 and Theorem 3.7 and Corollaries 3.8 and 3.9 are generalized when X is quasicomplete. Finally, the above mentioned results are generalized to $\sigma(\mathcal{P})$-measurable functions in Remark 11.15 when X is sequentially complete.

Let X be an lcHs and $\mathbf{m} : \sigma(\mathcal{P}) \to X$ be σ-additive. If $x^* \in X^*$, then q_{x^*} given by $q_{x^*}(x) = |x^*(x)|$, $x \in X$ belongs to Γ and by Proposition 10.12(iv), $||\mathbf{m}_{q_{x^*}}||(A) = v(x^*\mathbf{m})(A)$ for $A \in \sigma(\mathcal{P})$. Using this observation, we give the following

Definition 11.1. Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $f : T \to \boldsymbol{K}$ or $[-\infty,\infty]$ be $\mathbf{m}$-measurble. Then, for each $x^* \in X^*$, by Definition 10.3 there exists $N_{x^*} \in \sigma(\mathcal{P})$ with $v(x^*\mathbf{m})(N_{x^*}) = 0$ such that $f\chi_{T\setminus N_{x^*}}$ is $\sigma(\mathcal{P})$-measurable. We say that f is $x^*\mathbf{m}$-integrable if $f\chi_{T\setminus N_{x^*}}$ is $x^*\mathbf{m}$-integrable and in that case, we define

$$\int_A f d(x^*\mathbf{m}) = \int_A f\chi_{T\setminus N_{x^*}} d(x^*\mathbf{m})$$

for $A \in \sigma(\mathcal{P})$ and

$$\int_T f d(x^*\mathbf{m}) = \int_{N(f)\setminus N_{x^*}} f d(x^*\mathbf{m}).$$

(Note that $N(f)\backslash N_{x^*} = N(f\chi_{T\backslash N_{x^*}}) \in \sigma(\mathcal{P})$.**)**

It is easy to check that the above integrals are well defined.

The following definition generalizes the second part of Definition 3.1 to lcHs-valued σ-additive $\mathbf{m}$ on $\mathcal{P}$.

Definition 11.2. Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable. Then f is said to be (KL) $\mathbf{m}$-integrable in T if it is $x^*\mathbf{m}$-integrable for each $x^* \in X^*$ and if, for each $A \in \sigma(\mathcal{P}) \cup \{T\}$, there exists a vector $x_A \in X$ such that

$$x^*(x_A) = \int_A f d(x^*\mathbf{m})$$

for each $x^* \in X^*$. In that case, we define

$$(KL)\int_A f d\mathbf{m} = x_A, \ \ A \in \sigma(\mathcal{P}) \cup \{T\}. \qquad (11.2.1)$$

By the Hahn-Banach theorem the integral in (11.2.1) is well defined for each $A \in \sigma(\mathcal{P}) \cup \{T\}$. Also note that the above definition includes the definiton of (KL) $\mathbf{m}$-integrability given in [L] when f is $\sigma(\mathcal{P})$-measurable.

Proposition 11.3. If $f : T \to [-\infty, \infty]$ is (KL) $\mathbf{m}$-integrable in T, then f is finite $\mathbf{m}$-a.e. in T.

Proof. Let $A = \{t \in T : |f(t)| = \infty\}$. As f is $\mathbf{m}$-measurable, for each $q \in \Gamma$, there exist B_q, N_q, M_q such that $A = B_q \cup N_q, N_q \subset M_q, B_q, M_q \in \sigma(\mathcal{P})$ and $||\mathbf{m}||_q(M_q) = 0$. Then by Proposition 10.14(ii)(c)

$$||\mathbf{m}||_q(A) = ||\mathbf{m}||_q(B_q) = \sup_{x^* \in U_q^o} v(x^*\mathbf{m})(B_q). \qquad (11.3.1)$$

On the other hand, for each $x^* \in X^*$, by Definition 11.1 there exists $N_{x^*} \in \sigma(\mathcal{P})$ with $v(x^*\mathbf{m})(N_{x^*}) = 0$ such that $f\chi_{T\backslash N_{x^*}}$ is $\sigma(\mathcal{P})$-measurable and $x^*\mathbf{m}$-integrable. Then $f\chi_{T\backslash N_{x^*}}$ and hence f are $(x^*\mathbf{m})$-a.e. finite in T. Consequently, by (11.3.1) we have $||\mathbf{m}||_q(A) = 0$ for all $q \in \Gamma$ and hence A is $\mathbf{m}$-null.

The second part of the following theorem and Remark 11.5 below play a key role in the sequel.

Theorem 11.4. Let $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable and let X be an lcHs. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then:

(i) f is (KL) $\mathbf{m}$-integrable in T, then f is (KL) $\mathbf{m}_q$-integrable in T with values in X_q (in the sense that the integral of f assuems values in X_q for each $q \in \Gamma$ (resp. (KL) $\mathbf{m}_{q_E}$-integrable in T) with values in X_{q_E} for each $E \in \mathcal{E}$).

(ii) If f is (KL) $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$ (i.e., the integrals of f assume values in $\widetilde{X_q}$) for each $q \in \Gamma$ (resp. (KL) $\mathbf{m}_{q_E}$-integrable in T with values in $\widetilde{X_{q_E}}$ for each $E \in \mathcal{E}$), then f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$ and (KL) $\int_A f d\mathbf{m} = \varprojlim (\mathrm{KL}) \int_A f d\mathbf{m}_q$ (resp. $= \varprojlim (\mathrm{KL}) \int_A f d\mathbf{m}_{q_E}$) for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Proof. (i) Suppose f is (KL) $\mathbf{m}$-integrable in T. Let $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then, for each $x^* \in X^*$, f is $x^*\mathbf{m}$-integrable and there exists $x_A \in X$ such that $x^*(x_A) = \int_A f d(x^*\mathbf{m})$. Let $q \in \Gamma$ and let $y^* \in (X_q)^*$. Then $y^*\Pi_q \in X^*$ and hence f is $y^*\Pi_q\mathbf{m}$-integrable and $(y^*\Pi_q)(x_A) = \int_A f d(y^*\Pi_q\mathbf{m}) = \int_A f d(y^*\mathbf{m}_q)$. Hence f is (KL) $\mathbf{m}_q$-integrable in T with values in X_q. Since $q_E \in \Gamma$ for $E \in \mathcal{E}$, f is (KL) $\mathbf{m}_{q_E}$-integrable in T with values in X_{q_e} for $E \in \mathcal{E}$.

(ii) Suppose

$$f \text{ is (KL) } \mathbf{m}_q\text{-integrable in } T \text{ with values in } \widetilde{X_q} \text{ for each } q \in \Gamma. \qquad (11.4.1)$$

Let $A \in \sigma(\mathcal{P}) \cup \{T\}$ and let $q \in \Gamma$. Then there exists $y_A^{(q)} \in \widetilde{X_q}$ such that

$$y^*(y_A^{(q)}) = \int_A f d(y^*\mathbf{m}_q) \qquad (11.4.2)$$

for $y^* \in (X_q)^*$ so that

$$y_A^{(q)} = (\mathrm{KL})\int_A f d\mathbf{m}_q. \qquad (11.4.3)$$

By the proof of Theorem 5.4, Ch. II of [Scha], there exists $x_A \in \tilde{X}$ such that $x_A = \underleftarrow{\lim}\, y_A^{(q)}$. (See Notations 10.16.)

Let $x^* \in X^*$ be arbitrary. Then q_{x^*} given by $q_{x^*}(x) = |x^*(x)|$, $x \in X$, belongs to Γ and clearly, $\widetilde{q_{x^*}} = |x^*(x)|$ for $x \in \tilde{X}$. (See Notation 10.15.) In the representation of the dual of $\tilde{X}$ given in §22, 6.(6) of [KÖ], take $\Psi_{x^*} \in (X_{\widetilde{q_{x^*}}})^* = (\widetilde{X_{q_{x^*}}})^*$ as given in Proposition 10.12(i) and the zero functional in $(\widetilde{X_q})^*$ for each $q \in \Gamma\backslash\{q_{x^*}\}$. Then, taking $E = \{x^*\} \in \mathcal{E}$, by (11.4.2) and by Proposition 10.12(i) we have $x^*(x_A) = \Psi_{x^*}(y_A^{(q_{x^*})}) = \int_A f d(\Psi_{x^*}\mathbf{m}_{q_{x^*}}) = \int_A f d(x^*\mathbf{m})$. Hence f is (KL) $\mathbf{m}$-integrable in T with $(\mathrm{KL})\int_A f d\mathbf{m} = x_A \in \tilde{X}$ so that $(\mathrm{KL})\int_A f d\mathbf{m} = \underleftarrow{\lim}\,(\mathrm{KL})\int_A f d\mathbf{m}_q$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

If condition (11.4.1) holds for $\{q_E : E \in \mathcal{E}\}$, then it holds all $q \in \Gamma$ by Proposition 10.14(i) and by Notation 10.10 since $q = q_{U_q^o}$ and $U_q^o \in \mathcal{E}$ by Proposition 6, §4, Ch. 3 of [Ho]. Hence f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$.

Remark 11.5. By Theorems 12.2 and 12.3 below, (ii) of the above theorem can be strengthened as follows:

If X is a quasicomplete lcHs and if f is (KL) $\mathbf{m}_q$-integrable (resp. (KL) $\mathbf{m}_{q_E}$-integrable) in T with values in $\widetilde{X_q}$ (resp. $\widetilde{X_{q_E}}$) for each $q \in \Gamma$ (resp. $E \in \mathcal{E}$), then f is (KL) m-integrable in T with values in X and $(\mathrm{KL})\int_A f d\mathbf{m} = \underleftarrow{\lim}\,(\mathrm{KL})\int_A f d\mathbf{m}_q$ (resp. $= \underleftarrow{\lim}\,(\mathrm{KL})\int_A f d\mathbf{m}_{q_E}$) belongs to X for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Notation 11.6. Let X be an lcHs and let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then $\mathcal{I}(\mathbf{m})$ denotes the class of all $\mathbf{m}$-measurable scalar functions on T which are (KL) $\mathbf{m}$-integrable in T with $(\mathrm{KL})\int_A f d\mathbf{m} \in X$ for all $A \in \sigma(\mathcal{P}) \cup \{T\}$.
The following lemma is needed to generalize Theorem 3.5(viii) to lcHs.

Lemma 11.7. Let X, Y be lcHs over the same scalar field $\boldsymbol{K}$ and let $L(X,Y)$ be the vector space of all continuous linear mappings from X into Y. If $\mathbf{m} : \mathcal{P} \to X$ is σ-additive and $u \in L(X,Y)$, then

$u\mathbf{m} : \mathcal{P} \to Y$ is σ-additive. If $f : T \to \boldsymbol{K}$or $[-\infty, \infty]$ is $\mathbf{m}$-measurable, then f is also $u\mathbf{m}$-measurable.

Proof. As u is linear and continuous, $u\mathbf{m}$ is σ-additive. Let $\mathcal{F}$ be the family of equicontinuous subsets of Y^*. Let $F \in \mathcal{F}$. Then, given $\epsilon > 0$, there exists a neighbourhood W of 0 in Y such that $\sup_{y^* \in F} |y^*(y)| < \epsilon$ for $y \in W$. As u is continuous and linear, there exists a neighbourhood U of 0 in X such that $u(U) \subset W$. Then $\sup_{y^* \in F} |y^*(ux)| < \epsilon$ for $x \in U$. Let u^* be the adjoint of u. Then $\sup_{y^* \in F} |(u^*y^*)(x)| < \epsilon$ for $x \in U$ and hence $u^*F \in \mathcal{E}$. As f is $\mathbf{m}$-measurable, there exists $N_{u^*F} \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_{q_{u^*F}}(N_{u^*F}) = 0$ such that $f\chi_{T \setminus N_{u^*F}}$ is $\sigma(\mathcal{P})$-measurable. Then by Proposition 10.12(iii) we have $||u\mathbf{m}||_{q_F}(N_{u^*F}) = \sup_{y^* \in F} v(y^*u\mathbf{m})(N_{u^*F}) = \sup_{y^* \in F} v(u^*y^*\mathbf{m})(N_{u^*F}) = \sup_{x^* \in u^*F} v(x^*\mathbf{m})(N_{u^*F}) = ||\mathbf{m}||_{q_{u^*F}}(N_{u^*F}) = 0$ and hence f is $(u\mathbf{m})_{q_F}$-measurable. Then, in the light of Remark 10.5, f is $u\mathbf{m}$-measurable.

We generalize below (i)-(iv) and (viii) of Theorem 3.5 to lcHs.

Theorem 11.8. Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then:

(i) A $\mathcal{P}$-simple function $s = \sum_1^r \alpha_i \chi_{A_i}$ with $(\alpha_i) \subset \boldsymbol{K}$, $(A_i)_1^r \subset \mathcal{P}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, is (KL) $\mathbf{m}$-integrable in T and (KL)$\int_A s d\mathbf{m} = \sum_1^r \alpha_i \mathbf{m}(A_i \cap A)$ for $A \in \sigma(\mathcal{P})$. We write $\int_A s d\mathbf{m}$ instead of (KL)$\int_A s d\mathbf{m}$. Consequently, $||\mathbf{m}||_q(A) = \sup\{q(\int_A s d\mathbf{m}) = |\int_A s d\mathbf{m}_q|_q :$ $s\,\mathcal{P}$-simple,
$|s(t)| \le \chi_A(t), t \in T\}$ for $q \in \Gamma$.

(ii) If $f : T \to \boldsymbol{K}$or $[-\infty, \infty]$ is (KL) $\mathbf{m}$-integrable in T, then $\boldsymbol{\gamma}(\cdot) = $ (KL) $\int_{(\cdot)} f d\mathbf{m}$ is σ-additive (in τ) on $\sigma(\mathcal{P})$.

(iii) If $\boldsymbol{\gamma}$ is as in (ii), $q \in \Gamma$ and $E \in \mathcal{E}$, then:
  (a) $||\boldsymbol{\gamma}||_q(A) = \sup_{x^* \in U_q^o} \int_A |f| dv(x^*\mathbf{m}),\ A \in \sigma(\mathcal{P})$.
  (b) $||\boldsymbol{\gamma}||_{q_E}(A) = \sup_{x^* \in E} \int_A |f| dv(x^*\mathbf{m}),\ A \in \sigma(\mathcal{P})$.
  (c) $\lim_{||\mathbf{m}||_q(A) \to 0} \boldsymbol{\gamma}_q(A) = \lim_{||\mathbf{m}||_q(A) \to 0} ||\boldsymbol{\gamma}||_q(A) = 0,\ A \in \sigma(\mathcal{P})$.

(iv) $\mathcal{I}(\mathbf{m})$ is a vector space over $\boldsymbol{K}$with respect to pointwise addition and scalar multiplication. For $A \in \sigma(\mathcal{P})$ fixed, the mapping $f \to$ (KL) $\int_A f d\mathbf{m}$ is linear on $\mathcal{I}(\mathbf{m})$ with values in X. Consequently, if s in (i) is with (A_i) not necessarily mutually disjoint, then also $\int_A s d\mathbf{m} = \sum_1^r \alpha_i \mathbf{m}(A \cap A_i),\ A \in \sigma(\mathcal{P})$.

(v) Let Y, $L(X, Y)$, u and f be as in Lemma 11.7. If f is $\mathbf{m}$-measurable and (KL) $\mathbf{m}$-integrable in T, then f is $u\mathbf{m}$-measurable and (KL) $u\mathbf{m}$-integrable in T and $u(\text{(KL)} \int_A f d\mathbf{m}) = $ (KL) $\int_A f du\mathbf{m},\ A \in \sigma(\mathcal{P}) \cup \{T\}$.

Proof. (i) and (iv) are obvious and (ii) is due to the Orlicz-Pettis theorem (for example, see [McA]).

(iii)(a) (resp.(b)) By (ii) $\boldsymbol{\gamma}$ is σ-additive on $\sigma(\mathcal{P})$ and hence by Definition 11.1, and by Propositions 2.11 and 10.14(ii)(c) (resp. and 10.12(iii)) the result holds.

(iii)(c) If $||\mathbf{m}||_q(A) = 0$, $A \in \sigma(\mathcal{P})$, then by Proposition 10.14(ii)(c), $v(x^*\mathbf{m})(A) = 0$ for $x^* \in U_q^o$ and hence by (a), $||\boldsymbol{\gamma}||_q(A) = 0$. As $||\mathbf{m}||_q$ and $||\boldsymbol{\gamma}||_q$ are σ-subadditive on $\sigma(\mathcal{P})$ and as $||\boldsymbol{\gamma}||_q$ is further continuous on $\sigma(\mathcal{P})$ (treating $\boldsymbol{\gamma}_q = \Pi_q \circ \boldsymbol{\gamma} : \sigma(\mathcal{P}) \to \widetilde{X_q}$ and by Proposition 2.3), the proof of Theorem 3.5(iii)(b) holds here.

(v) By Lemma 11.7, $u\mathbf{m}$ is σ-additive on $\sigma(\mathcal{P})$ and f is $u\mathbf{m}$-measurable. Let $A \in \sigma(\mathcal{P}) \cup \{T\}$ and let (KL) $\int_A f d\mathbf{m} = x_A \in X$. For $y^* \in Y^*$, $u^*y^* \in X^*$ and hence f is $u^*y^*\mathbf{m}$-integrable and

$y^*(ux_A) = u^*y^*(x_A) = \int_A fd(u^*y^*\mathbf{m}) = \int_A fd(y^*u\mathbf{m})$. Hence the result holds.

The following theorem generalizes (v)-(vii) of Theorem 3.5 to quasicomplete lcHs-valued σ-additive vector measures.

Theorem 11.9. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then:

(i)(a) If $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ is $\mathbf{m}$-measurable and $\mathbf{m}$-essentially bounded in $A \in \mathcal{P}$, then f is (KL) $\mathbf{m}$-integrable in A with values in $\tilde{X}$ (in fact, with values in X by Remark 11.5) and, for each $q \in \Gamma$,

$$q((\mathrm{KL}) \int_B f d\mathbf{m}) = |(\mathrm{KL}) \int_B f d\mathbf{m}_q|_q \le (\mathrm{ess\ sup}_{t\in A}|f(t)|) \cdot ||\mathbf{m}||_q(B)$$

for $B \in A \cap \mathcal{P}\,(= \sigma(\mathcal{P}) \cap A)$.

(i)(b) If f is $\mathbf{m}$-essentially bounded in T and if $\mathcal{P}$ is a σ-ring $\mathcal{S}$, then f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$ (in fact, in X by Remark 11.5) and, for each $q \in \Gamma$,

$$\begin{aligned} q((\mathrm{KL}) \int_A f d\mathbf{m}) = |(\mathrm{KL}) \int_A f d\mathbf{m}_q|_q &\le (\mathrm{ess\ sup}_{t\in T}|f(t)|) \cdot ||\mathbf{m}||(A) \\ &\le (\mathrm{ess\ sup}_{t\in T}|f(t)|) \cdot ||\mathbf{m}||(T) \end{aligned}$$

for $A \in \mathcal{S} \cup \{T\}$.

(ii) If φ is an $\mathbf{m}$-essentially bounded scalar function on T and if $f \in \mathcal{I}(\mathbf{m})$, then $\varphi f \in \mathcal{I}(\mathbf{m})$. Consequently, for $f \in \mathcal{I}(\mathbf{m})$ and for $A \in \sigma(\mathcal{P}) \cup \{T\}$, $f\chi_A \in \mathcal{I}(\mathbf{m})$ and $(\mathrm{KL})\int_A f d\mathbf{m} = (\mathrm{KL}) \int_T f\chi_A d\mathbf{m}$.

(iii) **(Domination principle)**. If f is an $\mathbf{m}$-measurable scalar function on T and if $g \in \mathcal{I}(\mathbf{m})$ with $|f| \le |g|$ $\mathbf{m}$-a.e. in T, then $f \in \mathcal{I}(\mathbf{m})$. Consequently, an $\mathbf{m}$-measurable function $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ is (KL) $\mathbf{m}$-integrable in T (with values in X) if and only if $|f|$ is so. Moreover, for an $\mathbf{m}$-measurable scalar function f on T the following statements are equivalent: (a) $f \in \mathcal{I}(\mathbf{m})$; (b) $|f| \in \mathcal{I}(\mathbf{m})$; (c) $\bar{f} \in \mathcal{I}(\mathbf{m})$; (d) $\mathrm{Re}\, f \in \mathcal{I}(\mathbf{m})$ and $\mathrm{Im}\, f \in \mathcal{I}(\mathbf{m})$; (e) $(\mathrm{Re}\, f)^+$, $(\mathrm{Im}\, f)^+$, $(\mathrm{Re}\, f)^-$ and $(\mathrm{Im}\, f)^-$ belong to $\mathcal{I}(\mathbf{m})$. Moreover, if $f_1, f_2 : T \to \boldsymbol{R}$ belong to $\mathcal{I}(\mathbf{m})$, then $\max(f_1, f_2)$ and $\min(f_1, f_2)$ belong to $\mathcal{I}(\mathbf{m})$.

Proof. (i)(a) Let $A \in \mathcal{P}$ and let $\alpha = \mathrm{ess\ sup}_{t\in A}|f(t)|$. Then by (10.7.1) there exists an $\mathbf{m}$-null set $M \in \sigma(\mathcal{P})$ such that $\alpha = \sup_{t\in T\backslash M} |f(t)|$. Then $f\chi_{A\backslash M}$ is an $\mathbf{m}$-measurable bounded function on T. Let $q \in \Gamma$. Then there exists $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$ such that $f\chi_{A\backslash M\backslash N_q}$ is $\sigma(\mathcal{P})$-measurable so that by Proposition 10.9 there exists a sequence $(s_n^{(q)})_{n=1}^\infty \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f\chi_{A\backslash M\backslash N_q}$ and $|s_n^{(q)}| \nearrow |f\chi_{A\backslash M\backslash N_q}|$ pointwise in T. Let $\Sigma_A = A \cap \mathcal{P}$. Then Σ_A is a σ-algebra of sets in A and $f\chi_{A\backslash M\backslash N_q}$ is Σ_A-measurable. As $\mathbf{m}_q$ is σ-additive on Σ_A with values in $X_q \subset \widetilde{X_q}$, by Proposition 2.2 $||\mathbf{m}||_q$ is continuous on Σ_A. Let $F = \bigcup_{n=1}^\infty N(s_n^{(q)})$ (see Notation 2.7). Then $F \subset A\backslash N_q$ and $F \in \Sigma_A$. Then by the Egoroff-Lusin theorem (Proposition 2.12) there exist $N \subset F$, $N \in \Sigma_A$ with $||\mathbf{m}||_q(N) = 0$ and a sequence $(F_k)_1^\infty \subset \Sigma_A$ with $F_k \nearrow F\backslash N$ such that $s_n^{(q)} \to f$ uniformly in each F_k. Then by an argument similar to that in the proof of Theorem 3.5(v) there exists $x_B^{(q)} \in \widetilde{X_q}$ such that $(\mathrm{KL})\int_B f d\mathbf{m}_q = x_B^{(q)} = \lim_n \int_B s_n^{(q)} d\mathbf{m}_q$ and

$$\left|(\mathrm{KL}) \int_B f d\mathbf{m}_q\right|_q = \left|\lim_n \int_B s_n^{(q)} d\mathbf{m}_q\right|_q \le \alpha||\mathbf{m}||_q(B) \qquad (11.9.1)$$

for $B \in \Sigma_A$ and this holds for each $q \in \Gamma$. Consequently, by Theorem 11.4(ii), f is (KL) $\mathbf{m}$-integrable in A with values in $\tilde{X}$ (and hence with values in X by Remark 11.5). Thus, if $(\mathrm{KL})\int_B f d\mathbf{m} = z_B \in X$ for $B \in \Sigma_A$, then $\Pi_q(z_B) = (\mathrm{KL}) \int_B f d\mathbf{m}_q$ for $q \in \Gamma$ and for $B \in \Sigma_A$. Moreover, by (11.9.1) we have

$$q\left((\mathrm{KL})\int_B f d\mathbf{m}\right) = q(z_B) = |\Pi_q(z_B)|_q$$
$$= \left|(\mathrm{KL})\int_B f d\mathbf{m}_q\right|_q \le (\text{ess sup}_{t\in A}|f(t)|)\cdot ||\mathbf{m}||_q(B)$$

for $B \in \Sigma_A$. Hence (i)(a) holds.

(i)(b) This is immediate from (i)(a) as $||\mathbf{m}||_q$ is continuous on the σ-ring $\mathcal{S} = \sigma(\mathcal{P}) = \mathcal{P}$ and as $f\chi_{T\backslash M\backslash N_q}$ is $\sigma(\mathcal{P})$-measurable. (A direct proof is indicated in Remark 11.10 below.)

(ii) Let $\boldsymbol{\nu}(\cdot) = (\mathrm{KL})\int_{(\cdot)} f d\mathbf{m}$ on $\sigma(\mathcal{P})$. By Theorem 11.8(ii), $\boldsymbol{\nu}$ is σ-additive on $\sigma(\mathcal{P})$ and by Theorem 11.8(iii)(c), an $\mathbf{m}_q$-null set in $\sigma(\mathcal{P})$ is also $\boldsymbol{\nu}_q$-null. Hence, the $\mathbf{m}$-measurable function φ is also $\boldsymbol{\nu}$-measurable and $\boldsymbol{\nu}$-essentially bounded. Therefore, by (i)(b), φ is (KL) $\boldsymbol{\nu}$-integrable in T with values in X. For $A \in \sigma(\mathcal{P}) \cup T$, let $(\mathrm{KL})\int_A \varphi d\boldsymbol{\nu} = x_A \in X$. Then, for each $x^* \in X^*$, by Definition 11.1 there exists $N_{x^*} \in \sigma(\mathcal{P})$ with $v(x^*\boldsymbol{\nu})(N_{x^*}) = 0$ such that $\varphi\chi_{T\backslash N_{x^*}}$ is $\sigma(\mathcal{P})$-measurable and $x^*(x_A) = \int_A \varphi\chi_{T\backslash N_{x^*}} d(x^*\boldsymbol{\nu})$. As f is (KL) $\mathbf{m}$-integrable in T, there exists $M_{x^*} \in \sigma(\mathcal{P})$ with $v(x^*\mathbf{m})(M_{x^*}) = 0$ such that $f\chi_{T\backslash M_{x^*}}$ is $\sigma(\mathcal{P})$-measurable. Then by Theorem 11.8(iii)(b) and by Proposition 10.12(iv), $0 = v(x^*\boldsymbol{\nu})(N_{x^*}) = ||x^*\boldsymbol{\nu}||(N_{x^*}) = ||\Pi_{q_{x^*}}\boldsymbol{\nu}||(N_{x^*}) = ||\boldsymbol{\nu}||_{q_{x^*}}||(N_{x^*}) = \int_{N_{x^*}} |f| dv(x^*\mathbf{m})$ since $E = \{x^*\} \in \mathcal{E}$. Hence either $f = 0\, x^*\mathbf{m}$-a.e. in N_{x^*} or N_{x^*} is $x^*\mathbf{m}$-null. In either case, $\varphi\chi_{T\backslash N_{x^*}} f\chi_{T\backslash M_{x^*}}$ is $\sigma(\mathcal{P})$-measurable and is equal to φf $x^*\mathbf{m}$-a.e in T. Hence φf is $x^*\mathbf{m}$-integrable and $\int_A \varphi f d(x^*\mathbf{m}) = \int_A \varphi\chi_{T\backslash N_{x^*}} f\chi_{T\backslash M_{x^*}} d(x^*\mathbf{m})$. Then arguing as in the proof of Theorem 3.5(vi), we have $x^*(x_A) = \int_A \varphi d(x^*\boldsymbol{\nu}) = \int_A \varphi f d(x^*\mathbf{m})$ for $x^* \in X^*$. Hence $\varphi f \in \mathcal{I}(\mathbf{m})$. The second part is proved by an argument similar to that in the proof of the second part of Theorem 3.5(vi).

(iii) Let $h(t) = \frac{f(t)}{g(t)}$ for $t \in N(g)$ and $h(t) = 0$ otherwise. Then h is clearly $\mathbf{m}$-measurable, $|h(t)| \le 1$ $\mathbf{m}$-a.e. in T and $f = gh$ $\mathbf{m}$-a.e. in T. Then by (ii), f is (KL) $\mathbf{m}$-integrable in T with values in X. If $f : T \to \boldsymbol{K}$or $[_\infty, \infty]$ is (KL) $\mathbf{m}$-integrable in T, then by Theorem 11.3 there exists an $\mathbf{m}$-null set $N \in \sigma(\mathcal{P})$ such that f is finite in $T\backslash N$ and hence $g = f\chi_{T\backslash N} \in \mathcal{I}(\mathbf{m})$ and $|f| \le |g|$ $\mathbf{m}$-a.e. in T. Hence $|f|$ is (KL) $\mathbf{m}$-integrable in T. The other parts follow from the first and from the facts that $\max(f_1, f_2) = \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|)$ and $\min(f_1, f_2) = \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|)$ and that $\mathcal{I}(\mathbf{m})$ is a vector space.

Remark 11.10. A direct alternative proof of Theorem 11.9(i)(b) can be given without appealing to the Egoroff-Lusin theorem by arguing as in the alternative proof of the last part of Theorem 3.5(v) and then combining with the projective limit argument. Details are left to the reader.

Theorem 11.11 (Generalization of Theorem 3.7-(LDCT a.e. version)). Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. For each $q \in \Gamma$, let $f_n^{(q)}$, $n \in \boldsymbol{N}$ be $\mathbf{m}_q$-measurable on T with values in $\boldsymbol{K}$or in $[-\infty, \infty]$ and let $g : T \to \boldsymbol{K}$or $[-\infty, \infty]$ be $\mathbf{m}$-measurable and (KL) $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$. Suppose $|f_n^{(q)}(t)| \le |g(t)|$ $\mathbf{m}_q$-a.e. in T for each $n \in \boldsymbol{N}$ and for each $q \in \Gamma$ and let $f : T \to \boldsymbol{K}$or $[-\infty, \infty]$. If $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T, then f is $\mathbf{m}_q$-measurable, f, $f_n^{(q)}$, $n \in \boldsymbol{N}$ are (KL) $\mathbf{m}_q$-integrable in T and consequently, f is (KL) $\mathbf{m}$-integrable in T with values in X. Moreover, for each $q \in \Gamma$,

$$\lim_n \left|(\mathrm{KL})\int_A f d\mathbf{m}_q - (KL)\int_A f_n^{(q)} d\mathbf{m}_q\right|_q = 0 \text{ for } A \in \sigma(\mathcal{P}) \cup \{T\} \qquad (11.11.1)$$

the limit being uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed) and

$$\lim_n \sup_{x^* \in U_q^o} \int_T |f_n^{(q)} - f| dv(\Psi_{x^*}\mathbf{m}_q) = 0.$$

where Ψ_{x^*} is as in Proposition 10.14(ii)(a).

Proof. By hypothesis, by Theorem 11.4(ii) and by Remark 11.5, g is (KL) $\mathbf{m}$-integrable in T with values in X. Let $q \in \Gamma$ be given. By hypothesis there exists an $\mathbf{m}_q$-null set $N^{(q)} \in \sigma(\mathcal{P})$ such that $f_n^{(q)} \to f$ pointwise in $T \backslash N^{(q)}$. As g is (KL) $\mathbf{m}$-integrable in T, by Proposition 11.3 there exists an $\mathbf{m}$-null set $M \in \sigma(\mathcal{P})$ such that $M \supset \{t \in T : |g(t)| = \infty\}$ so that g is finite in $T\backslash M$. Let $M^{(q)} = M \cup N^{(q)}$. Then $M^{(q)} \in \sigma(\mathcal{P})$ and is $\mathbf{m}_q$-null. As $f_n^{(q)}$, $n \in \mathbb{N}$ are $\mathbf{m}_q$-measurable, for each n there exists $M_n^{(q)} \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(M_n^{(q)}) = 0$ such that $f_n \chi_{T\backslash M_n^{(q)}}$ is $\sigma(\mathcal{P})$-measurable. Let $M_q = M^{(q)} \bigcup \bigcup_{n=1}^{\infty} M_n^{(q)}$. Then $M_q \in \sigma(\mathcal{P})$ and $||\mathbf{m}||_q(M_q) = 0$ as $||\mathbf{m}||_q$ is σ-subadditive on $\sigma(\mathcal{P})$. As $f_n^{(q)}\chi_{T\backslash M_q} \to f\chi_{T\backslash M_q}$ pointwise in T and as $f_n^{(q)}\chi_{T\backslash M_q}$, $n \in \mathbb{N}$ are $\sigma(\mathcal{P})$-measurable, it follows that $f\chi_{T\backslash M_q}$ is $\sigma(\mathcal{P})$-measurable so that f is $\mathbf{m}_q$-measurable for each $q \in \Gamma$ and hence, f is $\mathbf{m}$-measurable. Consideing $\mathbf{m}_q : \mathcal{P} \to \widetilde{X_q}$, the hypothesis of domination and Theorem 3.5(vii) imply that f, $f_n^{(q)}$, $n \in \mathbb{N}$ are (KL) $\mathbf{m}_q$-integrable in T. Then by Theorem 11.4(ii) and by Remark 11.5, f is (KL) $\mathbf{m}$-integrable in T with values in X.

Let $q \in \Gamma$. If $\boldsymbol{\nu}(\cdot) = (\text{KL}) \int_{(\cdot)} g d\mathbf{m}$, then by Theorem 11.8(ii), $\boldsymbol{\nu}$ and consequently, $\Pi_q \circ \boldsymbol{\nu} = \boldsymbol{\nu}_q$ are σ-additive on $\sigma(\mathcal{P})$ ($\boldsymbol{\nu}_q$ assums values in the Banach space $\widetilde{X_q}$). Hence by Proposition 2.6 there exists a control measure $\mu_q : \sigma(\mathcal{P}) \to [0, \infty)$ for $\boldsymbol{\nu}_q$ so that, given $\epsilon > 0$, there exists $\delta > 0$ such that $||\boldsymbol{\nu}||_q(A) < \frac{\epsilon}{3}$ whenever $A \in \sigma(\mathcal{P})$ with $\mu_q(A) < \delta$. Let $F_q = \bigcup_{n=1}^{\infty} N(f_n^{(q)}) \cap (T\backslash M_q)$. Clearly, $F_q \in \sigma(\mathcal{P})$. Let $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then, arguing as in the proof of Theorem 3.7 with F_q, $F_k^{(q)}$, M_q, N_q, $||\boldsymbol{\nu}||_q$, μ_q, A, $f_n^{(q)}$ and $||\mathbf{m}||_q$ replacing F, F_k, M, N, $||\boldsymbol{\nu}||$, μ, E f_n and $||\mathbf{m}||$, respectively, and taking Ψ_{x^*} as in Proposition 10.14(ii)(a) and n_0 such that $||f_n^{(q)} - f||_{F_{k_0}^{(q)}} \cdot ||\mathbf{m}||_q(F_{k_0}^{(q)}) < \frac{\epsilon}{3}$ for $n \geq n_0$, we have for $x^* \in U_q^o$ and $n \geq n_0$,

$$\int_{A \cap (T \backslash F_{k_0}^{(q)})} |f_n^{(q)} - f| dv(\Psi_{x^*}\mathbf{m}_q) \leq 2||\boldsymbol{\nu}||_q(F_q \backslash N_q \backslash F_{k_0}^{(q)}) + 2||\boldsymbol{\nu}||_q(N_q) + 2||\boldsymbol{\nu}||_q(M_q) < \frac{2\epsilon}{3}$$

since $\int_B |f_n^{(q)} - f| dv(\Psi_{x^*}\mathbf{m}_q) \leq 2 \int_B |g| dv(\Psi_{x^*}\mathbf{m}_q) = 2 \int_B |g| dv(x^*\mathbf{m}) \leq 2||\boldsymbol{\nu}||_q(B)$ for $B \in \sigma(\mathcal{P})$ by Proposition 10.14(ii) and Theorem 11.8(iii)(a) and

$$\int_{A \cap F_{k_0}^{(q)}} |f_n^{(q)} - f| dv(\Psi_{x^*}\mathbf{m}_q) < \frac{\epsilon}{3}$$

since $v(\Psi_{x^*}\mathbf{m}_q)(F_{k_0}^{(q)}) = v(x^*\mathbf{m})(F_{k_0}^{(q)}) \leq ||\mathbf{m}||_q(F_{k_0}^{(q)})$ by Proposition 11.14(ii)(c). Consequently, by the above inequalities and by the fact that f is (KL) $\mathbf{m}$-integrable in T with values in X we have

$\Pi_q((\text{KL})\int_A f d\mathbf{m}) = (\text{KL})\int_A f d\mathbf{m}_q$ and

$$\begin{aligned}\left|(\text{KL})\int_A f_n^{(q)} d\mathbf{m}_q - (\text{KL})\int_A f d\mathbf{m}_q\right|_q &= \\ \left|(\text{KL})\int_A (f_n^{(q)} - f) d\mathbf{m}_q\right|_q& \\ \leq \sup_{x^* \in U_q^\circ} \int_A |f_n^{(q)} - f| dv(\Psi_{x^*}\mathbf{m}_q) < \epsilon & \qquad (11.11.2)\end{aligned}$$

for $n \geq n_0$ and for all $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then (11.11.2) implies (11.11.1) and that the limit in (11.11.1) is uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed in Γ).

Corollary 11.12. (Generalization of Corollary 3.8-(LBCT a-e-version)). Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{S} \to X$ be σ-additive, where $\mathcal{S}$ is a σ-ring. Suppose $f, f_n^{(q)} : T \to \mathbb{K}$ or $[-\infty, \infty]$ for $n \in \mathbb{N}$ and let $0 < K^{(q)} < \infty$ for $q \in \Gamma$. If $f_n^{(q)}$, $n \in \mathbb{N}$, are $\mathbf{m}_q$-measurable, $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and if $|f_n^{(q)}| \leq K^{(q)}$ $\mathbf{m}_q$-a.e. for all n, then f, $f_n^{(q)}$, $n \in \mathbb{N}$, are (KL) $\mathbf{m}_q$-integrable in T and consequently, f is (KL) $\mathbf{m}$-integrable in T with values in X. Then, for each $q \in \Gamma$,

$$\lim_n \left|(\text{KL})\int_A f d\mathbf{m}_q - (\text{KL})\int_A f_n^{(q)} d\mathbf{m}_q\right|_q = 0$$

for $A \in \mathcal{S} \cup \{T\}$, the limit being uniform with respect to $A \in \mathcal{S}$ (for q fixed).

Proof. The corollary is immediate from the above theorem and the second part of Theorem 3.5(v) since $\mathcal{S}$ is a σ-ring.

The following theorem plays a key role in Section 12.

Theorem 11.13 (Generalization of Corollary 3.9). Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. If f is an $\mathbf{m}$-measurable (KL) $\mathbf{m}_q$-integrable scalar function on T with values in $\widetilde{X_q}$ for each $q \in \Gamma$, then f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$. Moreover, for each $q \in \Gamma$, there exist a set $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$ and a sequence $(s_n^{(q)})_{n=1}^\infty \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f$ and $|s_n^{(q)}| \nearrow |f|$ pointwise in $T \backslash N_q$. Then for any such sequence $(s_n^{(q)})$, by (10.15)

$$\begin{aligned}&\lim_n q\left(\int_A s_n^{(q)} d\mathbf{m} - (\text{KL})\int_A f d\mathbf{m}\right) \\ &= \lim_n \left|\int_A s_n^{(q)} d\mathbf{m}_q - (\text{KL})\int_A f d\mathbf{m}_q\right|_q = 0\end{aligned}$$

for $A \in \sigma(\mathcal{P}) \cup \{T\}$, the limit being uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed). (By Remark 12.5 below, $(\text{KL})\int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.)
Consequently, for $A \in \sigma(\mathcal{P})$ and $q \in \Gamma$,

$$||\mathbf{m}||_q(A) = \sup\{|(\text{KL})\int_A h d\mathbf{m}_q|_q : h \in \mathcal{I}(\mathbf{m}_q),\ |h| \leq \chi_A\, \mathbf{m}_q\text{-a.e. in } T\}.$$

Proof. Let $q \in \Gamma$. By hypothesis, f is (KL) $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$. Then by Corollary 3.9 there exists $(s_n^{(q)}) \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f$ and $|s_n^{(q)}| \nearrow |f|\, \mathbf{m}_q$-a.e. in T and consequently,by Corollary 3.9 we have

$$\lim_n \left|(\mathrm{KL})\int_A f d\mathbf{m}_q - \int_A s_n^{(q)} d\mathbf{m}_q\right|_q = 0,\ A \in \sigma(\mathcal{P}) \cup \{T\}.5cm(11.13.1)$$

the limit being uniform with respect to $A \in \sigma(\mathcal{P})$. Thus

$$\left|(\mathrm{KL})\int_A f d\mathbf{m}_q\right|_q = \lim_n \left|\int_A s_n^{(q)} d\mathbf{m}_q\right|_q \text{ for } A \in \sigma(\mathcal{P}) \cup \{T\}. \qquad (11.13.2)$$

Since f is (KL) $\mathbf{m}-q$-integrable in T for each $q \in \Gamma$, by Theorem 11.4(ii) f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$ and by (11.13.1) we have

$$\lim_n q((KL)\int_A f d\mathbf{m} - \int_A s_n^{(q)} d\mathbf{m}) = 0$$

where we use Notation 10.15.

Consequently, by Theorem 11.8(i), by (11.13.2) and by Proposition 10.14(ii) we have

$$\begin{aligned}
||\mathbf{m}||_q(A) &= \sup\left\{\left|\int_A s d\mathbf{m}_q\right|_q : s \in \mathcal{I}_s,\ |s| \le \chi_A\right\} \\
&= \sup\left\{\left|\int_A s d\mathbf{m}_q\right|_q : s \in \mathcal{I}_s,\ |s| \le \chi_A\, \mathbf{m}_q\text{-a.e. in} T\right\} \\
&\le \sup\left\{\left|\int_A h d\mathbf{m}_q\right|_q : h \in \mathcal{I}(\mathbf{m}_q),\ |h| \le \chi_A\, \mathbf{m}_q\text{-a.e. in} T\right\} \\
&= \sup\left\{\sup_{x^* \in U_q^o}\left|\int_A h d(\Psi_{x^*}\mathbf{m}_q)\right| : h \in \mathcal{I}(\mathbf{m}_q),\ |h| \le \chi_A\, \mathbf{m}_q\text{-a.e. in} T\right\} \\
&\le \sup_{x^* \in U_q^o} v(\Psi_{x^*}\mathbf{m}_q)(A) = \sup_{x^* \in U_q^o} v(x^*\mathbf{m})(A) \\
&= ||\mathbf{m}||_q(A)
\end{aligned}$$

for $A \in \sigma(\mathcal{P})$ and for $q \in \Gamma$.

Corollary 11.14 (Simple function approximation). Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. If $f : T \to \boldsymbol{K}$is $\mathbf{m}$-measurable and (KL) $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$ for each $q \in \Gamma$, then f is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$ (by Remark 12.5 with values in X). Moreover, there exists $(s_n^{(q)})_{n=1}^{\infty} \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and such that $\lim_n \sup_{x^* \in U_q^o} \int_T |f - s_n^{(q)}| dv(\Psi_{x^*}\mathbf{m}_q) = \lim_n \sup_{x^* \in U_q^o} \int_T |f - s_n^{(q)}| dv(x^*\mathbf{m}) = 0$.

Proof. This follows from Propositions 10.9 and 10.14(ii), Theorem 11.13 and Proposition 10.14(ii) and from the fact that for a (KL) $\mathbf{m}_q$-integrable function g on T, $|(KL)\int_A g d\mathbf{m}_q|_q \le ||\boldsymbol{\gamma}||_q(A) \le \sup_{x^* \in U_q^o} \int_A |g| dv(\Psi_{x^*}\mathbf{m}_q) = \sup_{x^* \in U_q^o} \int_A |g| dv(x^*\mathbf{m})$ for $A \in \sigma(\mathcal{P})$ where $\boldsymbol{\gamma} : \sigma(\mathcal{P}) \to \widetilde{X_q}$ is given by $\boldsymbol{\gamma}(\cdot) = \int_{(\cdot)} g d\mathbf{m}_q$ is σ-additive by Theorem 3.5(ii).

Remark 11.15 (**(KL)-integrability with respect to a sequentially complete lcHs-valued m**). Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ be the collection of all (KL) $\mathbf{m}$-integrable $\sigma(\mathcal{P})$-measurable scalar functions (with values in X). If Theorem 11.9' is the same as Theorem 11.9 with X sequentially complete and all the functions considerd to be $\sigma(\mathcal{P})$-measurable or to belong to $\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$, then the proof of Theorem 3.5(v) holds here verbatim to prove (i)(a) and (i)(b) of Theorem 11.9' if we replace $|\cdot|$ by q and $||\mathbf{m}||$ by $||\mathbf{m}||_q, q \in \Gamma$ and if we use the sequential completeness of X. Consequently, by arguments similar to those in the proofs of (vi) and (vii) of Theorem 3.5 we can show that (ii) and (iii) of Theorem 11.9' are also valid. Let Theorem 11.11' be the same as Theorem 11.11 not only with the above changes but also taking $g \in \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$, replacing $(f_n^{(q)})$ by (f_n) for all $q \in \Gamma$ with f_n, $n \in \mathbb{N}$, $\sigma(\mathcal{P})$-measurable, $|f_n| \leq |g|$ $\mathbf{m}$-a.e. in T for all n and $f_n \to f$ $\mathbf{m}$-a.e. in T with f $\sigma(\mathcal{P})$-measurable. Then by Theorem 11.9'(iii), f_n, $n \in \mathbb{N}$, and f are (KL) $\mathbf{m}$-integrable in T. Then arguing as in the proof of Theorem 11.11, one can show that

$$\begin{aligned} q\left((\mathrm{KL})\int_A f d\mathbf{m} - (\mathrm{KL})\int_A f_n d\mathbf{m}\right) &= \left|(\mathrm{KL})\int_A (f-f_n) d\mathbf{m}_q\right|_q \\ &\leq \sup_{x^* \in U_q^o} \int_A |f-f_n| dv(\Psi_{x^*}\mathbf{m}_q) \\ &= \sup_{x^* \in U_q^o} \int_A |f-f_n| dv(x^*\mathbf{m}) < \epsilon \end{aligned}$$

for $n \geq n_0$ and for all $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then the corresponding version of LBCT (Corollary 11.12') too holds. (Note that Theorem 11.11' is essentially the version of LDCT given in Theorem 3.3 of [L].) If f is $\sigma(\mathcal{P})$-measurable, then by Proposition 10.9 there exists a sequence $(s_n) \subset \mathcal{I}_s$ such that $s_n \to f$ and $|s_n| \nearrow |f|$ pointwise in T. Thus, if Theorem 11.13' and Corollary 11.14' are the analogues of Theorem 11.13 and Corollary 11.14, respectively, with X sequentially complete, f $\sigma(\mathcal{P})$-measurable and $(s_n^{(q)})_{n=1}^\infty$ replaced by $(s_n)_1^\infty$ for all $q \in \Gamma$, they hold by Theorem 11.11'.

Remark 11.16. LDCT for (KL) $\mathbf{m}$-integrals with respect to a σ-additive vector measure defined on a δ-ring τ with values in a sequentially complete lcHs is given in [L] under the hypothesis that the dominated sequence converges pointwise, but, as observed in Remark 3.12 of [P3], its proof is incorrect and is corrected in the said Remark. For the case of σ-additive vector measures defined on σ-algebras with values in a sequentially complete lcHs, Theorems 11.8, 11.9' and 11.11' are obtained in [KK] (for real lcHs) and [OR] (for complex lcHs). Theorem 3.5 of [L], whose incorrect proof is corrected in Remark 3.12 of [P3], is easily deducible from Corollary 11.14'.

12. (BDS) m-INTEGRABILITY (m LcHs-VALUED)

Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. For an $\mathbf{m}$-measurable function f we define (BDS) $\mathbf{m}$-integrability or simply, $\mathbf{m}$-integrability of f with values in $\tilde{X}$ and show that f is $\mathbf{m}$-integrable in T if and only if it is (KL) $\mathbf{m}$-integrable in T and that (BDS)$\int_A f d\mathbf{m} =$ (KL)$\int_A f d\mathbf{m} \in X$, $A \in \sigma(\mathcal{P}) \cup \{T\}$ (Theorems 12.2 and 12.3). Also we generalize Theorems 4.4, 4.5 and 4.8 and Corollary 4.11. We define (BDS) $\mathbf{m}$-integrability for $\sigma(\mathcal{P})$-measurable functions im Remeark 12.11 when $\mathbf{m}$ assumes values in a sequentially complete lcHs and Theorem 12.2' in Remark 12.11 gives an analogue of Theorem 12.2 for these spaces.

Definition 12.1. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. An $\mathbf{m}$-measurable function $f : T \to \mathbb{K}$or $[-\infty, \infty]$ is said to be $\mathbf{m}$-integrable in T in the sense of Bartle-Dunford-Schwartz or (BDS) $\mathbf{m}$-integrable in T or simply, $\mathbf{m}$-integrable in T, if f is $\mathbf{m}_q$-integrable in T (considering $\mathbf{m}_q : \mathcal{P} \to X \subset \widetilde{X_q}$) for each $q \in \Gamma$ (see Definition 4.1). In that case, we define

$$(\mathrm{BDS})\int_A f d\mathbf{m} = \lim_{\longleftarrow} \int_A f d\mathbf{m}_q, \ \ A \in \sigma(\mathcal{P})$$

and

$$\text{(BDS)}\int_T f d\mathbf{m} = \lim_{\longleftarrow}\int_T f d\mathbf{m}_q = \lim_{\longleftarrow}\int_{N(f)\backslash N_q} f d\mathbf{m}_q$$

with values in $\tilde{X}$, where $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$ such that $f\chi_{T\backslash N_q}$ is $\sigma(\mathcal{P})$-measurable. (In the light of Theorem 12.3 below, these integrals indeed assume values in X as X is quasicomplete.)

Theorem 12.2. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable. Then:

(i) If f is (KL) $\mathbf{m}$-integrable in T, then f is $\mathbf{m}$-integrable in T and (BDS)$\int_A f d\mathbf{m} =$ (KL) $\int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

(ii) If f is $\mathbf{m}$-integrable in T, then f is (KL) $\mathbf{m}$-integrable in T and (KL)$\int_A f d\mathbf{m} =$ (BDS) $\int_A f d\mathbf{m} \in \tilde{X}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

(Moreover, as X is quasicomplete, by Theorem 12.3 below (BDS)$\int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$ whenever f is $\mathbf{m}$-integrable in T.) **hereafter we shall denote either of the integrals of f over $A \in \sigma(\mathcal{P}) \cup \{T\}$ by $\int_A f d\mathbf{m}$.**

Proof. (i) As f is (KL) $\mathbf{m}$-integrable in T, then, for $A \in \sigma(\mathcal{P}) \cup \{T\}$, there exists $x_A \in X$ such that $x^*(x_A) = \int_A f d(x^*\mathbf{m})$ for $x^* \in X^*$ so that (KL)$\int_A f d\mathbf{m} = x_A$. Then by Theorem 11.4(i), for each $q \in \Gamma$, we have

$$\begin{aligned}\Pi_q(x_A) = \Pi_q((\text{KL})\int_A f d\mathbf{m}) &= (\text{KL})\int_A f d(\Pi_q \circ \mathbf{m}) \quad (\text{by Theorem 11.8(v)}) \\ &= (\text{KL})\int_A f d\mathbf{m}_q = (\text{BDS})\int_A f d\mathbf{m}_q\end{aligned}$$

by Theorem 4.2. Then by Definition 12.1 and Theorem 11.4(i), f is $\mathbf{m}$-integrable in T and

$$(\text{BDS})\int_A f d\mathbf{m} = \lim_{\longleftarrow}(BDS)\int_A f d\mathbf{m}_q = \lim_{\longleftarrow}(\text{KL})\int_A f d\mathbf{m}_q = (\text{KL})\int_A f d\mathbf{m}$$

belongs to X for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Conversely, let f be $\mathbf{m}$-integrable in T. Then, for each $q \in \Gamma$, by Definitions 12.1 and 4.1 there exist $(s_n^{(q)})_{n=1}^{\infty} \subset \mathcal{I}_s$ and an $\mathbf{m}_q$-null set $N_q \in \sigma(\mathcal{P})$ such that $s_n^{(q)} \to f$ pointwise in $T\backslash N_q$ and such that $\lim_n \int_A s_n^{(q)} d\mathbf{m}_q = x_A^{(q)}$ (say) exists in $\widetilde{X_q}$ for each $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then by Theorem 4.2, f is (KL) $\mathbf{m}_q$-integrable in T and (KL)$\int_A f d\mathbf{m}_q = x_A^{(q)} \in \widetilde{X_q}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$. Consequently, by Theorem 11.4(ii) and by Definition 12.1

$$(\text{KL})\int_A f d\mathbf{m} = \lim_{\longleftarrow} x_A^{(q)} = (\text{BDS})\int_A f d\mathbf{m} \in \tilde{X}$$

for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Theorem 12.3. Let X be a quasicomplete lcHs and $\mathbf{m} : \sigma(\mathcal{P}) \to X$ be σ-additive. If $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ is $\mathbf{m}$-measurable and $\mathbf{m}$-integrable in T, then $\int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Proof. Let $A \in \sigma(\mathcal{P}) \cup \{T\}$. Let $\Phi = \{s \in \mathcal{I}_s : |s| \le |f| \,\text{in}\, T\}$ and let $G_A = \{\int_A s d\mathbf{m} : s \in \Phi\}$. Let $x^* \in X^*$. Then by Theorem 12.2(ii), f is (KL) $\mathbf{m}$-integrable in T and hence f is $x^*\mathbf{m}$-integrabe in T and therefore,

$$\sup_{s\in\Phi} |\int_A s d(x^*\mathbf{m})| \le \int_A |f| dv(x^*\mathbf{m}) = M_{x^*}(\text{say}) < \infty.$$

Thus G_A is weakly bounded. Then by Theorem 3.18 of [Ru], G_A is τ-bounded in X.

Taking $\omega_n^{(q)} = s_n^{(q)}\chi_{T\backslash N_q}$ in Proposition 10.9, we have $(\omega_n^{(q)})_{n=1}^{\infty} \subset \mathcal{I}_s$, $|\omega_n^{(q)}| \leq |f|$ in T for $n \in \mathbb{N}$, $\omega_n^{(q)} \to f$ and $|\omega_n^{(q)}| \nearrow |f|$ pointwise in $T\backslash N_q$. Then by Theorems 11.13 and 12.2(ii) and by Notation and Convention 10.15 we have

$$\lim_n q\left(\int_A \omega_n^{(q)} d\mathbf{m} - (\mathrm{KL})\int_A f d\mathbf{m}\right) = \lim_n q\left(\int_A \omega_n^{(q)} d\mathbf{m} - \int_A f d\mathbf{m}\right) = 0.$$

Hence, given $\epsilon > 0$, for each $q \in \Gamma$ there exists $s_q \in \Phi$ such that $q(\int_A s_q d\mathbf{m} - \int_A f d\mathbf{m}) < \epsilon$ and therefore, $\int_A f d\mathbf{m}$ belongs to the $\tilde{\tau}$-closure of G_A (in $\tilde{X}$). Consequently, there exists a net $(x_\alpha) \subset G_A$ such that $x_\alpha \to \int_A f d\mathbf{m}$ in $\tilde{\tau}$. Then (x_α) is $\tilde{\tau}$-Cauchy.

On the other hand, as $G_A \subset X$ is τ-bounded and as X is quasicomplete, the τ-closure of G_A is τ-complete. Since $\tilde{\tau}|_X = \tau$, it follows that (x_α) is also τ-Cauchy and hence there exists x_0 in the τ-closure of G_A (so that $x_0 \in X$) such that $x_\alpha \to x_0$ in τ and hence in $\tilde{\tau}$. Since $\tilde{\tau}$ is Hausdorff, we conclude that $\int_A f d\mathbf{m} = x_0 \in X$.

Remark 12.4. The proof of Theorem 1.35 of [T] based on the bipolar theorem is incorrect as $\int f d\mu$ is an element of the completion of the lcHs E (in the notation of [T]). In [P5] we provide a correct proof of the said theorem.

Remark 12.5. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then, in the light of Theorems 12.2 and 12.3, $\mathcal{I}(\mathbf{m})$ **in Notation 11.6 is the same as the class of all m-measurable scalar functions on T which are m-integrable in T and Theorems 11.4, 11.8, 11.9, 11.11 and 11.13 and Corollaries 11.12 and 11.14 and Remarks 11.5 and 11.10 hold for functons m-integrable in T. Moreover, the integrals of f in Theorem 11.13 and Corollary 11.14 also assume values in X itself.** (Compare Remark 4.3.)

We now generalize Theorems 4.4, 4.5 and 4.8 to lcHs in Theorems 12.6, 12.7 and 12.8, repectively.

Theorem 12.6. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $f : T \to \mathbb{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable. For each $q \in \Gamma$, let $(s_n^{(q)})_{n=1}^{\infty} \subset \mathcal{I}_s$ be such that $s_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and let $\gamma_n^{(q)} : \sigma(\mathcal{P}) \to \widetilde{X_q}$ be given by $\gamma_n^{(q)}(\cdot) = \int_{(\cdot)} s_n^{(q)} d\mathbf{m}_q$, $n \in \mathbb{N}$ Then:

(a) The following statements are equivalent:
   (i) $\lim_n \gamma_n^{(q)}(A) = \gamma^{(q)}(A)$ exists in $\widetilde{X_q}$ for each $A \in \sigma(\mathcal{P})$.
   (ii) $\gamma_n^{(q)}$, $n \in \mathbb{N}$ (q fixed), are uniformly σ-additive on $\sigma(\mathcal{P})$.
   (iii) $\lim_n \gamma_n^{(q)}(A)$ exists in $\widetilde{X_q}$ uniformly with respect to $A \in \sigma(\mathcal{P})$ (for q fixed).

(b) If anyone of (i), (ii) or (iii) in (a) holds for each $q \in \Gamma$, then f is $\mathbf{m}_q$ integrable in T with values in $\widetilde{X_q}$ and $\int_A f d\mathbf{m}_q = \gamma^{(q)}(A)$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$ and for $q \in \Gamma$. In that case, f is $\mathbf{m}$-integrable in T with values in X and

$$\int_A f d\mathbf{m} = \varprojlim \gamma^{(q)}(A)$$

for each $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Proof. (a) and the first part of (b) hold by Theorem 4.4 applied to $\mathbf{m}_q$, $q \in \Gamma$. The last part of (b) holds by Definition 12.1 and by Theorem 12.3.

Theorem 12.7. Let X, $\mathbf{m}$ and f be as in Theorem 12.6 and let $\mathcal{P}$ be a σ-ring $\mathcal{S}$. Then f is $\mathbf{m}$-integrable in T if and only if, for each $q \in \Gamma$, there exists a sequence $(f_n^{(q)})_{n=1}^\infty$ of bounded $\mathbf{m}_q$-measurable functions on T such that $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and such that $\lim_n \int_A f_n^{(q)} d\mathbf{m}_q$ exists in $\widetilde{X_q}$ for each $A \in \sigma(\mathcal{P})$. In that case, f is $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$ and $\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q$, $A \in \sigma(\mathcal{P})$ and the limit is uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed). Moreover,

$$\int_A f d\mathbf{m} = \underset{\longleftarrow}{\lim} \int_A f d\mathbf{m}_q, \ \in X \tag{12.7.1}$$

for $A \in \sigma(\mathcal{P})$. Moreover,

$$\int_T f d\mathbf{m} = \underset{\longleftarrow}{\lim} \int_{N(f)\setminus N_q} f d\mathbf{m}_q \in X \tag{12.7.2}$$

where N_q is as in Definition 12.1.

Proof. The condition is necessary by Definitions 4.1 and 12.1. Let $q \in \Gamma$. As $\mathcal{S}$ is a σ-ring, by Theorem 4.2 and by the last part of Theorem 3.5(v) bounded $\mathbf{m}_q$-measurable functions on T are $\mathbf{m}_q$-integrable in T. Let $\lim_n \int_A f_n^{(q)} d\mathbf{m}_q = x_A^{(q)} \in \widetilde{X_q}$ and let $x_A = \underset{\longleftarrow}{\lim}\, x_A^{(q)}$ for $A \in \sigma(\mathcal{P})$ and let $\lim_n \int_{N(f)\setminus N_q} f_n^{(q)} d\mathbf{m}_q = x_T^{(q)}$ and let $x_T = \underset{\longleftarrow}{\lim} x_T^{(q)}$, where $N_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(N_q) = 0$ such that $f\chi_{T\setminus N_q}$ is $\sigma(\mathcal{P})$-measurable. Let $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then $x_A \in \tilde{X}$. By Theorem 4.5, f is $\mathbf{m}_q$-integrable in T and $\int_A f d\mathbf{m}_q = x_A^{(q)}$, and $\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q$, $A \in \sigma(\mathcal{P})$ and the limit is uniform with respct to $A \in \sigma(\mathcal{P})$. Consequently, $x_A = \underset{\longleftarrow}{\lim} \int_A f d\mathbf{m}_q$ and hence by Definition 12.1, f is $\mathbf{m}$-integrable in T and $\int_A f d\mathbf{m} = x_A \in \tilde{X}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$. But by Theorem 12.3, $\int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$ and hence (12.7.1) and (12.7.2) hold.

Then by Theorems 12.2, 12.3 and 11.4(ii), f is $\mathbf{m}$-integrable in T, (12.7.1) and (12.7.2) hold and $x_A \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Theorem 12.8 (Closure theorem). Let X, $\mathbf{m}$, and f be as in Theorem 12.6. For each $q \in \Gamma$, let $(f_n^{(q)})_{n=1}^\infty \subset \mathcal{I}(\mathbf{m}_q)$ (Note that $\mathbf{m}_q : \mathcal{P} \to X \subset \widetilde{X_q}$). If $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$, then f is $\mathbf{m}$-measurable. Let $\gamma_n^{(q)}(A) = \int_A f_n^{(q)} d\mathbf{m}_q$ for $A \in \sigma(\mathcal{P})$. Then, for each $q \in \Gamma$, the following statements are equivalent:

(i) $\lim_n \gamma_n^{(q)}(A) = \gamma(A)$ exists in $\widetilde{X_q}$ for each $A \in \sigma(\mathcal{P})$.

(ii) $\gamma_n^{(q)}$, $n \in \mathbb{N}$, (q fixed), are uniformly σ-additive on $\sigma(\mathcal{P})$.

(iii) $\lim_n \gamma_n^{(q)}(A) = \gamma(A) \in \widetilde{X_q}$ exists uniformly with respect to $A \in \sigma(\mathcal{P})$ (q fixed).

If anyone of (i), (ii) or (iii) holds for each $q \in \Gamma$, then f is $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$ for each $q \in \Gamma$ and

$$\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q \text{ for } A \in \sigma(\mathcal{P}) \tag{12.8.1}$$

the limit being uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed). Moreover, f is $\mathbf{m}$-integrable in T with values in X and

$$\int_A f d\mathbf{m} = \underset{\longleftarrow}{\lim} \int_A f d\mathbf{m}_q \text{ for } A \in \sigma(\mathcal{P}) \cup \{T\}. \tag{12.8.2}$$

Proof. Clearly, f is $\mathbf{m}_q$-measurable for each $q \in \Gamma$ and hence f is $\mathbf{m}$-measurable. By hypothesis and by Theorem 4.8, (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii) for $q \in \Gamma$ and if anyone of (i),(ii) or (iii) holds for each $q \in \Gamma$, then by the last part of the said theorem, f is $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$ and (12.8.1) (with the limit being unform with respect to $A \in \sigma(\mathcal{P})$) holds for $q \in \Gamma$.

Consequently, by Definition 12.1 and by Theorem 12.3, f is $\mathbf{m}$-integrable in T with values in X and (12.8.2) holds.

Remark 12.9. If we replace the simple functions in Definition 4.1 by functions $\mathbf{m}_q$-integrable in T, then Theorem 12.8 above says that the process described in Definition 12.1 yields functions which are already in $\mathcal{I}(\mathbf{m})$ and no new functions are rendered $\mathbf{m}$-integrable in T. Hence Theorem 12.8 is called the closure theorem (compare Theorem 9 of [DP1]).

The following result generalizes Corollary 4.11 to quasicomplete lcHs-valued vector measures.

Corollary 12.10. Let X be a quasicomplete lcHs, $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and $\mathbf{m} : \mathcal{S} \to X$ be σ-additive. If X, $f_n^{(q)}$, $n \in \mathbb{N}$, q and f are as in Theorem 12.8 and if $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$ and if, for each $q \in \Gamma$,

$$\lim_{||\mathbf{m}||_q(A)\to 0} \int_A f_n^{(q)} d\mathbf{m}_q = 0,\ A \in \mathcal{S}$$

uniformly for $n \in \mathbb{N}$, then f is $\mathbf{m}_q$-integrable in T with values in $\widetilde{X_q}$ and

$$\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q,\ A \in \mathcal{S} \qquad (12.10.1)$$

the limit being uniform with respect to $A \in \mathcal{S}$ (for q fixed). Consequently, f is $\mathbf{m}$-integrable in T with values in X and

$$\int_A f d\mathbf{m} = \lim_{\longleftarrow} \int_A f d\mathbf{m}_q,\ A \in \mathcal{S} \cup \{T\}. \qquad (12.10.2)$$

Proof. Let $\gamma_n^{(q)}(\cdot) = \int_{(\cdot)} f_n^{(q)} d\mathbf{m}_q$, $n \in \mathbb{N}$. Then by hypothesis and by Corollary 4.11, f is $\mathbf{m}_q$-integrable in T and (12.10.1) holds (with the limit being uniform with respect to $A \in \mathcal{S}$). Then by Definition 12.1 and by Theorem 12.3, f is $\mathbf{m}$-integrable in T with values in X and (12.10.2) holds.

Remark 12.11 (**m-integrability of $\sigma(\mathcal{P})$-measurable functions with respect to a sequentially complete lcHs-valued m**).

Definition 12.1'. Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. A $\sigma(\mathcal{P})$-measurable function $f : T \to \mathbb{K}$ or $[-\infty, \infty]$ is said to be $\mathbf{m}$-integrable in T (in the sense of Bartle-Dunford-Schwartz) if there exists a sequence $(s_n) \subset \mathcal{I}_s$ such that $s_n(t) \to f(t)$ pointwise in T and such that $\lim_n \int_A s_n d\mathbf{m}$ exists in X for each $A \in \sigma(\mathcal{P})$. In that case, we define (BDS)$\int_A f d\mathbf{m} = \lim_n \int_A s_n d\mathbf{m}$, $A \in \sigma(\mathcal{P})$ and (BDS)$\int_T f d\mathbf{m} = \lim_n \int_{N(f)} s_n d\mathbf{m}$.

Theorem 12.2'. If X is a sequentially complete lcHs and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, then a $\sigma(\mathcal{P})$-measurable function $f : T \to \mathbb{K}$ or $[-\infty, \infty]$ is $\mathbf{m}$-integrable in T (with values in X) if and only if it is (KL) $\mathbf{m}$-integrable in T and in that case, (BDS)$\int_A f d\mathbf{m} =$ (KL) $\int_A f d\mathbf{m}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$.

Proof. If f is $\mathbf{m}$-integrable in T, then by hypothesis and by Proposition 2.13, it follows that f is (KL) $\mathbf{m}$-integrable in T and that (BDS)$\int_A f d\mathbf{m} =$ (KL) $\int_A f d\mathbf{m}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$. Conversely, if f is (KL) $\mathbf{m}$-integrable in T, take $(s_n) \subset \mathcal{I}_s$ as in the last part of Proposition 10.9. Then given $\epsilon > 0$ and $q \in \Gamma$, by Theorem 11.11', $q(\int_A s_n d\mathbf{m} - \int_A s_r d\mathbf{m}) \le q(\int_A s_n d\mathbf{m} -$ (KL) $\int_A f d\mathbf{m}) +$

$q((\mathrm{KL})\int_A f d\mathbf{m} - \int_A s_r d\mathbf{m}) < \epsilon$ for n, r sufficiently large. Thus $(\int_A s_n d\mathbf{m})_1^\infty$ is Cauchy in X and as X is sequentially complete, there exists $x_A \in X$ such that $x_A = \lim_n \int_A s_n d\mathbf{m}$ so that f is (BDS) $\mathbf{m}$-integrable in T and $(\mathrm{BDS})\int_A f d\mathbf{m} = x_A$. Then by LDCT for scalar measures, $x^*(x_A) = \int_A f d(x^*\mathbf{m})$ for $x^* \in X^*$ so that $(\mathrm{KL})\int_A f d\mathbf{m} = x_A = (\mathrm{BDS})\int_A f d\mathbf{m}$. Hence Theorem 12.2' holds. Consequently, when X is a sequentially complete lcHs, $\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ in Remark 11.15 is the same as the family of all $\sigma(\mathcal{P})$-measurable $\mathbf{m}$-integrable scalar functions on T and, for $f \in \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$, $(\mathrm{KL})\int_A f d\mathbf{m} = (\mathrm{BDS})\int_A f d\mathbf{m}$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$. Hence, **hereafter, when X is sequentially complete, we shall denote either of the integrals by $\int_A f d\mathbf{m}$ for $f \in \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ and for $A \in \sigma(\mathcal{P}) \cup \{T\}$.** Then in that case, Remark 12.5' is the same as Remark 12.5 in which $\mathcal{I}(\mathbf{m})$ is replaced by $\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ and X is sequentially complete, reference to Theorems 11.9, 11.11 and 11.13 is changed to Theorems 11.9', 11.11' and 11.13', respectively (in which no reference is made to Theorem 11.4 and reference to Theorem 11.8 remains unchanged) and reference to Corollaries 11.12 and 11.14 is changed to Corollaries 11.12' and 11.14', respectively.

Remark 12.12. If $\mathbf{m} : \mathcal{P} \to X$ is σ-additive and X is quasicomplete, by Theorem 11.4(ii), an $\mathbf{m}$-measurable function which is (KL) $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$, is (KL) $\mathbf{m}$-integrable in T with values in $\tilde{X}$, but in the light of Theorems 12.2 and 12.3, f is also $\mathbf{m}$-integrable in T and $(\mathrm{KL})\int_A f d\mathbf{m} = \int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P}) \cup \{T\}$. **Hence the concept of m-integrability of m-measurable functions is indispensable in the study of integration in quasicomplete lcHs. However, if f is $\sigma(\mathcal{P})$-measurable and X is sequentially complete, the concept of m-integrability is not needed. But in this case too we use the m-integrability in order to treat simultaneously integration of an m-measurable (resp. $\sigma(\mathcal{P})$-measurable) function with respect to a quasicomplete (resp. sequentially complete) lcHs-valued m.**

Remark 12.13. If X is quasicomplete and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, then the definitions of $\mathbf{m}$-integrability and the $\mathbf{m}$-integral for a $\sigma(\mathcal{P})$-measurable function f as given in Definitions 12.1 and 12.1' coincide due to the last part of Remark 10.16.

13. THE LOCALLY CONVEX SPACES $\mathcal{L}_p\mathcal{M}(\mathbf{m})$, $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ AND $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$, $1 \le p < \infty$

We generalize the results in Section 5 to an lcHs-valued σ-additive vector measure on $\mathcal{P}$ and this section plays a key role in the study of the $\mathcal{L}_p$-spaces when X is quasicomplete (resp. sequentially complete).

Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then $\mathbf{m}_q = \Pi_q \circ \mathbf{m} : \mathcal{P} \to X_q \subset \widetilde{X_q}$ is σ-additive for $q \in \Gamma$.

Definition 13.1. Let X be an lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $g : T \to \boldsymbol{K}$or $[-\infty, \infty]$ be $\mathbf{m}$-measurable, $1 \le p < \infty$ and $A \in \sigma(\mathcal{P})$. For $q \in \Gamma$, we define

$$(\mathbf{m}_q)^\bullet_p(g, A) = \sup\left\{\left|\int_A s d\mathbf{m}_q\right|_q^{\frac{1}{p}} : s \in \mathcal{I}_s,\ |s| \le |g|^p\ \mathbf{m}_q\text{-a.e. in } A\right\}$$

and

$$(\mathbf{m}_q)^\bullet_p(g, T) = \sup_{A \in \sigma(\mathcal{P})} (\mathbf{m}_q)^\bullet_p(A).$$

By Definition 5.4, for $A \in \sigma(\mathcal{P})_q$ with $A = B_q \cup N_q$, $B_q \in \sigma(\mathcal{P})$, $N_q \subset M_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(M_q) = 0$ we define

$$(\mathbf{m}_q)^\bullet_p(g, A) = (\mathbf{m}_q)^\bullet_p(g, B_q)$$

and it is well defined as shown in the paragraph preceding Remark 5.5.

Theorem 13.2. Let g, p and q be as in Definition 13.1. For $A \in \sigma(\mathcal{P})$,

$$\begin{aligned}(\mathbf{m}_q)^\bullet_p(g, A) &= \sup_{x^* \in U_q^o} \left(\int_A |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\ &= \sup_{x^* \in U_q^o} (\Psi_{x^*}\mathbf{m}_q)^\bullet_p(g, A) \\ &= \sup\left\{\left|(\text{KL})\int_A f d\mathbf{m}_q\right|^{\frac{1}{p}} : f \in \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}_q), |f| \le |g|^p\, \mathbf{m}_q\text{-a.e. in } A\right\} \\ &= \sup\left\{\left|(\text{KL})\int_A f d\mathbf{m}_q\right|_q^{\frac{1}{p}} : f \in \mathcal{I}(\mathbf{m}_q),\ |f| \le |g|^p\, \mathbf{m}_q\text{-a.e. in } A\right\}\end{aligned}$$

where $\mathcal{I}(\mathbf{m}_q)$ is as in Notation 3.2, $\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}_q)$ is as in Remark 11.15 for $\mathbf{m}_q$ and Ψ_{x^*} is as in Proposition 10.14.

Consequently, for $A \in \sigma(\mathcal{P})$

$$\left|(\text{KL})\int_A |f|^p d\mathbf{m}_q\right|_q^{\frac{1}{p}} \le (\mathbf{m}_q)^\bullet_p(f, A)$$

if f is an $\mathbf{m}_q$-measurable scalar function with $|f|^p \in I(\mathbf{m}_q)$. Moreover, for $A \in \sigma(\mathcal{P})$

$$\left|(\text{KL})\int_A f d\mathbf{m}_q\right|_q \le (\mathbf{m}_q)^\bullet_1(f, A) \qquad (13.2.1)$$

if $f \in \mathcal{I}(\mathbf{m}_q)$.

Proof. By Proposition 10.14(ii)(b), $\{\Psi_{x^*} : x^* \in U_q^o\}$ is a norm determining subset of the closed unit ball of $(X_q)^*$ and for $x^* \in U_q^o$, $x^*(\Pi_q \circ \mathbf{m}) = \Psi_{x^*}\mathbf{m}_q = x^*\mathbf{m}$ by (ii)(a) of the said proposition. Then by Lemma 5.2(ii) for $\mathbf{m}_q$ we have

$$(\Psi_{x^*}\mathbf{m}_q)^\bullet_p(g, A) = \left(\int_A |g|^p dv(\Psi_{x^*}\mathbf{m}_q)\right)^{\frac{1}{p}} = \left(\int_A |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \qquad (13.2.2)$$

for $x^* \in U_q^o$. Arguing as in the proof of Theorem 5.3 using Lemma 5.2(i) for $\mathbf{m}_q$, and by Proposition 10.14(ii)(b) and by (13.2.2) we have

$$\begin{aligned}(\mathbf{m}_q)^\bullet_p(g, A) &= \sup\left\{\left(\sup_{x^* \in U_q^o}\left|\int_A s\,d(\Psi_{x^*}\mathbf{m}_q)\right|_q^{\frac{1}{p}}\right) : s \in \mathcal{I}_s,\ |s| \le |g|^p\, \mathbf{m}_q\text{-a.e.in } A\right\} \\ &= \sup_{x^* \in U_q^o} (\Psi_{x^*}\mathbf{m}_q)^\bullet_p(g, A) = \sup_{x^* \in U_q^o}\left(\int_A |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}\end{aligned}$$

for $A \in \sigma(\mathcal{P})$.

Let $q \in \Gamma$. Note that $\mathcal{I}_s \subset \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}_q) \subset \mathcal{I}(\mathbf{m})$. Given $f \in \mathcal{I}(\mathbf{m}_q)$, by Proposition 2.10 there exists a sequence $(s_n^{(q)}) \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and $|s_n^{(q)}| \nearrow |f|$ $\mathbf{m}_q$-a.e. in T and hence by Corollary 3.9 applied to $\mathbf{m}_q : \mathcal{P} \to \widetilde{X_q}$ we have $|(KL)\int_A f d\mathbf{m}_q|_q = \lim_n |\int_A s_n^{(q)} d\mathbf{m}_q|_q$. This proves the other equalities in the first part. The second part is evident from the first.

Remark 13.3. The above proof is more general and elementary than that of Lemma II.2.2 of [KK]. Also compare Remark 5.5.

The following generalizes Theorem 5.6 to lcHs-valued σ-additive vector measures on $\mathcal{P}$.

Theorem 13.4. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Let $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable and let $|f|^p$ be (KL) $\mathbf{m}$-integrable in T. Let $\gamma(\cdot) = \text{(KL)} \int_{(\cdot)} |f|^p d\mathbf{m}$. Then $\gamma : \sigma(\mathcal{P}) \to X$ is σ-additive. For $q \in \Gamma$, $(\mathbf{m}_q)_p^\bullet(f, A) = (||\gamma||_q(A))^{\frac{1}{p}}$ for $A \in \sigma(\mathcal{P})$. Consequently, $(\mathbf{m}_q)_p^\bullet(f, T) = (||\gamma||_q(T))^{\frac{1}{p}} < \infty$ and $(\mathbf{m}_q)_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$.

Proof. By (ii) of Theorem 11.8, γ is σ-additive on $\sigma(\mathcal{P})$ and hence, $\gamma_q = \Pi_q \circ \gamma : \sigma(\mathcal{P}) \to X_q \subset \widetilde{X_q}$ is σ-additive. Then by Theorems 11.8(iii)(a) and 13.2, $(\mathbf{m}_q)_p^\bullet(f, A) = (||\gamma||_q(A))^{\frac{1}{p}}$, $A \in \sigma(\mathcal{P})$. By Propositions 2.2 and 2.3 other results hold.

Theorem 13.5. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive, $1 \le p < \infty$ and $f : T \to \boldsymbol{K}$ or $[-\infty, \infty]$.

(i) If X is quasicomplete, f is $\mathbf{m}$-measurable and $c_0 \not\subset \widetilde{X_q}$ for each $q \in \Gamma$, then $|f|^p$ is $\mathbf{m}$-integrable in T with values in X if and only if $(\mathbf{m}_q)_p^\bullet(f, T) < \infty$ for each $q \in \Gamma$.

(ii) If f is $\sigma(\mathcal{P})$-measurable and $c_0 \not\subset X$, then $|f|^p$ is $\mathbf{m}$-integrable in T (and hence (KL) $\mathbf{m}$-integrable in T) if and only if $(\mathbf{m}_q)_p^\bullet(f, T) < \infty$ for each $q \in \Gamma$.

Proof. (i) Since X is quasicomplete by hypothesis, the condition is necessary by Theorems 13.4 and 12.2 (note that this part holds for any quasicomplete lcHs). Conversely, let $(\mathbf{m}_q)_p^\bullet(f, T) < \infty$ for each $q \in \Gamma$. For $x^* \in X^*$, let $q_{x^*}(x) = |x^*(x)|$, $x \in X$. Then by Theorem 13.2 and by hypothesis,

$$\int_A |f|^p dv(x^*\mathbf{m}) \le (\mathbf{m}_{q_x^*})_p^\bullet(f, T) < \infty \qquad (13.5.1)$$

for each $A \in \sigma(\mathcal{P})$.

Let $q \in \Gamma$. Then by Proposition 10.9 there exists a sequence $(s_n^{(q)}) \subset \mathcal{I}_s$ such that $0 \le s_n^{(q)} \nearrow |f|^p$ $\mathbf{m}_q$-a.e. in T. If $u_n^{(q)} = s_n^{(q)} - s_{n-1}^{(q)}$ for $n \ge 1$, where $s_0^{(q)} = 0$, then as in the proof of Theorem 5.8 we have $\sum_1^\infty \int_A |u_n^{(q)}| dv(y^*\mathbf{m}_q) = \int_A |f|^p dv(y^*\Pi_q\mathbf{m}) < \infty$ by (13.5.1) for $y^* \in (X_q)^*$, since $y^*\Pi_q \in X^*$. As $c_0 \not\subset \widetilde{X_q}$ by hypothesis, arguing as in the proof of Theorem 5.8 we observe that there exists a vector $x_A \in \widetilde{X_q}$ such that $x_A = \lim_n \int_A s_n^{(q)} d\mathbf{m}_q$ and this holds for each $A \in \sigma(\mathcal{P})$. Then by Definition 4.1, $|f|^p$ is $\mathbf{m}_q$-integrable in T. Since q is arbitrary in Γ and since X is quasicomplete by hypothesis, by Definition 12.1 and by Theorem 12.3, $|f|^p$ is $\mathbf{m}$-integrable in T with values in X.

(ii) The condition is necessary by Theorem 13.4 (this part holds for any lcHs X). Conversely, let f be $\sigma(\mathcal{P})$-measurable and $c_0 \not\subset X$ with $(\mathbf{m}_q)_p^\bullet(f, T) < \infty$ for each $q \in \Gamma$. Then by the last part of Proposition 10.9 there exists $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $0 \le s_n \nearrow |f|^p$ pointwise in T. Then arguing as in the proof of Theorem 5.8 and using (13.5.1), we have $\sum_1^\infty |x^*(\int_A u_n d\mathbf{m})| < \infty$ for $A \in \sigma(\mathcal{P})$ and for each $x^* \in X^*$, where $u_n = s_n - s_{n-1}$ for $n \ge 1$ and $s_0 = 0$. Then by Theorem 4 of [Tu] there exists a vector $x_A \in X$ such that $x_A = \sum_1^\infty \int_A u_n d\mathbf{m} = \lim_n \int_A s_n d\mathbf{m}$ for $A \in \sigma(\mathcal{P})$

and hence by Definition 12.1' in Remark 12.11, $|f|^p$ is $\mathbf{m}$-integrable in T. Moreover, by LDCT for scalar measures we have

$$x^*(x_A) = \lim_n \int_A s_n d(x^*\mathbf{m}) = \int_A |f|^p d(x^*\mathbf{m}),\ A \in \sigma(\mathcal{P})$$

for each $x^* \in X^*$ and hence $|f|^p$ is (KL) $\mathbf{m}$-integrable in T.

Definition 13.6. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. We define $\mathcal{L}_p\mathcal{M}(\mathbf{m}) = \{f : T \to \boldsymbol{K}, f\ \mathbf{m}\text{-measurable with } (\mathbf{m}_q)^\bullet_p(f,T) < \infty \text{ for each } q \in \Gamma\}$; $\mathcal{I}_p(\mathbf{m}) = \{f : T \to \boldsymbol{K}, f\ \mathbf{m}\text{-measurable and } |f|^p \text{ (KL)}$
$\mathbf{m}\text{-integrable in } T\}$ and $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p\mathcal{M}(\mathbf{m}) \bigcap \mathcal{I}_p(\mathbf{m})$. Let $\mathcal{M}(\sigma(\mathcal{P})) = \{f : T \to \boldsymbol{K}, f\ \sigma(\mathcal{P})\text{-measurable}\}$. Then we define:
$\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p\mathcal{M}(\mathbf{m}) \bigcap \mathcal{M}(\sigma(\mathcal{P}))$; $\mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{I}_p(\mathbf{m}) \bigcap \mathcal{M}(\sigma(\mathcal{P}))$ and $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) \bigcap \mathcal{M}(\sigma(\mathcal{P}))$. (Note that in the case of Banach spaces, (KL) $\mathbf{m}$-integrablity and $\mathbf{m}$-integrability coincide and hence $\mathcal{I}_p(\mathbf{m})$ and $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ are the same as those given in 5.9 when X is a Banach space.)

Theorem 5.10 is generalized to lcHs-valued $\mathbf{m}$ as follows.

Theorem 13.7. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Then:

(i) $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$.
(ii) If X is quasicomplete, then $\mathcal{I}_1(\mathbf{m}) = \mathcal{I}(\mathbf{m})$ (see Notation 11.6).
(iii) If X is sequentially complete, then $\mathcal{I}_1(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$. (See Remark 11.15.)
(iv) If X is quasicomplete and $c_0 \not\subset \widetilde{X_q}$ for each $q \in \Gamma$, then $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{M}(\mathbf{m})$.
(v) If $c_0 \not\subset X$, then $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$.

Proof. (i) holds by Theorem 13.4. (ii) (resp. (iii)) is due to Theorem 11.9(iii) (resp. Theorem 11.9'(iii) in Remark 11.15). (iv) and (v) are due to (i) and (ii) of Theorem 13.5.

If X is an lcHs and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, then by Definition 13.1 and by Theorem 13.2, it is clear that **Theorems 5.11-5.13 hold for m-measurable functions on T and hence for $\sigma(\mathcal{P})$-measurable functions on T with values in $\boldsymbol{K}$ or in $[-\infty, \infty]$ and for $q \in \Gamma$, if we replace $\mathbf{m}^\bullet_p(g, \cdot)$ by $(\mathbf{m}_q)^\bullet_p(g, \cdot)$, $||\mathbf{m}||$ by $||\mathbf{m}||_q$, $\mathbf{m}^\bullet_p(ag, \cdot)$ by $(\mathbf{m}_q)^\bullet_p(ag, \cdot)$ and $\mathbf{m}^\bullet_p(f+g, \cdot)$ by $(\mathbf{m}_q)^\bullet_p(f+g, \cdot)$. Hence, when these results are used, we simply refer like by Theorem 5.11 for $\mathbf{m}_q$, etc.**

(i)-(iv) of the following theorem generalize Theorem 5.14 to an lcHs-valued $\mathbf{m}$ on $\mathcal{P}$ with Notation 5.15 being suitably interpreted here.

Theorem 13.8. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive, $1 \leq p < \infty$ and $\xi^{(p)}_\Gamma = \{(\mathbf{m}_q)^\bullet_p : q \in \Gamma\}$. Then:

(i) $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$) is a vector space over $\boldsymbol{K}$
(ii) $\xi^{(p)}_\Gamma$ is a family of seminorms on $\mathcal{L}_p\mathcal{M}(\mathbf{m})$. If $\tau^{(p)}_{\mathbf{m}}$ is the locally convex topology generated by the family $\xi^{(p)}_\Gamma$ on $\mathcal{L}_p\mathcal{M}(\mathbf{m})$, then by $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ we mean the locally convex space $(\mathcal{L}_p\mathcal{M}(\mathbf{m}), \tau^{(p)}_{\mathbf{m}})$.
(iii) If X is quasicomplete, then $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ is a linear subspace of $\mathcal{L}_p\mathcal{M}(\mathbf{m})$. In that case, by $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ we mean the locally convex space $(\mathcal{L}_p\mathcal{I}(\mathbf{m}),$
$\tau^{(p)}_{\mathbf{m}}|_{\mathcal{L}_p\mathcal{I}(\mathbf{m})})$.

(iv) If X is sequentially complete, then $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$ is a linear subspace of $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$. In that case, by $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$ we mean the locally convex space $(\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m}), \tau_{\mathbf{m}}^{(p)}|_{\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})})$.

(v) For $f,g \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$, we write $f \sim g$ if $f = g$ $\mathbf{m}$-a.e. in T (see Definition 10.4). Then $'\sim'$ is an equivalence relation and we denote $\mathcal{L}_p\mathcal{M}(\mathbf{m})/\sim$ by $L_p\mathcal{M}(\mathbf{m})$; when X is quasi-complete, $\mathcal{L}_p\mathcal{I}(\mathbf{m})/\sim$ by $L_p\mathcal{I}(\mathbf{m})$ and when X is sequentially complete, $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})/\sim$ by $L_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$. Then $f \sim g$ if and only if $(\mathbf{m}_q)_p^\bullet(f-g,T) = 0$ for all $q \in \Gamma$. If $\mathcal{F} = \mathcal{L}_p\mathcal{M}(\mathbf{m})$(resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$) and $N_{\mathcal{F}} = \{f \in \mathcal{F} : f \sim 0\}$, then $N_{\mathcal{F}}$ is a closed linear subspace of $\mathcal{F}$. Hence $L_p\mathcal{M}(\mathbf{m})$, $L_p\mathcal{I}(\mathbf{m})$, $L_p\mathcal{M}(\sigma(P),\mathbf{m})$ and $L_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$ are lcHs.

Proof. By Theorems 5.11(ii) and 5.13(ii) for $\mathbf{m}_q$, $q \in \Gamma$ and by Definition 13.6, (i) and (ii) hold. Arguing as in the proof of Theorem 5.14 and using Theorems 11.8(iv) and 11.9(iii) (resp. and 11.9'(iii) in Remark 11.15) for $\mathbf{m}_q$, $q \in \Gamma$ in place of (iv) and (vii) of Theorem 3.5 and by Theorem 13.7(i), (iii) (resp. (iv)) holds.

(v) Clearly, $'\sim'$ is an equivalence relation. By Definition 10.4, $f \sim g$ if and only if there exists $M \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(M) = 0$ for all $q \in \Gamma$ such that $f(t) - g(t) = 0$ fot $t \in T\backslash M$. Let $f \sim g$ and let $q \in \Gamma$. Then by Theorem 13.2

$$((\mathbf{m}_q)_p^\bullet(f-g,T))^p = \sup_{x^* \in U_q^o} \int_T |f-g|^p dv(x^*\mathbf{m}) = \sup_{x^* \in U_q^o} \int_M |f-g|^p dv(x^*\mathbf{m}) = 0$$

since by Theorem 10.14(ii)(c), $v(x^*\mathbf{m})(M) \le ||\mathbf{m}||_q(M) = 0$ for $x^* \in U_q^o$. Hence $(\mathbf{m}_q)_p^\bullet(f-g,T) = 0$ for all $q \in \Gamma$. Conversely, if $(\mathbf{m}_q)_p^\bullet(f-g,T) = 0$ for all $q \in \Gamma$, let $A = \{t \in T : f(t) - g(t) \neq 0\}$. For $q \in \Gamma$, A is of the form $A = B_q \cup N_q$, where $B_q \in \sigma(\mathcal{P})$, $N_q \subset M_q \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_q(M_q) = 0$ if f,g are $\mathbf{m}$-measurable and $A = B_q \in \sigma(\mathcal{P})$ if f,g are $\sigma(\mathcal{P})$-measurable. Then by hypothesis and by Theorem 13.2, $((\mathbf{m}_q)_p^\bullet(f-g,B_q))^p = \sup_{x^* \in U_q^o} \int_{B_q} |f-g|^p dv(x^*\mathbf{m}) = 0$ and hence $v(x^*\mathbf{m})(B_q) = 0$ for $x^* \in U_q^0$. Then by Proposition 10.14(ii)(c) we have $||\mathbf{m}||_q(A) = ||\mathbf{m}||_q(B_q) = 0$ (in both the cases). Since $q \in \Gamma$ is arbitrary, it follows that A is $\mathbf{m}$-null. Using this characterization, it is easy to check $N_{\mathcal{F}}$ is a closed linear subspace of $\mathcal{F}$ and hence the last part of (v) holds.

Definition 13.9. Definition 5.16 is suitably modified for $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable) scalar functions on T to define (i) convergence in measure (resp. Cauchy in measure) in T or in $A \in \sigma(\mathcal{P})$ with respect to $\mathbf{m}_q$ for $q \in \Gamma$; (ii) almost uniform convergence (resp. Cauchy for almost uniform convergence) in T or in $A \in \sigma(\mathcal{P})$ with respect to $\mathbf{m}_q$ for $q \in \Gamma$; and (iii) convergence to f in (mean^p) with respect to $\mathbf{m}_q$ for $q \in \Gamma$ and for $1 \le p < \infty$.

Then for functions in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. for $\mathbf{m}$-measurable scalar functions on T) (iv)-(viii) of Theorem 5.18 (resp. Theorem 5.19) hold with $\mathbf{m}$ being replaced by $\mathbf{m}_q$, $q \in \Gamma$. Hence when these results are used, we simply refer like by (vii) of Theorem 5.18 for $\mathbf{m}_q$, etc.

Remark 13.10. If the topology τ of X is generated by another family Γ_1 of seminorms on X, then it is easy to show that the topology $\tau_{\mathbf{m}}^{(p)}$ is the same as the locally convex topology generated by $\xi_{\Gamma_1}^{(p)}$ (see Definition 13.8(ii)) on the space $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ for $1 \le p < \infty$.

14. COMPLETENESS OF $\mathcal{L}_p\mathcal{M}(\mathbf{m}), \mathcal{L}_p\mathcal{I}(\mathbf{m}),\ \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$ AND $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$, X A FRECHÉT SPACE

If X is a Frechét space and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, then we show that $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$ are complete pseudo-metrizable locally convex spaces so that $L_p\mathcal{M}(\mathbf{m})$ and $L_p(\sigma(\mathcal{P}),\mathbf{m})$ are Frechét spaces for $1 \le p < \infty$. Similar to the case of Banach space-valued measures in [P4], we introduce two new locally convex spaces $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_P(\sigma(\mathcal{P}),\mathbf{m})$) for $1 \le p < \infty$, when X is quasicomplete (resp. sequentially complete) and show that (i) $\mathcal{L}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m})$ (resp. (i') $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$) and (ii) $\mathcal{L}_p(m)$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. (ii') $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$). Consequently, $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}),\mathbf{m})$) is a Frechét space for $1 \le p < \infty$ whenever X is a Frechét space. See Remarks 14.9 and 14.10 for further information.

The following theorem is obtained by adapting the proof of Theorem 6.1 with the use of Theorem 13.2 in place of Theorem 5.3.

Theorem 14.1. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Let $f, f_n, n \in \mathbb{N}$, be $\mathbf{m}$-measurable on T with values in $\mathbb{K}$ or in $[-\infty,\infty]$ and let $q \in \Gamma$. Then:

(i) **(The Fatou property of $(\mathbf{m}_q)^\bullet_p(\cdot, A)$).** If $|f_n| \nearrow |f|$ $\mathbf{m}_q$-a.e. in $A \in \widetilde{\sigma(\mathcal{P})}_q$, then

$$(\mathbf{m}_q)^\bullet_p(f, A) = \sup_n (\mathbf{m}_q)^\bullet_p(f_n, A) = \lim_n (\mathbf{m}_q)^\bullet_p(f_n, A).$$

(ii) **(Generalizad Fatou's lemma).** For $A \in \widetilde{\sigma(\mathcal{P})}_q$,

$$(\mathbf{m}_q)^\bullet_p(\liminf_{n\to\infty} |f_n|, A) \le \liminf_{n\to\infty} (\mathbf{m}_q)^\bullet_p(f_n, A).$$

Theorem 14.2. Let X be a Frechét space with its topology generated by the seminorms $(q_n)_1^\infty$, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Then:

(i) $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is a complete pseudo-metrizable locally convex space so that $L_p\mathcal{M}(\mathbf{m})$ (resp. $L_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is a Frechét space.

(ii) If $c_0 \not\subset \widetilde{X_q}$ for each $q \in \Gamma$ (resp. if $c_0 \not\subset X$), then $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$) is a complete pseudo-metrizable locally convex space so that $L_p\mathcal{I}(\mathbf{m})$ (resp. $L_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$) is a Frechét space.

Proof. (i) By Theorem 13.8 and Remark 13.10, $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is pseudo-metrizable with respect to $\tau_{\mathbf{m}}^{(p)}$ which is generated by $((\mathbf{m}_{q_n})^\bullet_p(\cdot, T))_{n=1}^\infty$. Suppose $(f_r)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $(f_r)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is Cauchy in $\tau_{\mathbf{m}}^{(p)}$. Proceeding as in the proof of Theorem 6.3(i), we can choose a subsequence $(f_{1,r})_{r=1}^\infty$ of $(f_r)_1^\infty$ such that $(\mathbf{m}_{q_1})^\bullet(f_{1,k+1} - f_{1,k}, T) < \frac{1}{2^k}$, $k \in \mathbb{N}$. Let

$$g_{1,k} = \sum_{r=1}^{k} |f_{1,r+1} - f_{1,r}|$$

and

$$g_1 = \sum_{r=1}^{\infty} |f_{1,r+1} - f_{1,r}|.$$

Then $g_{1,k}, k \in \mathbb{N}$ and g_1 are $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable) and $g_{1,k} \nearrow g_1$. Moreover, by Theorem 5.13(ii) for $\mathbf{m}_{q_1}$, $(\mathbf{m}_{q_1})^\bullet_p(g_{1,k}, T) < 1$ for all $k \in \mathbb{N}$ so that by Theorem 14.1(i),

$(\mathbf{m}_{q_1})^\bullet_p(g_1,T) = \sup_k(\mathbf{m}_{q_1})^\bullet_p(g_{1,k},T) \le 1$. Then by Theorem 5.12(ii) for $\mathbf{m}_{q_1}$, g_1 is finite $\mathbf{m}_{q_1}$-a.e. in T and hence there exists $N_1 \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_{q_1}(N_1) = 0$ such that g_1 is finite in $T\backslash N_1$. Arguing similarly with $(f_{1,r})_{r=1}^\infty$ and $\mathbf{m}_{q_2}$, there exist a subsequence $(f_{2,r})_{r=1}^\infty$ of $(f_{1,r})_{r=1}^\infty$ and a set $N_2 \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_{q_2}(N_2) = 0$ such that

$$g_2 = \sum_{r=1}^\infty |f_{2,r+1} - f_{2,r}|$$

is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable), $(\mathbf{m}_{q_2})^\bullet_p(g_2,T) \le 1$ and g_2 is finite in $T\backslash N_2$. Proceeding successively, in the $(k+1)^{th}$-step we shall have a subsequence $(f_{k+1,r})_{r=1}^\infty$ of $(f_{k,r})_{r=1}^\infty$ and a set $N_{k+1} \in \sigma(\mathcal{P})$ with $||\mathbf{m}||_{q_{k+1}}(N_{k+1}) = 0$ such that

$$g_{k+1} = \sum_{r=1}^\infty |f_{k+1,r+1} - f_{k+1,r}|$$

is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable), $(\mathbf{m}_{q_{k+1}})^\bullet_p(g_{k+1},T) \le 1$ and g_{k+1} is finite in $T\backslash N_{k+1}$. Then the diagonal sequence $(f_{k,k})_{k=1}^\infty$ is a subsequence of each subsequence $(f_{k,r})_{r=1}^\infty$ starting with the term $f_{k,k}$; and if

$$g = \sum_{k=1}^\infty |f_{k+1,k+1} - f_{k,k}|,$$

then g is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable). Moreover, $(\mathbf{m}_{q_n})^\bullet_p(g,T) < \infty$ for all n since $(\mathbf{m}_{q_n})^\bullet_p(g_n,T) \le 1$ and $(\mathbf{m}_{q_n})^\bullet_p(f_{i,i},T) < \infty$ for $i = 1,2,...,n$.

Let $N = \bigcap_1^\infty N_n$. Then $N \in \sigma(\mathcal{P})$ and $||\mathbf{m}||_{q_n}(N) = 0$ for all n. Given $t \in T\backslash N$, there exists n_0 such that $t \in T\backslash N_{n_0}$ so that $g_{n_0}(t)$ is finite. As each f_r is finite valued in T and as all but a finite number of terms of $(f_{k,k})_1^\infty$ belong to $(f_{n_0,r})_{r=1}^\infty$, it follows that $g(t)$ is finite. Hence g is finite in $T\backslash N$. By Remark 10.5 and by the hypothesis that $(q_n)_1^\infty$ generate the topology of X, it follows that N is $\mathbf{m}$-null and hence g is finite $\mathbf{m}$-a.e. in T.

Then the series

$$\sum_{k=1}^\infty (f_{k+1,k+1} - f_{k,k})$$

is absolutely convergent in $T\backslash N$. As $(f_{k,k})_1^\infty$ is a subsequence of $(f_r)_1^\infty$, let $f_{n_k} = f_{k,k}$. Let $h_k = f_{n_k}$ for $k \ge 1$ and $h_0 = 0$. Define

$$f(t) = \begin{cases} \sum_{k=0}^\infty (h_{k+1}(t) - h_k(t)) = \lim_k h_k(t), & \text{for } t \in T\backslash N \\ 0 & \text{otherwise.} \end{cases}$$

Then f is $\mathbb{K}$-valued in T, is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable) and is $\mathbf{m}$-a.e. pointwise limit of $(h_k)_1^\infty$. Let $\epsilon > 0$ and let n_0 be given. By hypothesis, there exists r_0 such that $(\mathbf{m}_{q_{n_0}})^\bullet_p(f_r - f_\ell, T) < \epsilon$ for $r, \ell \ge r_0$ so that $(\mathbf{m}_{q_{n_0}})^\bullet_p(h_k - h_\ell, T) < \epsilon$ for $n_k, n_\ell \ge r_0$. Let $F = \bigcup_1^\infty N(h_k)\backslash N$. Then $F \in \widetilde{\sigma(\mathcal{P})}_{q_{n_o}}$ (resp. $F \in \sigma(\mathcal{P})$) and $N(f) \subset F$. Then, arguing as in the proof of Theorem 6.3(i) and using Theorem 14.1(ii) in place of Theorem 6.1(ii), we have $(\mathbf{m}_{q_{n_0}})^\bullet_p(f - h_k, T) < \epsilon$ for $n_k \ge r_0$. Then by the triangular inequality, $(\mathbf{m}_{q_{n_0}})^\bullet_p(f,T) < \infty$. As n_0 is arbitrary in $\mathbb{N}$, we conclude that $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $f \in L_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$). Moreover, $\lim_k(\mathbf{m}_{q_{n_0}})^\bullet_p(f - h_k, T) = 0$. Consequently,

$$(\mathbf{m}_{q_{n_0}})^\bullet_p(f - f_r, T) \le (\mathbf{m}_{q_{n_0}})^\bullet_p(f_r - f_{n_k}, T) + (\mathbf{m}_{q_{n_0}})^\bullet_p(f - h_k, T) \to 0$$

as $n_k, r \to \infty$ and hence $\lim_r(\mathbf{m}_{q_{n_0}})^\bullet_p(f_r - f, T) = 0$. As n_0 is arbitrary in $\mathbb{N}$ it follows that $f_r \to f$ in $\tau_{\mathbf{m}}^{(p)}$. Hence $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is complete. Consequently, by the last part of

Theorem 13.8(v) and by Theorem 5.7 of Ch. 2 and Lemma 11.3 of Ch. 3 of [KN], $L_p\mathcal{M}(\mathbf{m})$ (resp. $L_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is a Frechét space.

(ii) is immediate from (i) and from (iv) (resp. (v)) of Theorem 13.7.

The following corollary is immediate from the proof of the last part of Theorem 14.2(i) and it holds even if X is not a Frechét space.

Corollary 14.3. Let X be an lcHs, $\mathbf{m}: \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. If $(f_n)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) is Cauchy in $(\mathbf{m}_q)_p^\bullet(\cdot,T)$ for each $q \in \Gamma$, and if there exist a scalar function f on T, a subsequence (f_{n_k}) of (f_n) and an $\mathbf{m}$-null set $N \in \sigma(\mathcal{P})$ such that $f_{n_k} \to f$ in $T\backslash N$ (resp. and such that $f(t) = 0$ for $t \in N$), then f is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable), $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $f \in \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) and $\lim_n(\mathbf{m}_q)_p^\bullet(f_n - f, T) = 0$ for each $q \in \Gamma$.

Similar to the second part of Definition 6.5 we give the following

Definition 14.4. Let X be an lcHs, $\mathbf{m}: \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Let $\mathcal{L}_p(\mathbf{m}) = \{f \in \mathcal{L}_p\mathcal{M}(\mathbf{m}) : (\mathbf{m}_q)_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma\}$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p(\mathbf{m}) \bigcap \mathcal{M}(\sigma(\mathcal{P})$ (see Definition 13.6)) provided with the relative topology induced by $\tau_{\mathbf{m}}^{(p)}$. If $'\sim'$ is as given in Theorem 13.8(v), then $\mathcal{L}_p(\mathbf{m})/\sim$ is denoted by $L_p(\mathbf{m})$ and $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})/\sim$ by $L_p(\sigma(\mathcal{P}),\mathbf{m})$. (Note that in the light of Theorem 4.2, the above definition of $\mathcal{L}_p(\mathbf{m})$ is the same as that in Definition 6.5 when X is a Banach space.)

Since $(\mathbf{m}_q)_p^\bullet(f,\cdot)$ is subadditive and positively homogeneous, $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$) is a linear subspace of $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. of $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m})$) so that $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$) is a locally convex space and $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}),\mathbf{m})$) is an lcHs.

The following result generalizes Theorem 6.7 to quasicomplete lcHs-valued $\mathbf{m}$.

Theorem 14.5. Let X be a quasicomplete lcHs, $\mathbf{m}: \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Let f be an $\mathbf{m}$-measurable scalar function on T such that $(\mathbf{m}_q)_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. Then $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$, and hence $f \in \mathcal{L}_p(\mathbf{m})$. Moreover, $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$.

Proof. Let $(\mathbf{m}_q)_p^\bullet(f,\cdot)$ be continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. Then by Theorem 6.7, $(\mathbf{m}_q)_p^\bullet(f,T) < \infty$ and $|f|^p$ is $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$, where $\mathbf{m}_q$ has values in $\widetilde{X_q}$. Then, in particular, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and moreover, by Theorems 4.2, 11.4(ii), 12.2 and 12.3, $|f|^p$ is $\mathbf{m}$-integrable in T with values in X so that $|f|^p \in I(\mathbf{m})$. Consequently, by Theorem 13.7(i), $f \in \mathcal{L}_p\mathcal{I}(\mathbf{m})$. This also proves that $\mathcal{L}_p(\mathbf{m}) \subset \mathcal{L}_p\mathcal{I}(\mathbf{m})$.

Conversely, let $f \in \mathcal{L}_p\mathcal{I}(\mathbf{m})\ (= \mathcal{I}_p(\mathbf{m})$ by Theorem 13.7(i)). Then, by Remark 12.5, $|f|^p$ is $\mathbf{m}$-integrable in T with values in X and hence by Theorem 13.4, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $(\mathbf{m}_q)_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. Hence $f \in \mathcal{L}_p(\mathbf{m})$. Therefore, $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$.

The following theorem is an analogue of Theorem 4.8 for sequentially complete lcHs and plays a key role in the study of $\mathcal{L}_p$-spaces of a sequentially complete lcHs-valued $\mathbf{m}$ (see Theorem 14.7 below).

Theorem 14.6. Let X be a sequentially complete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $(f_n)_1^\infty \subset \mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ (see Remarks 11.15 and 12.11) and f be a $\sigma(\mathcal{P})$-measurable scalar function on T. If $f_n \to f$ $\mathbf{m}$-a.e. in T and if $\gamma_n(\cdot) = (\mathrm{KL}) \int_{(\cdot)} f_n d\mathbf{m}$, $n \in \mathbb{N}$, then the following are equivalent:

(i) $\lim_n \Pi_q \circ \gamma_n(A)$ exists in $\widetilde{X_q}$ for each $A \in \sigma(\mathcal{P})$ and for each $q \in \Gamma$.
(ii) $\Pi_q \circ \gamma_n$, $n \in \mathbb{N}$, are uniformly σ-additive on $\sigma(\mathcal{P})$ for each q fixed in Γ.

If anyone of (i) or (ii) holds, then $\lim_n \gamma_n(A) = \gamma(A)$ (say) exists in X, $\lim_n \Pi_q \circ \gamma_n(A) = \Pi_q \circ \gamma(A) \in X_q$, $\gamma : \sigma(\mathcal{P}) \to X$ is σ-additive, f is $\mathbf{m}$-integrable in T and

$$\gamma(A) = \int_A f d\mathbf{m} = (\mathrm{KL}) \int_A f d\mathbf{m} = \lim_n (\mathrm{KL}) \int_A f_n d\mathbf{m},\ A \in \sigma(\mathcal{P}).$$

Proof. By Theorem 11.8(ii), γ_n, $n \in \mathbb{N}$, are σ-additive on $\sigma(\mathcal{P})$ and hence $\Pi_q \circ \gamma_n$, $n \in \mathbb{N}$, are σ-additive on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. Then, as $\lim_n \Pi_q \circ \gamma_n(A) \in \widetilde{X_q}$, by VHSN (Theorem 2.5), (i) implies (ii).

Conversely, let (ii) hold. Then by Theorem 4.8, $\lim_n \Pi_q \circ \gamma_n(A) = \widetilde{x_{A,q}} \in \widetilde{X_q}$ exists uniformly with respect to $A \in \sigma(\mathcal{P})$ (for $q \in \Gamma$ fixed). Thus, given $\epsilon > 0$, there exists $n_0(q) \in \mathbb{N}$ such that $|\Pi_q \circ \gamma_n(A) - \Pi_q \circ \gamma_r(A)|_q = q(\gamma_n(A) - \gamma_r(A)) < \epsilon$ for $n, r \geq n_0(q)$. Since q is arbitrary in Γ, it follows that $(\gamma_n(A))_{n=1}^\infty$ is τ-Cauchy in X. Consequently, as X is sequentially complete, there exists $x_A \in X$ such that $\gamma_n(A) \to x_A$ in τ. Then

$$|\widetilde{x_{A,q}} - \Pi_q(x_A)|_q \leq |\widetilde{x_{A,q}} - \Pi_q \circ \gamma_n(A)|_q + |\Pi_q(x_A - \gamma_n(A))|_q \to 0$$

as $n \to \infty$ and hence $\widetilde{x_{A,q}} = \Pi_q(x_A) \in X_q$. Thus $\lim_n \Pi_q \circ \gamma_n(A) = \Pi_q(x_A) \in X_q$ for each $q \in \Gamma$. Hence, particularly, (i) holds.

Let (i) or (ii) hold. Then (ii) holds and as shown above $\lim_n \gamma_n(A) = x_A \in X$ exists in τ for each $A \in \sigma(\mathcal{P})$.

Now, for $x^* \in X^*$,

$$\begin{aligned} x^*\gamma(A) &= \lim_n x^*\gamma_n(A) = \lim_n x^*\Big(\int_A f_n d\mathbf{m}\Big) \\ &= \lim_n \int_A f_n d(x^*\mathbf{m}) \qquad \text{(by Theorem 11.8(v))} \\ &= \lim_n \int_A f d(x^*\mathbf{m}) \qquad \text{(by Proposition 2.13)} \end{aligned}$$

and hence f is (KL) $\mathbf{m}$-integrable in T so that f is $\mathbf{m}$-integrable in T by Theorem 12.2' (see Remark 12.11) and $\gamma(A) = (\mathrm{KL}) \int_A f d\mathbf{m} = \int_A f d\mathbf{m} \in X$ for $A \in \sigma(\mathcal{P})$. Then by Theorem 11.8(ii), $\gamma : \sigma(\mathcal{P}) \to X$ is σ-additive.

For a sequentially complete lcHs-valued $\mathbf{m}$, (i) (resp. (ii)) in the following theorem is an analogue of Theorem 14.5 (resp. of Theorem 11.4 combined with Remark 11.5). The following theorem plays a key role in Section 15 for generalizing the results in Sections 7, 8 and 9 to σ-additive vector measures with values in such spaces.

Theorem 14.7. Let X be a sequentially complete lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Let f be a $\sigma(\mathcal{P})$-measurable scalar function on T. Then:

(i) If $(\mathbf{m}_q)_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$, then $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ as well as $|f|^p$ is $\mathbf{m}$-integrable in T with values in X. Moreover, $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$.

(ii) $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ if and only if $|f|^p$ is $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$. Consequently, $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) = \bigcap_{q\in\Gamma} \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$ where $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q) = \mathcal{M}(\sigma(\mathcal{P})) \bigcap \mathcal{L}_p(\mathbf{m}_q)$.

Proof. (i) Let $(\mathbf{m}_q)^\bullet_p(f, \cdot)$ be continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. Then one can prove that $f \in \mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$ by appealing to Theorem 6.7 for $\mathbf{m}_q, q \in \Gamma$, as in the proof of Theorem 14.5. As $|f|^p$ is $\sigma(\mathcal{P})$-measurable, by the last part of Proposition 10.9 there exists a sequence $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $0 \le s_n \nearrow |f|^p$ pointwise in T. Let $q \in \Gamma$. Then by hypothesis and by Theorem 7.5 for $\mathbf{m}_q$, $|f|^p$ is $\mathbf{m}_q$-integrable in T and hence, by Theorem 4.2, $|f|^p$ is (KL) $\mathbf{m}_q$-integrable in T. Therefore, by Theorem 3.7 for $\mathbf{m}_q$, $(\gamma_n(A))_1^\infty$ is Cauchy in X_q where $\gamma_n(A) = \int_A s_n d\mathbf{m}$ for $A \in \sigma(\mathcal{P})$. Since q is arbitrary in Γ, then by Theorem 14.6, $|f|^p$ is (KL)$\mathbf{m}$-integrable in T with values in X. Therefore, $f \in \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m})$. Then by Theorem 13.7(i), $f \in \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$. This also proves that $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$.

Converesely, if $f \in \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ $(= \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m})$ by Theorem 13.7(i)), then $|f|^p$ is (KL) $\mathbf{m}$-integrable in T and hence $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ by Theorem 13.4 and Definition 14.4. Therefore, $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$.

(ii) If $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, then $|f|^p$ is (KL) $\mathbf{m}$-integrable in T so that by Theorem 11.8(v), $|f|^p$ is (KL) $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$ as $\mathbf{m}_q = \Pi_q \circ \mathbf{m}$. Then the condition is necessary by Theorem 4.2. Conversely, if $|f|^p$ is $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$, then by Theorem 7.5, $(\mathbf{m}_q)^\bullet_p(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$ and hence by (i), $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$. The last part holds by the above part and by Theorem 4.2.

Theorem 14.8. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Then:

(i) If X is quasicomplete, then $\mathcal{L}_p(\mathbf{m})$ $(=\mathcal{L}_p\mathcal{I}(\mathbf{m})$ by Theorem 14.5) is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$.

(ii) If X is a Frechét space, then $\mathcal{L}_p(\mathbf{m})$ $(= \mathcal{L}_p\mathcal{I}(\mathbf{m})$ by Theorem 14.5) is a complete pseudo-metrizable locally convex space. Consequently, $L_p(\mathbf{m})$ $(= L_p\mathcal{I}(\mathbf{m}))$ is a Frechét space.

(iii) If X is sequentially complete, then $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ $(=\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ by Theorem 14.7) is closed in $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$.

(iv) If X is a Frechét space, then $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ $(=\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$ by Theorem 14.7) is a complete pseudo-metrizable locally convex space. Consequently, $L_p(\sigma(\mathcal{P}), \mathbf{m})$ $(= L_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}))$ is a Frechét space.

Proof. (i) (resp. (iii)) Let f be an element in the closure of $\mathcal{L}_p(\mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. in the closure of $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m})$). Let $q \in \Gamma$. For each $n \in \mathbb{N}$, there exists $f_n^{(q)} \in \mathcal{L}_p(\mathbf{m})$ (resp. $f_n^{(q)} \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) such that $(\mathbf{m}_q)^\bullet_p(f - f_n^{(q)}, T) < \frac{1}{n}$. Given $\epsilon > 0$, choose n_0 such that $\frac{2}{n_0} < \epsilon$. Then $(\mathbf{m}_q)^\bullet_p(f - f_n^{(q)}, T) < \frac{\epsilon}{2}$ for $n \ge n_0$. Let $(E_k)_1^\infty \subset \sigma(\mathcal{P})$ such that $E_k \searrow \emptyset$. Since $f_{n_0}^{(q)} \in \mathcal{L}_p(\mathbf{m})$ (resp. $\in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$), there exists k_0 such that $(\mathbf{m}_q)^\bullet_p(f_{n_0}^{(q)}, E_k) < \frac{\epsilon}{2}$ for $k \ge k_0$. Then

$$\begin{aligned}(\mathbf{m}_q)^\bullet_p(f, E_k) &\le (\mathbf{m}_q)^\bullet_p(f - f_{n_0}^{(q)}, E_k) \\ &+ \ (\mathbf{m}_q)^\bullet_p(f_{n_0}^{(q)}, E_k) \le (\mathbf{m}_q)^\bullet_p(f - f_{n_0}^{(q)}, T) + (\mathbf{m}_q)^\bullet_p(f_{n_0}^{(q)}, E_k) < \epsilon\end{aligned}$$

for $k \ge k_0$. Hence $(\mathbf{m}_q)^\bullet_p(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$. As $q \in \Gamma$ is arbitrary, we conclude that $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and hence (i) (resp. (iii)) holds.

(ii) (resp. (iv)) The first part is immediate from (i) (resp. (iii)) and from the first part of Theorem 14.2(i). Then $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is a Frechét space by Theorem 5.7, Ch. 2 and Lemma 11.3, Ch. 3 of [KN] and by the fact that $L_p(\mathbf{m})$ (rsp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is an LcHs by Theorem

13.8(v).

Remark 14.9. One can prove directly that $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$) is a complete pseudo-metrizable space for $1 \le p < \infty$ whenever X is a Frechét space. In fact, let $(f_r)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ be Cauchy in $(\mathbf{m}_{q_n})_p^\bullet(\cdot,T)$ for each $n \in \mathbb{N}$, where q_n, $n \in \mathbb{N}$ generate the topology of X. Arguning as in the proof of Theorem 14.2(i), we obtain a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_r)_1^\infty$ and an $\mathbf{m}$-measurable scalar function f on T such that $f_{n_k} \to f$ $\mathbf{m}$-a.e. in T and such that $\lim_k (\mathbf{m}_{q_n})_p^\bullet(f_{n_k} - f, T) = 0$, for each $n \in \mathbb{N}$. Then, given $n \in \mathbb{N}$ and $\epsilon > 0$, there exists k_0 such that $(\mathbf{m}_{q_n})_p^\bullet(f_{n_k} - f, T) < \frac{\epsilon}{2}$ for $k \ge k_0$. Let $E_\ell \searrow \emptyset$ in $\sigma(\mathcal{P})$. Then there exists an ℓ_0 such that $(\mathbf{m}_{q_n})_p^\bullet(f_{n_{k_0}}, E_\ell) < \frac{\epsilon}{2}$ for $\ell \ge \ell_0$, since $f_{n_{k_0}} \in \mathcal{L}_p(\mathbf{m})$. Then $(\mathbf{m}_{q_n})_p^\bullet(f, E_\ell) \le (\mathbf{m}q_n)_p^\bullet(f - f_{n_{k_0}}, T) + (\mathbf{m}_{q_n})_p^\bullet(f_{n_{k_0}}, E_\ell) < \epsilon$ for $\ell \ge \ell_0$ and hence $(\mathbf{m}_{q_n})_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$. Since n is arbitrary in $\mathbb{N}$, $f \in \mathcal{L}_p(\mathbf{m})$ by Theorem 14.5. Moreover, by an argument similar to that in the last part of the proof of Theorem 14.2(i), $\lim_r (\mathbf{m}_{q_n})_p^\bullet(f_r - f, T) = 0$. Hence $\mathcal{L}_p(\mathbf{m})$ is complete and consequenctly, $L_p(\mathbf{m})$ is a Frechét space. Similarly, the completeness of $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$ is proved, where we have to use Theorem 14.8 in place of Theorem 14.5.

Remark 14.10. When $p = 1$ and when the domain of $\mathbf{m}$ is a σ-algebra, the second part of Theorem 14.8(iv) is obtained in Theorem 4.1, Ch. IV of [KK] for a real Frechét space X, using the concept of closed vector measures. Later, when X is a complex Frechét space admitting a continuous norm, a simple direct proof (without using closed measures) is given for the said result in [Ri3]. Recently, for an arbitrary complex Frechét space X, a direct proof of the above result is given in [FNR] and the proof in [FNR] uses the complex version of Theorem 4.1, Ch. II of [KK] and the diagonal sequence argument. As the reader can observe, our proof is also direct and moreover, is much more stronger than the proofs in the literature, since not only the domain of the vector measure $\mathbf{m}$ is assumed to be just a δ-ring but also p is arbitrary in $[1,\infty)$. As for the problem of completeness of $\mathcal{L}_p$-spaces for vector measures defined on δ-rings, the reader may note that the concept of closed vector measures is inutil.

15. CHARACTERIZATIONS OF $\mathcal{L}_p$-SPACES, CONVERGENCE THEOREMS AND RELATIONS BETWEEN $\mathcal{L}_P$-SPACES

In this section, using Theorem 14.5 (resp. Theorem 14.7) we generalize the results in Sections 7, 8 and 9 to a quasicomplete (resp. sequentially complete) lcHs-valued σ-additive vector measure $\mathbf{m}$ on $\mathcal{P}$. Similar to that in Definition 6.5, we introduce the space $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}),\mathbf{m})$) and show that $\mathcal{L}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}),\mathbf{m})$) for $1 \le p < \infty$.

Theorem 15.1 (Generalizations of Theorem 7.1). Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Let $f : T \to \mathbb{K}$ (resp. $\sigma(\mathcal{P})$-measurable) and $(f_n^{(q)})_1^\infty \subset \mathcal{L}_p(\mathbf{m}_q)$ (resp. $\subset \mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m}_q)$) for each $q \in \Gamma$. Suppose $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$. Then $(\mathbf{m}_q)_p^\bullet(f_n^{(q)} - f, T) \to 0$ as $n \to \infty$ for each $q \in \Gamma$ if and only if $(\mathbf{m}_q)_p^\bullet(f_n^{(q)},\cdot)$, $n \in \mathbb{N}$, are uniformly continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. In that case, $f \in \mathcal{L}_p(\mathbf{m}_q)$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m}_q)$) for each $q \in \Gamma$ and $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$). Moreover, when $p = 1$,

$$\left| q\Big(\int_A f d\mathbf{m}\Big) - \lim_n \left| \int_A f_n^{(q)} d\mathbf{m}_q \right|_q \right| \le \lim_n \left| \int_A f d\mathbf{m}_q - \int_A f_n^{(q)} d\mathbf{m}_q \right|_q = 0 \quad (15.1.1)$$

for $A \in \sigma(\mathcal{P}) \cup \{T\}$ and for each $q \in \Gamma$ so that the limit in (15.1.1) is uniform with respect to $A \in \sigma(\mathcal{P})$ (for $q \in \Gamma$ fixed).

Proof. By hypothesis, f is $\mathbf{m}_q$-measurable for each $q \in \Gamma$ and hence f is $\mathbf{m}$-measurable (resp. by hypothesis, f is $\sigma(\mathcal{P})$-measurable). As $\mathbf{m}_q = \Pi_q \circ \mathbf{m} : \mathcal{P} \to \widetilde{X_q}$ is σ-additive, the first part is immediate from Theorem 7.1. Moreover, in that case, $f \in \mathcal{L}_p(\mathbf{m}_q)$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) for each q and hence, by Theorem 7.5, $|f|^p$ is $\mathbf{m}_q$-integrable in T for each $q \in \Gamma$. Then by Definition 12.1 and by Theorem 12.3 $|f|^p$ is $\mathbf{m}$-integrable in T with values in X and hence $f \in \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$ by Theorems 12.2, 13.7(i) and 14.5 (resp. by Theorem 14.7(ii), $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$).

Now let $p = 1$. Given $q \in \Gamma$ and $\epsilon > 0$, there exists n_0 such that $(\mathbf{m}_q)^\bullet_1(f - f_n^{(q)}, T) < \epsilon$ for all $n \geq n_0$. Then by Theorem 12.2 (resp. by Theorem 12.2') and by (13.2.1) we have

$$\begin{aligned} &\left| q\Big(\int_A f d\mathbf{m}\Big) - \left|\int_A f_n^{(q)} d\mathbf{m}_q\right|_q \right| \\ &\leq \left|\int_A f d\mathbf{m}_q - \int_A f_n^{(q)} d\mathbf{m}_q\right|_q \\ &= \left|\int_A (f - f_n^{(q)}) d\mathbf{m}_q\right|_q \leq (\mathbf{m}_q)^\bullet_1(f - f_n^{(q)}, T) < \epsilon \qquad (15.1.2) \end{aligned}$$

for $n \geq n_0$ and for all $A \in \sigma(\mathcal{P}) \cup \{T\}$. Then (15.1.2) implies (15.1.1) and that the limit in (15.1.1) is uniform with respect to $A \in \sigma(\mathcal{P})$.

Theorem 15.2 (Generalizations of Theorem 7.2). Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Let $f : T \to \mathbb{K}$ be $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable). A sequence $(f_n^{(q)})_1^\infty \subset \mathcal{L}_p(\mathbf{m}_q)$ (resp. $\subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) converges to f in (mean^p) with respect to $\mathbf{m}_q$ for each $q \in \Gamma$ if and only if $(f_n^{(q)})$ converges in measure in T with respect to $\mathbf{m}_q$ for each $q \in \Gamma$ and $(\mathbf{m}_q)^\bullet_p(f_n^{(q)}, \cdot)$, $n \in \mathbb{N}$, are uniformly continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$. In that case, $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$). When $p = 1$, results similar to those in the last part of Theorem 15.1 hold.

Proof. The first part is immediate from Theorem 7.2. Then by the said theorem, $f \in \mathcal{L}_p(\mathbf{m}_q)$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) for each $q \in \Gamma$. Then as shown in the proof of Theorem 15.1, we conclude that $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$)and the results asserted for $p = 1$ hold.

The a.e. convergence version in the following theorem generalizes Theorem 11.11 and Corollary 11.12 (resp. Theorem 11.11' and Corollary 11.12' in Remark 11.15) for general $p \in [1, \infty)$.

Theorem 15.3. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Then:

(i) **(LDCT and LBCT for $\mathcal{L}_p(\mathbf{m})$, X quasicomplete).** Let X be quasicomplete and $f_n^{(q)}$, $n \in \mathbb{N}$, be $\mathbf{m}_q$-measurable scalar functions on T for each $q \in \Gamma$. Let $g \in \mathcal{L}_p(\mathbf{m}_q)$ for each $q \in \Gamma$ and let $|f_n^{(q)}| \leq |g|$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$ (resp. let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let $K^{(q)}$ be a finite constant such that $|f_n^{(q)}| \leq K^{(q)}$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$) and for all n. If $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T where f is a scalar function on T or if f is an $\mathbf{m}$-measurable scalar function on T and if $f_n^{(q)} \to f$ in measure in T with respect to $\mathbf{m}_q$ for each $q \in \Gamma$, then f, $f_n^{(q)}$, $n \in \mathbb{N}$ belong to $\mathcal{L}_p(\mathbf{m}_q)$ and $\lim_n (\mathbf{m}_q)^\bullet_p(f_n^{(q)} - f, T) = 0$ for each

$q \in \Gamma$. Consequently, $f \in \mathcal{L}_p(\mathbf{m})$. When $p = 1$,

$$\left| q\Big(\int_A f d\mathbf{m}\Big) - \lim_n \left| \int_A f_n^{(q)} d\mathbf{m}_q \right|_q \right| \leq \lim_n \left| \int_A f d\mathbf{m}_q - \int_A f_n^{(q)} d\mathbf{m}_q \right|_q = 0 \quad (15.3.1)$$

for $A \in \sigma(\mathcal{P}) \cup \{T\}$ (resp. for $A \in \mathcal{S} \cup \{T\}$) for each $q \in \Gamma$ so that the limit in (15.3.1) is uniform with respect to $A \in \sigma(\mathcal{P})$ (resp. $A \in \mathcal{S}$).

(ii) **(LDCT and LBCT for $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, X sequentially complete)**. Let X be sequentially complete and let $(f_n^{(q)})_1^\infty$, g, f be $\sigma(\mathcal{P})$-measurable scalar functions and $K^{(q)}$ constants satisfying the the other hypothesis in (i). Then $f, f_n^{(q)}$, $n \in \mathbb{N}$, belong to $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$ and $\lim_n (\mathbf{m}_q)_p^\bullet(f_n^{(q)} - f, T) = 0$ for each $q \in \Gamma$ and $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$. Then, for $p = 1$, the remaining assertions in (i) hold here verbatim.

Proof. By hypothesis, (i) is immediate from Theorems 15.1 and 15.2, in the light of Theorems 5.11(iv) (for $\mathbf{m}_q$), 11.9(i)(b), 11.9(iii), 14.5 and 12.2 and Remark 12.5. By hypothesis, (ii) follows from Theorems 15.1 and 15.2 in view of Theorems 5.11(iv) (for $\mathbf{m}_q$), 11.9'(b), 11.9'(iii) (in Remark 11.15), 14.7(i) and 12.2' (in Remark 12.11).

The following is motivated by the first part of Definition 6.5.

Definition 15.4. Let X be a quasicomplete (resp. sequentially complete) lcHs and let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. For $1 \leq p < \infty$, let $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$= closure of $\mathcal{I}_s$ in the locally convex space $(\mathcal{L}_p\mathcal{M}(\mathbf{m}), \tau_{\mathbf{m}}^{(p)})$ (resp. $\mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m})$= closure of $\mathcal{I}_s$ in the locally convex space $(\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}), \mathbf{m}), \tau_{\mathbf{m}}^{(p)})$). Let $L_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m})/\sim$ and $L_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m})/\sim$. By $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ we mean the locally convex space $(\mathcal{L}_p\mathcal{I}_s(\mathbf{m}), \tau_{\mathbf{m}}^{(p)}|_{\mathcal{L}_p\mathcal{I}_s(\mathbf{m})})$ and $L_p\mathcal{I}_s(\mathbf{m})$ is the lcHs with the corresponding quotient topology. Similarly for $\mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m})$ and $L_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m})$.

The following result generalizes Theorem 7.5 to a quasicomplete (resp. a sequentially complete) lcHs-valued vector measure.

Theorem 15.5.(Characterizations of $\mathcal{L}_p\mathcal{I}(\mathbf{m})$)(resp. $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m})$). Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Let $f : T \to \mathbb{K}$ be $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable). Then the following statements are equivalent:

(i) $f \in \mathcal{I}_p(\mathbf{m})$ (resp. (i') $f \in \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m})$).

(ii) $(\mathbf{m}_q)_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$.

(iii) **(Simple function approximation)**. For each $q \in \Gamma$, there exists a sequence $(s_n^{(q)})_1^\infty \subset \mathcal{I}_s$ such that $s_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T and such that $\lim_n (\mathbf{m}_q)_p^\bullet(s_n^{(q)} - f, T) = 0$ (resp. (iii') there exists a sequence $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $s_n \to f$ pointwise in T and such that $\lim_n (\mathbf{m}_q)_p^\bullet(s_n - f, T) = 0$).

Consequently,

$$\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m}). \qquad (15.5.1)$$

(resp.

$$\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{I}_p(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}). \qquad (15.5.1'))$$

If $c_0 \not\subset \tilde{X}_q$ for each $q \in \Gamma$, then

$$\mathcal{L}_p\mathcal{M}(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m}). \qquad (15.5.2)$$

(resp. if $c_0 \not\subset X$, then $\mathcal{L}_p\mathcal{M}(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{I}_p(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$. (15.5.2′))

Proof. (i)⇔(ii) by Theorems 13.7(i) and 14.5 (resp. (i')⇔(ii) by Theorems 13.7(i) and 14.7(i)).

(ii)⇒(iii) by Proposition 10.9 and by Theorems 14.5 and 15.3(i) (resp. (ii)⇒(iii') by the last part of Proposition 10.9 and by Theorems 14.7 and 15.3(ii)).

(iii)⇒(ii) (resp. (iii')⇒(ii)) Let $\epsilon > 0$ and $q \in \Gamma$. Let $A_k \searrow \emptyset$ in $\sigma(\mathcal{P})$. Arguing as in the proof of (iii)⇒(ii) of Theorem 7.5 with $\mathbf{m}_q$ in place of $\mathbf{m}$, we can show that there exists $k_0(q)$ such that $(\mathbf{m}_q)^\bullet_p(f, A_k) < \epsilon$ for $k \geq k_0(q)$ and hence (ii) holds.

Thus (i)⇔(ii)⇔(iii) (resp. (i')⇔(ii)⇔(iii')).

Since $\mathcal{I}_s \subset \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m})$ by Theorem 13.7(i), $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) \subset$ closure of $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$= closure of $\mathcal{L}_p(\mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ =$\mathcal{L}_p(\mathbf{m})$ by Theorems 14.5 and 14.8(i). On the other hand, $\mathcal{I}_s(\mathbf{m})$ is dense in $\mathcal{L}_p(\mathbf{m})$ by (iii) and by Theorem 14.5 and hence $\mathcal{L}_p(\mathbf{m}) \subset \mathcal{L}_p\mathcal{I}_s(\mathbf{m})$. Therefore, $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$. Consequently, (15.5.1) holds by Theorems 13.7(i) and 14.5. A similar argument invoking (iii') and Theorems 13.7(i), 14.7(i) and 14.8(iii) proves (15.5.1'). (15.5.2) (resp. (15.5.2')) holds by (15.5.1) and by Theorem 13.7(iv) (resp. by (15.5.1') and by Theorem 13.7(v)).

The following result which generalizes Theorem 7.7 is immediate from Theorem 15.5.

Theorem 15.6. Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Then $\mathcal{I}_s$ is dense in $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{I}_s$ is dense in $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$ and moreover, given $f \in \mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$ there exists a sequence $(s_n)_1^\infty \subset \mathcal{I}_s$ such that $s_N \to f$ in $\tau_{\mathbf{m}}^{(p)}$).

Similar to Definition 6.10 we can introduce $\mathcal{L}_\infty(\mathbf{m})$ and $\mathcal{L}_\infty(\sigma(\mathcal{P}),\mathbf{m})$ for an lcHs-valued σ-additive vector measure $\mathbf{m}$ on $\mathcal{P}$.

Definition 15.7. Let X be a quasicomplete lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then we define $\mathcal{L}_\infty(\mathbf{m}) = \{f : T \to \mathbb{K}, f\ \mathbf{m}\text{-essentially bounded}$
$\mathbf{m}\text{-measurable function}\}$ and $||f||_\infty = \text{ess sup}_{t\in T}|f(t)|$ for $f \in \mathcal{L}_\infty(\mathbf{m})$. Then $L_\infty(\mathbf{m}) = \mathcal{L}_\infty(\mathbf{m})/\sim$. In the above, if X is sequentially complete, then we define $\mathcal{L}_\infty(\sigma(\mathcal{P}),\mathbf{m}) = \{f \in \mathcal{M}(\sigma(\mathcal{P}) : f\mathbf{m}\text{-essentially bounded}\}$, $||f||_\infty = \text{ess sup} t \in T|f(t)|$ for $f \in \mathcal{L}_\infty(\sigma(\mathcal{P}),\mathbf{m})$. Then $L_\infty(\sigma(\mathcal{P}),\mathbf{m}) = \mathcal{L}_\infty(\sigma(\mathcal{P}),\mathbf{m})/\sim$. As in convention 7.10, the members of $L_\infty(\mathbf{m})$ and $L_\infty(\sigma(\mathcal{P}),\mathbf{m})$ are treated as functions in which two functions which are equal $\mathbf{m}$-a.e. in T are identified.

In the light of (10.7.1) and the σ-subadditivity of $||\mathbf{m}||_q$ on $\sigma(\mathcal{P})$ for $q \in \Gamma$, the proof of Theorem 6.11 holds here to obtain

Theorem 15.8. Let X be a quasicomplete (resp. sequentially complete) lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then $\mathcal{L}_\infty(\mathbf{m}, ||\cdot||_\infty)$ (resp. $\mathcal{L}_\infty(\sigma(\mathcal{P}),\mathbf{m}), ||\cdot||_\infty)$) is a complete seminormed space so that $L_\infty(m)$ (resp. $L_\infty(\sigma(\mathcal{P}),\mathbf{m})$) is a Banach space.

Notation 15.9. In the light of Theorem 15.5 **we shall hereafter use the symbol $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})$) to denote not only the space given in Definition 14.4 but also anyone of the spaces $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ or $\mathcal{I}_p(\mathbf{m})$ (resp. $\mathcal{L}_p\mathcal{I}_s(\sigma(\mathcal{P}),\mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\sigma(\mathcal{P}),\mathbf{m})$ or $\mathcal{I}_p(\sigma(\mathcal{P}),\mathbf{m})$. The quotient $\mathcal{L}_p(\mathbf{m})/\sim$ is denoted by $L_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}),\mathbf{m})/\sim$ is denoted by $L_p(\sigma(\mathcal{P}),\mathbf{m})$)**

and as in Convention 7.10, the members of the latter are treatd as functions in which two functions which are equal m-a.e. in T are identified.

The following result generalizes Theorem 7.12 and is proved by an argument similar to that in the proof of the said theorem, in which we use (iv) and (vi) of Theorem 5.18 and Theorem 5.19 with respect to $\mathbf{m}_q$, Remark 12.5, Theorems 11.4, 15.5 and 14.8(i), Corollary 14.3 and inequality (13.2.1). Details are left to the reader.

Theorem 15.10. Let X be a quasicomplete lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. An $\mathbf{m}$-measurable scalar function f on T belongs to $\mathcal{L}_p(\mathbf{m})$ if and only if, for each $q \in \Gamma$, there exists a sequence $(s_n^{(q)})_1^\infty \subset \mathcal{I}_s$ (resp. $(f_n^{(q)})_1^\infty \subset \mathcal{L}_p(\mathbf{m})$) such that $s_n^{(q)} \to f$ (resp. $f_n^{(q)} \to f$) in measure in T with respect to $\mathbf{m}_q$ and $(s_n^{(q)})$ (resp. $(f_n^{(q)})$) is Cauchy in (mean^p) with respcect to $\mathbf{m}_q$. When $p = 1$, $f \in \mathcal{L}_1(\mathbf{m})$ and

$$\lim_n \left| \int_A (f - s_n^{(q)}) d\mathbf{m}_q \right|_q = 0 \text{ (resp. } \lim_n \left| \int_A (f - f_n^{(q)}) d\mathbf{m}_q \right|_q = 0)$$

for $A \in \sigma(\mathcal{P}) \cup \{T\}$ and for $q \in \Gamma$, the limit being uniform with respect to $A \in \sigma(\mathcal{P})$ (for q fixed).

The following result generalizes Theorem 7.11 to Frechét space-valued σ-additive vector measures on $\mathcal{P}$.

Theorem 15.11. Let X be a Frechét space with its topology generated by the seminorms $(q_n)_1^\infty$ and let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. For $1 \leq p < \infty$, let $\mathcal{L}_p^r(\mathbf{m}) = \{f \in \mathcal{L}_p(\mathbf{m}) : f \text{ real valued}\}$ (resp. $\mathcal{L}_p^r(\sigma(\mathcal{P}), \mathbf{m}) = \{f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) : f \text{ real valued}\}$). Let $|f|_{\mathbf{m}}^{(p)} = \sum_1^\infty \frac{1}{2^n} \frac{(\mathbf{m}_{q_n})_p^\bullet(f,T)}{1+(\mathbf{m}_{q_n})_p^\bullet(f,T)}$ for $f \in L_p(\mathbf{m})$ (resp. $f \in L_p(\sigma(\mathcal{P}), \mathbf{m})$)(15.11.1). Then:

(i) $|\cdot|_{\mathbf{m}}^{(p)}$ is a complete quasinorm on $L_p(\mathbf{m})$ (resp. on $L_p(\sigma(\mathcal{P}), \mathbf{m})$) in the sense of Definition 2, Section 2, Ch. I of [Y] and generates the quotient topology induced by $\tau_{\mathbf{m}}^{(p)}$ on $L_p(\mathbf{m})$ so that $(L_p(\mathbf{m}), |\cdot|_{\mathbf{m}}^{(p)})$ (resp. $(L_p(\sigma(\mathcal{P}), \mathbf{m}), |\cdot|_{\mathbf{m}}^{(p)})$) is the Frechét space $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$).

(ii) $(L_p^r(\mathbf{m}), |\cdot|_{\mathbf{m}}^{(p)})$ (resp. $(L_p^r(\sigma(\mathcal{P}), \mathbf{m}), |\cdot|_{\mathbf{m}}^{(p)})$) is a Frechét lattice (i.e., an F-lattice in the sense of Definition in Section 3, Ch. XII and Definition 1 in Section 9, Ch. I of [Y]) under the partial order $f \leq g$ if and only if $f(t) \leq g(t)$ $\mathbf{m}$-a.e. in T for $f, g \in L_p^r(\mathbf{m})$ (resp. for $f, g \in L_p^r(\sigma(\mathcal{P}), \mathbf{m})$).

(iii) $L_\infty(\mathbf{m})$ (resp. $L_\infty(\sigma(\mathcal{P}), \mathbf{m})$) is a Banach lattice.

Proof. We shall prove the results for $L_p(\mathbf{m})$ and $L_p^r(\mathbf{m})$ only. In the light of Theorem 14.8(iv) and Theorem 11.9'(iii), the proof of the other case is similar.

(i) $L_p(\mathbf{m})$ is a Frechét space by Theorem 14.8(ii) and consequently, $L_p^r(\mathbf{m})$ is a Frechét space over $\mathbb{R}$. Arguing as in the proof of the first part of Proposition 2, §6, Ch. 2 of [Ho] and using the fact that $(\mathbf{m}_{q_n})_p^\bullet(\cdot, T)$, $n \in \mathbb{N}$, are seminorms, one can show that $|\cdot|_{\mathbf{m}}^{(p)}$ given in (15.11.1) is a quasinorm on $L_p(\mathbf{m})$. Let the sequence $f_r \to f$ in $L_p(\mathbf{m})$. Given $\epsilon > 0$, choose k_0 such that $\frac{1}{2^{k_0}} < \frac{\epsilon}{2}$. By hypothesis, there exists r_0 such that $(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T) < \frac{\epsilon}{2}$ for $r \geq r_0$ and for $n = 1, 2, ..., k_0$. Then

$$\begin{aligned} |f_r - f|^{(p)}_{\mathbf{m}} &= \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T)}{1 + (\mathbf{m}_{q_n})^\bullet_p(f_r - f, T)} \\ &< \sum_{n=1}^{k_0} \frac{1}{2^n} \frac{(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T)}{1 + (\mathbf{m}_{q_n})^\bullet_p(f_r - f, T)} + \frac{\epsilon}{2} \\ &< \sum_{n=1}^{k_0} \frac{1}{2^n} \cdot \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

for $r \geq r_0$. Conversely, let $|f_r - f|^{(p)}_{\mathbf{m}} \to 0$. Let $n \in \mathbb{N}$ be given and let $\frac{1}{2} > \epsilon > 0$. Then there exists r_0 such that $|f_r - f|^{(p)}_{\mathbf{m}} < \frac{\epsilon}{2^{n+1}}$ for $r \geq r_0$. Then $(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T) < \frac{\epsilon}{2}(1 + (\mathbf{m}_{q_n})^\bullet_p(f_r - f, T))$ so that $(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T) < \epsilon$ for $r \geq r_0$. Hence $(\mathbf{m}_{q_n})^\bullet_p(f_r - f, T) \to 0$ as $r \to 0$. Since n is arbitrary, it follows that $f_r \to f$ in the topology of $L_p(\mathbf{m})$ (see Remark 13.10). Hence the topology of $L_p(\mathbf{m})$ is generated by $|\cdot|^{(p)}_{\mathbf{m}}$.

If $(f_r)_1^\infty$ is Cauchy in $|\cdot|^{(p)}_{\mathbf{m}}$, then replacing $f_r - f$ by $f_r - f_k$ in the above argument (in the converse part), it follows that (f_r) is Cauchy in $(\mathbf{m}_{q_n})^\bullet_p(\cdot, T)$ for each n and hence by Theorem 14.8(ii), there exists $f \in L_p(\mathbf{m})$ such that $f_r \to f$ in $L_p(\mathbf{m})$ and hence in $|\cdot|^{(p)}_{\mathbf{m}}$. This completes the proof of (i). (The compatiblity of the topology generated by $|\cdot|^{(p)}_{\mathbf{m}}$ with that of $L_p(\mathbf{m})$ follows from Remarks 1.38(c) of [Ru], but not the completeness of $|\cdot|^{(p)}_{\mathbf{m}}$.)

(ii) If $|f| \leq |g|$ in $L^r_p(\mathbf{m})$, then by Theorem 13.2 $(\mathbf{m}_{q_n})^\bullet_p(f, T) \leq (\mathbf{m}_{q_n})^\bullet_p(g, T)$ for each n and consequently, $|f|^{(p)}_{\mathbf{m}} \leq |g|^{(p)}_{\mathbf{m}}$. As $L^r_p(\mathbf{m})$ is a vector space and as $L^r_1(\mathbf{m})$ is a lattice by Theorem 11.9(iii), it follows that $L^r_p(\mathbf{m})$ is a lattice. Clearly, $f \leq g$ implies $f + h \leq g + h$ and $\alpha f \leq \alpha g$ for $f, g, h \in L^r_p(\mathbf{m})$ and $\alpha \geq 0$. Hence $L^r_p(\mathbf{m})$ is a vector lattice. Consequently, by (i), $L^r_p(\mathbf{m})$ is a Frechét lattice with respect to $|\cdot|^{(p)}_{\mathbf{m}}$.

(iii) In the light of Theorem 15.8, the result is obvious.

The following theorem generalizes Theorems 8.5, 8.6, 8.7 and 8.10 and Corollary 8.11 to a quasicomplete (resp. sequentially complete) lcHs-valued vector measure.

Theorem 15.12. Let X be an lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Let X be quasicomplete (resp. sequentially complete). Then:

(i) **(Generalizations of Theorem 8.5)**. Let $(f_n^{(q)})_{n=1}^\infty \subset \mathcal{L}_p(\mathbf{m}_q)$ (resp. $\subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) for each $q \in \Gamma$ and $f : T \to \mathbb{K}$ be $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable). Then $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and $\lim_n (\mathbf{m}_q)^\bullet_p(f_n^{(q)} - f, T) = 0$ for each $q \in \Gamma$ if and only if the following conditions hold:
   (a) $f_n^{(q)} \to f$ in measure in each $E \in \mathcal{P}$ with respect to $\mathbf{m}_q$ for each $q \in \Gamma$.
   (b) $(\mathbf{m}_q)^\bullet_p(f_n^{(q)}, \cdot)$, $n \in \mathbb{N}$, are uniformly $\mathbf{m}_q$-continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$.
   (c) For each $\epsilon > 0$ and $q \in \Gamma$, there exists $A_\epsilon^{(q)} \in \mathcal{P}$ such that $(\mathbf{m}_q)^\bullet_p(f_n^{(q)}, T \backslash A_\epsilon^{(q)}) < \epsilon$ for all $n \in \mathbb{N}$. (See Notation 8.1.)

In such case, for $p=1$, $\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q$, $A \in \sigma(\mathcal{P})$ and the limit is uniform with respect to $A \in \sigma(\mathcal{P})$ for $q \in \Gamma$ fixed.

(ii) (**Generalizations of Theorem 8.6**). Let $(f_n^{(q)})_{n=1}^\infty \subset \mathcal{L}_p(\mathbf{m}_q)$ (resp.$\subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) for each $q \in \Gamma$ and let $f : T \to \boldsymbol{K}$(resp. be $\sigma(\mathcal{P})$-measurable). Suppose $f_n^{(q)} \to f$ $\mathbf{m}_q$-a.e. in T for each $q \in \Gamma$. Then $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and $\lim_n (\mathbf{m}_q)_p^\bullet(f_n^{(q)} - f, T) = 0$ for each $q \in \Gamma$ if and only if the following conditions are satisfied:

(a) $(\mathbf{m}_q)_p^\bullet(f_n^{(q)}, \cdot)$, $n \in \mathbb{N}$ are uniformly $\mathbf{m}_q$-continuous on $\sigma(\mathcal{P})$ for each $q \in \Gamma$.

(b) For each $\epsilon > 0$ and $q \in \Gamma$, there exists $A_\epsilon^{(q)} \in \mathcal{P}$ such that $(\mathbf{m}_q)_p^\bullet(f_n^{(q)}, T \backslash A_\epsilon^{(q)}) < \epsilon$ for all n.

In such case, for $p=1$, $\int_A f d\mathbf{m}_q = \lim_n \int_A f_n^{(q)} d\mathbf{m}_q$ for $A \in \sigma(\mathcal{P})$ and the limit is uniform with respect to $A \in \sigma(\mathcal{P})$ for $q \in \Gamma$ fixed.

(iii) LDCT and LBCT as given in (i) and (ii) of Theorem 15.3 are deducible from (i) and (ii) above.

(iv) (**Generalizations of Theorem 8.10**). For each $q \in \Gamma$, let $(f_\alpha^{(q)})$, $\alpha \in (D^{(q)}, \geq_q)$, be a net of $\mathbf{m}_q$-measurable (resp. $\sigma(\mathcal{P})$-measurable) scalar functions on T and let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-masurable (resp. $\sigma(\mathcal{P})$-measurable). Let $g^{(q)} \in \mathcal{L}_p(\mathbf{m}_q)$ (resp. $\in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$) and let $|f_\alpha^{(q)}| \leq |g^{(q)}|$ $\mathbf{m}_q$-a.e. in T for each $\alpha \in (D^{(q)}, \geq_q)$ and for each $q \in \Gamma$. Then $f_\alpha^{(q)} \to f$ in measure in T with resepect to each $\mathbf{m}_q$, $q \in \Gamma$, if and only if $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and $\lim_{\alpha \in (D^{(q)}, \geq_q)} (\mathbf{m}_q)_p^\bullet(f_\alpha^{(q)} - f, T) = 0$ for each $q \in \Gamma$. In such case, for $p=1$, $\int_A f d\mathbf{m}_q = \lim_\alpha \int_A f_\alpha^{(q)} d\mathbf{m}_q$ for $A \in \sigma(\mathcal{P})$ where the limit is uniform with respect to $A \in \sigma(\mathcal{P})$ for $q \in \Gamma$ fixed.

(v) (**Generalizations of Corollary 8.11**). Let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let $0 < K_q < \infty$ for each $q \in \Gamma$. If $(f_\alpha^{(q)})$ is a net as in (iv), if $|f_\alpha^{(q)}| \leq K_q$ $\mathbf{m}_q$-a.e. in T for each $\alpha \in (D^{(q)}, \geq_q)$ and for each $q \in \Gamma$ and if $f : T \to \boldsymbol{K}$ is $\mathbf{m}$-measurable (resp. $\sigma(\mathcal{P})$-measurable), then $f_\alpha^{(q)} \to f$ in measure in T with respect to $\mathbf{m}_q$ for each $q \in \Gamma$ if and only if $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and $\lim_\alpha (\mathbf{m}_q)_p^\bullet(f_\alpha^{(q)} - f, T) = 0$ for each $q \in \Gamma$. In such case, for $p=1$, $\int_A f d\mathbf{m}_q = \lim_\alpha \int_A f_\alpha^{(q)} d\mathbf{m}_q$ for $A \in \sigma(\mathcal{P})$ where the limit is uniform with respect to $A \in \sigma(\mathcal{P})$ for $q \in \Gamma$ fixed.

Proof. In the light of Remark 12.5 (resp. Remark 12.5' in Remark 12.11) and Notation 15.9, the above results hold by Theorems 11.4 and 12.3 (resp. by Theorem 14.7), by Lemmas 8.2 and 8.4 and by the respective results in Section 8 which they generalize.

The following theorem generalizes the results in Section 9 to a quasicomplete (resp. sequentially complete) lcHs-valued vector measure.

Theorem 15.13. Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Then the following statements hold:

(i) $\mathcal{L}_p(\mathbf{m}) = \bigcap_{q \in \Gamma} \mathcal{L}_p(\mathbf{m}_q)$ (resp. (i') $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) = \bigcap_{q \in \Gamma} \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}_q)$).

(ii) If $1 \leq r < p < s \leq \infty$, then $\mathcal{L}_r(\mathbf{m}) \bigcap \mathcal{L}_s(\mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$. (resp. (ii') $\mathcal{L}_r(\sigma(\mathcal{P}), \mathbf{m}) \bigcap \mathcal{L}_s(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$).

(iii) If $f : T \to \boldsymbol{K}$ is $\mathbf{m}$-measurable (resp. (iii') is $\sigma(\mathcal{P})$-measurable), then $\mathcal{I}_f = \{p : 1 \leq p < \infty, f \in \mathcal{L}_p(\mathbf{m})\}$ (resp. $\mathcal{I}_f(\sigma(\mathcal{P})) = \{f : 1 \leq p < \infty, f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})\}$) is either void or an interval, where singletons are considered as intervals.

(iv) Let $A \in \widetilde{\sigma(\mathcal{P})}$ (see Definition 11.3) (resp. (iv') $A \in \sigma(\mathcal{P})$)such that χ_A is $\mathbf{m}$-integrable in T. Then the set $\mathcal{I}_f(A) = \{p : 1 \leq p < \infty, f\chi_A \in \mathcal{L}_p(\mathbf{m})\}$ (resp. $\mathcal{I}_f(\sigma(\mathcal{P}), A) = \{p :$

$1 \leq p < \infty, f\chi_A \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})\})$ is either void or an interval containing 1 ($\mathcal{I}_f(A)$(resp. $\mathcal{I}_f(\sigma(\mathcal{P}), A)$)=$\{1\}$ is permitted).

(v) Let $\mathcal{P}$ be a σ-ring $\mathcal{S}$. Then:

(a) If $1 \leq r < s < \infty$, then $\mathcal{L}_s(\mathbf{m}) \subset \mathcal{L}_r(\mathbf{m})$ (resp. (a') $\mathcal{L}_s(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_r(\sigma(\mathcal{P}), \mathbf{m})$) and the topology of $\mathcal{L}_s(\mathbf{m})$ (resp. of $\mathcal{L}_s(\sigma(\mathcal{P}), \mathbf{m})$) is finer than that of $\mathcal{L}_r(\mathbf{m})$ (resp. of $\mathcal{L}_r(\sigma(\mathcal{P}), \mathbf{m})$).

(b) If $f : T \to \boldsymbol{K}$ is $\mathbf{m}$-measurable (resp. (b') $\sigma(\mathcal{P})$-measurable), then the set $\mathcal{I}_f = \{p : 1 \leq p < \infty, f \in \mathcal{L}_p(\mathbf{m})\}$ (resp. $\mathcal{I}_f(\sigma(\mathcal{P}))$ is either void or an interval containing 1 ($\mathcal{I}_f$ (resp. $\mathcal{I}_f(\sigma(\mathcal{P}))$=$\{1\}$ is permitted).

Proof. (i) holds by Remark 12.5 and by (i) and (ii) of Theorem 11.4 (resp. (i') holds by Theorem 14.7(ii)). (ii) and (iii) hold by (i) and by Theorem 9.2. (ii') and (iii') hold by (i'), by Theorem 9.2 and by the fact that $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) = \mathcal{L}_p(\mathbf{m}) \bigcap \mathcal{M}(\mathcal{P})$. (iv) is due to (i) and Theorem 9.3. (iv') is due to (i'), Theorem 9.3 and the definition of $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$. (v)(a) (resp. (v)(a')) is due to Corollary 9.4 and the fact that $(\mathbf{m}_q)^\bullet_r(f,T) \leq (\mathbf{m}_q)^\bullet_s(f,T) \cdot (||\mathbf{m}||_q(N(f))^{\frac{1}{r}-\frac{1}{s}}$ for each $q \in \Gamma$ and for $f \in \mathcal{L}_s(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) by Theorem 9.3. (v)(b) (resp. (v)(b')) follows from (v)(a) and (iii)(resp. (v)(a') and (iii')).

16. SEPARABILITY OF $\mathcal{L}_p(\mathbf{m})$ AND $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}), 1 \leq p < \infty$, $\mathbf{m}$ LCH-VALUED

In this section we give some sufficient conditions for the separability of $\mathcal{L}_p(\mathbf{m})$ (resp. of $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) for $1 \leq p < \infty$, when $\mathbf{m} : \mathcal{P} \to X$ is σ-additive and X is quasicomplete (resp. sequentially) complete. For such p, we also include a generalization of Propositions 2 and 3(ii) of [Ri2] to $L_p(\mathbf{m})$ and to $L_p(\sigma(\mathcal{P}), \mathbf{m})$ when $\mathcal{P}$ is a σ-ring.

Definition 16.1. Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. We identify $\mathcal{P}$ with the subset $\mathcal{F} = \{\chi_A : A \in \mathcal{P}\}$ of $\mathcal{L}_p(\mathbf{m})$ (resp. of $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and endow $\mathcal{P}$ with the relative topology $\tau^{(p)}_{\mathbf{m}}|_{\mathcal{P}}$. Then we write $(\mathcal{P}, \tau^{(p)}_{\mathbf{m}})$.

For $A, B \in \mathcal{P}$, we define $\rho(\mathbf{m})^{(p)}_q(A,B) = (\mathbf{m}_q)^\bullet_p(\chi_A - \chi_B, T)$ for $q \in \Gamma$. By Theorem 13.2 and Proposition 10.14(ii)(c) we have

$$\rho(\mathbf{m})^{(p)}_q(A,B) = \sup_{x^* \in U^o_q} (\int_T \chi_{A\Delta B} dv(x^*\mathbf{m}))^{\frac{1}{p}} = \sup_{x^* \in U^o_q} (v(x^*\mathbf{m})(A\Delta B))^{\frac{1}{p}} = (||\mathbf{m}||_q(A\Delta B))^{\frac{1}{p}}$$

for $A, B \in \mathcal{P}$. Moreover, by Theorems 5.11(iv) and 5.13(i) for $\mathbf{m}_q$, $\rho(\mathbf{m})^{(p)}_q$ is a pseudo-metric on $\mathcal{P}$. Thus, the topology $\tau^{(p)}_{\mathbf{m}}|_{\mathcal{P}}$ is generated by the pseudo-metrics $\{\rho(\mathbf{m})^{(p)}_q, q \in \Gamma\}$.

Theorem 16.2. Let X be a quasicomplete (resp. sequentially complete) lcHs, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \leq p < \infty$. Then:

(i) If $(\mathcal{P}, \tau^{(p)}_{\mathbf{m}})$ is separable, then $\mathcal{L}_p(\mathbf{m})$ and $L_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$) are separable. Moreover, for $1 \leq r < \infty$, $\tau^{(r)}_{\mathbf{m}}|_{\mathcal{P}} = \tau^{(p)}_{\mathbf{m}}|_{\mathcal{P}}$ and conseuently, $(\mathcal{P}, \tau^{(r)}_{\mathbf{m}})$ is separable whenever $(\mathcal{P}, \tau^{(p)}_{\mathbf{m}})$ is separable and in that case, $\mathcal{L}_r(\mathbf{m})$ and $L_r(\mathbf{m})$ (resp. $\mathcal{L}_r(\sigma(\mathcal{P}), \mathbf{m})$ and $L_r(\sigma(\mathcal{P}), \mathbf{m})$) are separable for each $r \in [1, \infty)$.

(ii) If X is further metrizable, then $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable if and only if $(\mathcal{P}, \tau^{(p)}_{\mathbf{m}})$ is separable.

Proof. (i) By hypothesis there exists a countable subset D of $\mathcal{P}$ such that D is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{P}$. Let $W = \{\sum_{j=1}^{r} \alpha_j \chi_{F_j} : \alpha_j = a_j + ib_j, a_j, b_j$ real rational, $(F_j)_1^r \subset D, r \in \mathbb{N}\}$. Then W is countable. Let $s = \sum_{j=1}^{k} \beta_j \chi_{A_j} \in \mathcal{I}_s$, $\beta_j \neq 0$ for all j. Then $c = \sum_{j=1}^{k} |\beta_j| > 0$. Let U be a $\tau_{\mathbf{m}}^{(p)}$-neighborhood of 0 in $\mathcal{L}_p(\mathbf{m})$ (resp. in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$). Then there exist an $\epsilon \in (0,1)$ and $q_1, q_2, ..., q_n$ in Γ such that $\{f \in \mathcal{L}_p(\mathbf{m})(\text{resp.} f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})) : (\mathbf{m}_{q_i})_p^{\bullet}(f, T) < \epsilon, i = 1, 2, ..., n\} \subset U$. Let $M = 1 + \sup_{1 \leq i \leq n, 1 \leq j \leq k} (||\mathbf{m}||_{q_i}(A_j))^{\frac{1}{p}}$. Since D is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{P}$, there exist $(F_j)_1^k \subset D$ such that

$$(\mathbf{m}_{q_i})_p^{\bullet}(\chi_{F_j} - \chi_{A_j}, T) = (||\mathbf{m}||_{q_i}(A_j \Delta F_j))^{\frac{1}{p}} < \frac{\epsilon}{3c} \qquad (16.2.1)$$

for $i = 1, 2, ..., n$ and $j = 1, 2, ...k$. Choose $\alpha_j = a_j + ib_j, a_j, b_j$ real rational for $j = 1, 2, ..., k$ such that $\sum_{j=1}^{k} |\beta_j - \alpha_j| < \frac{\epsilon c}{3M}$. Let $\omega = \sum_{j=1}^{k} \alpha_j \chi_{F_j}$. Then $\omega \in W$. Now by Theorem 5.13(i) for $\mathbf{m}_q$ and by (16.2.1) we have

$$\begin{aligned}(||\mathbf{m}||_{q_i}(F_j))^{\frac{1}{p}} &\leq (||\mathbf{m}||_{q_i}(F_j \backslash A_j))^{\frac{1}{p}} + (||\mathbf{m}||_{q_i}(F_j \cap A_j))^{\frac{1}{p}} \\ &\leq (||\mathbf{m}||_{q_i}(F_j \Delta A_j))^{\frac{1}{p}} + (||\mathbf{m}||_{q_i}(A_j))^{\frac{1}{p}} \\ &< \frac{\epsilon}{3c} + (||\mathbf{m}||_{q_i}(A_j))^{\frac{1}{p}} \qquad (16.2.2)\end{aligned}$$

for $j = 1, 2, ..., k$ and $i = 1, 2, ..., n$.

Again by Theorem 5.13(i) for $\mathbf{m}_q$ and by (16.2.1) and (16.2.2) we have

$$\begin{aligned}(\mathbf{m}_{q_i})_p^{\bullet}(s - \omega, T) &\leq \sum_{j=1}^{k} (\mathbf{m}_{q_i})_p^{\bullet}(\beta_j \chi_{A_j} - \alpha_j \chi_{F_j}, T) \\ &\leq \sum_{j=1}^{k} |\beta_j - \alpha_j| (\mathbf{m}_{q_i})_p^{\bullet}(\chi_{F_j}, T) + \sum_{j=1}^{k} |\beta_j| (\mathbf{m}_{q_i})_p^{\bullet}(\chi_{A_j} - \chi_{F_j}, T) \\ &= \sum_{j=1}^{k} |\beta_j - \alpha_j| (||\mathbf{m}||_{q_i}(F_j))^{\frac{1}{p}} + \sum_{j=1}^{k} |\beta_j| (\frac{\epsilon}{3c}) \\ &< \sum_{j=1}^{k} (|\beta_j - \alpha_j| \frac{\epsilon}{3c}) + \sum_{j=1}^{k} |\beta_j - \alpha_j| (||\mathbf{m}_{q_i}||(A_j))^{\frac{1}{p}} + \frac{\epsilon}{3} \\ &< \frac{\epsilon c}{3c} + \frac{\epsilon c}{3M} M + \frac{\epsilon}{3} = \epsilon\end{aligned}$$

for $i = 1, 2, ..., n$ since $M > 1$ and $0 < \epsilon < 1$ and hence $\omega \in s + U$. This shows that W is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{I}_s$. Since $\mathcal{I}_s$ is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{L}_p(\mathbf{m})$ (resp. in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) by Theorem 15.6, it follows that W is dense in $\mathcal{L}_p(\mathbf{m})$ (resp. in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and hence $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable. Then $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$ is separable by Problem H, Ch. 4 of [Ke].

Let $1 \leq r < \infty$ and let $q \in \Gamma$. Since $\rho(\mathbf{m})_q^{(r)}(A, B) = (||\mathbf{m}||_q(A \Delta B))^{\frac{1}{r}}$, it follows that $\rho(\mathbf{m})_q^{(r)}(A, B_\alpha) \to 0$ if and only if $\rho(\mathbf{m})_q^{(p)}(A, B_\alpha) \to 0$ for $A \in \mathcal{P}$ and for a net $(B_\alpha) \subset \mathcal{P}$. Hence the second part of (i) holds.

(ii) If X is metrizable, $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) is pseudo-metrizable and hence if $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable then by Theorem 11, Ch. 4 of [Ke], $(\mathcal{P}, \tau_{\mathbf{m}}^{(p)})$ is separable. The converse holds by (i).

The following theorem gives a sufficient condition for the separability of $(\mathcal{P}, \tau_{\mathbf{m}}^{(p)})$.

Theorem 16.3. Let X be a quasicomplete (resp. sequentially complete) lcHs and let $\mathcal{S}$ be a σ-ring of subset of T. Suppose $\mathbf{m} : \mathcal{P} \to X$ (resp. $\mathbf{n} : \mathcal{S} \to X$) is σ-additive. If there exists a countable family $\mathcal{R}$ such that (i) $\mathcal{P}$ is the δ-ring (resp. (ii) $\mathcal{S}$ is the σ-ring) generated by $\mathcal{R}$, then $(\mathcal{P}, \tau_{\mathbf{m}}^{(p)})$ (resp. $(\mathcal{S}, \tau_{\mathbf{n}}^{(p)})$) is separable for $1 \le p < \infty$.

Proof. In the light of Theorem 5C of [H], without loss of generality we shall assume that $\mathcal{R}$ is a countable subring of $\mathcal{P}$ (resp. of $\mathcal{S}$). Let $1 \le p < \infty$, p fixed.

(i) Let $A \in \mathcal{P}$ and let U be a $\tau_{\mathbf{m}}^{(p)}|_{\mathcal{P}}$-neighborhood of A in $\mathcal{P}$. Then there exist an $\epsilon > 0$ and $q_1, q_2, ..., q_k$ in Γ such that $\{B \in \mathcal{P} : \rho(\mathbf{m})_{q_j}^{(p)}(A, B) < \epsilon, j = 1, 2, ..., k\} \subset U$. By hypothesis and by Corollary to Proposition 10, §1 of [Din], there exists $F \in \mathcal{R}$ such that $A \subset F$. Then $F \cap \mathcal{P}$ is a σ-algebra of subsets of F and hence by Theorem 2.6 there exist control measures $\mu_F^{(j)} : F \cap \mathcal{P} \to [0, \infty)$ for $\mathbf{m}_{q_j} : F \cap \mathcal{P} \to \widetilde{X_{q_j}}$, $j = 1, 2, ..., k$. Hence there exists $\delta > 0$ such that $\mu_F^{(j)}(B) < \delta$ implies $||\mathbf{m}||_{q_j}(B) < \epsilon^p$ for $j = 1, 2, ..., k$. Since $F \cap \mathcal{P}$ is the σ-ring generated by the ring $F \cap \mathcal{R} (\subset \mathcal{R})$ (to prove this use Theorem 5E 0f [H]), by ex.13.8 of [H] there exists $B \in F \cap \mathcal{R}$ such that $\sum_{j=1}^{k-1} \mu_F^{(j)}(A \Delta B) < \delta$ and $\mu_F^{(k)}(A \Delta B) < \delta$ so that $\mu_F^{(j)}(A \Delta B) < \delta$ for $j = 1, 2, ..., k$. Consequently, $\rho(\mathbf{m})_{q_j}^{(p)}(A, B) = (\mathbf{m}_{q_j})_p^\bullet(\chi_A - \chi_B, T) = (||\mathbf{m}||_{q_j}(A \Delta B))^{\frac{1}{p}} < \epsilon$ for $j = 1, 2, ..., k$ and hence $B \in U$. This shows that the countable subring $\mathcal{R}$ is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{P}$ and hence $(\mathcal{P}, \tau_{\mathbf{m}}^{(p)})$ is separable for $p \in [1, \infty)$.

(ii) Let $A \in \mathcal{S}$ and let U be a $\tau_{\mathbf{n}}^{(p)}$-neighborhood of A in $\mathcal{S}$. Then there exist an $\epsilon > 0$ and $q_1, q_2, ..., q_k$ in Γ such that $\{B \in \mathcal{S} : \rho(\mathbf{n})_{q_j}^{(p)}(A, B) < \epsilon, j = 1, 2, ..., k\} \subset U$. Then by Theorem 2.6 there exist control measures $\mu_j : \mathcal{S} \to [0, \infty)$ for $\mathbf{n}_j : \mathcal{S} \to \widetilde{X_{q_j}}$, $j = 1, 2, ..., k$. Now using the hypothesis that $\mathcal{S}$ is the σ-ring generated by the countable subring $\mathcal{R}$ and arguing as in the proof of (i) we conclude that $(\mathcal{S}, \tau_{\mathbf{n}}^{(p)})$ is separable.

Definition 16.4. Let T be a locally compact Hausdorff space. Let $\mathcal{U}, \mathcal{C}$ and $\mathcal{C}_0$ be respectively the families of all open sets, all compact sets and all compact G_δs in T. Then $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_0(T)$) denotes $\sigma(\mathcal{U})$ (resp. $\sigma(\mathcal{C})$, $\sigma(\mathcal{C}_0)$). The members of $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_0(T)$) are called Borel (resp. σ-Borel, Baire) sets in T.

Theorem 16.5. Let T be a locally compact Hausdorff space with a countable base of open sets and let X be a quasicomplete (resp. sequentially complete) lcHs. Let $\mathbf{m} : \mathcal{S} \to X$ be σ-additive, where $\mathcal{S} = \mathcal{B}(T)$ or $\mathcal{B}_c(T)$ or $\mathcal{B}_0(T)$. Then $\mathcal{L}_p(\mathbf{m})$ and $L_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\mathcal{S}, \mathbf{m})$ and $L_p(\mathcal{S}, \mathbf{m})$) are separable for $1 \le p < \infty$. (Note that $\sigma(\mathcal{S}) = \mathcal{S}$.)

Proof. By hypothesis, $\mathcal{B}(T)$ is a countably generated σ-ring; by hypothesis and by Corollary to Proposition 2 (resp. and by Corollary to Proposition 16) of §14 of [Din], $\mathcal{B}_c(T)$ (resp. $\mathcal{B}_0(T)$) is a countably generated σ-ring. Hence the results hold by Theorems 16.3(ii) and 16.2(i).

To obtain some useful sufficient conditions for the separability of $\mathcal{L}_p(\mathbf{m})$ and $L_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$) for $\mathcal{P} = \delta(\mathcal{C})$ or $\delta(\mathcal{C}_0)$ (i.e. the δ-rings generated by $\mathcal{C}$ and $\mathcal{C}_0$, respectively) in a locally compact Hausdorff space T we give the following

Theorem 16.6. Let X be a Frechét space, $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and $1 \le p < \infty$. Suppose that $\mathbf{m}$ admits a σ-additive X-valued extension $\tilde{\mathbf{m}}$ on $\sigma(\mathcal{P})$ and that $\sigma(\mathcal{P})$ is countably generated. Then $\mathcal{L}_p(\mathbf{m})$, $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$, $L_p(\mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$ are separable.

Proof. By hypothesis and by Theorems 16.3 and 16.2(i), $\mathcal{L}_p(\tilde{\mathbf{m}})$ and $L_p(\tilde{\mathbf{m}})$ are separable. If $f \in \mathcal{L}_p(\mathbf{m})$, then f is $\mathbf{m}$-measurable and hence is $\tilde{\mathbf{m}}$-measurable. As $\tilde{\mathbf{m}}|_{\mathcal{P}} = \mathbf{m}$, $\int_A s d\tilde{\mathbf{m}} = \int_A s d\mathbf{m}$ for $A \in \sigma(\mathcal{P})$ and for $s \in \mathcal{I}_s$. Then by Definition 12.1 and by the fact $|f|^p$ is $\mathbf{m}$-integrable in T (see Notation 15.9), $|f|^p$ is $\tilde{\mathbf{m}}$-integrable in T with values in X so that $f \in \mathcal{L}_p(\tilde{\mathbf{m}})$. Moreover, $(\tilde{\mathbf{m}}_q)^\bullet_p(f,T) = \sup_{A\in\sigma(\mathcal{P}), x^*\in U_q^\circ}(\int_A |f|^p dv(x^*\tilde{\mathbf{m}}))^{\frac{1}{p}} = \sup_{A\in\sigma(\mathcal{P}), x^*\in U_q^\circ}(\int_A |f|^p dv(x^*\mathbf{m}))^{\frac{1}{p}} = (\mathbf{m}_q)^\bullet_p(f,T)$ for $q \in \Gamma$. Hence $\mathcal{L}_p(\mathbf{m})$ is a subspace of $\mathcal{L}_p(\tilde{\mathbf{m}})$ with $\tau^{(p)}_{\tilde{\mathbf{m}}}|_{\mathcal{L}_p(\mathbf{m})} = \tau^{(p)}_{\mathbf{m}}$. As X is metrizable, $\mathcal{L}_p(\tilde{\mathbf{m}})$ is pseudo-metrizable and consequently, by Theorem 11, Ch. 4, of [Ke], $\mathcal{L}_p(\mathbf{m})$ and $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ are separable and then by Problem H, Ch. 4 of [Ke], $L_p(\mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$ are spearable.

Let us recall the following definition from [P2].

Definition 16.7. Let X be a quasicomplete lcHs and let T be a locally compact Hausdorff space. Suppose $\mathcal{R}$ is a ring of sets in T such that $\mathcal{R} \supset \delta(\mathcal{C})$ or $\delta(\mathcal{C}_0)$. An X-valued σ-additive vector measure $\mathbf{m}$ on $\mathcal{R}$ is said to be $\mathcal{R}$-regular if, given $\epsilon > 0$ and $A \in \mathcal{R}$, there exist an open set $U \in \mathcal{R}$ and a compact $K \in \mathcal{R}$ such that $K \subset A \subset U$ and such that $||\mathbf{m}||_q(B) < \epsilon$ for each $B \in \mathcal{R}$ with $B \subset U \backslash K$ and for each $q \in \Gamma$. (In the light of Proposition 2.2, this is equivalent to the $\mathcal{R}$-regularity given in Definition 5 of [P2].)

Theorem 16.8. Let T be a locally compact Hausdorff space and let X be a Frechét space. Let $\mathbf{m} : \delta(\mathcal{C}_0) \to X$ be σ-additive. If the range of $\mathbf{m}$ is relatively weakly compact, then $\mathbf{m}$ has a unique σ-additive X-valued $\mathcal{B}_0(T)$-regular (resp. $\mathcal{B}_c(T)$-regular) extension $\mathbf{m}_0$ on $\mathcal{B}_0(T)$ (resp. $\mathbf{m}_c$ on $\mathcal{B}_c(T)$). Let $\mathbf{m}_c' = \mathbf{m}_c|_{\delta(\mathcal{C})}$. If T has a countable base of open sets, then $\mathcal{L}_p(\mathbf{m})$, $\mathcal{L}_p(\sigma(\mathcal{C}_0), \mathbf{m})$, $L_p(\mathbf{m})$ and $L_p(\sigma(\mathcal{C}_0), \mathbf{m})$ (resp. $\mathcal{L}_p(\mathbf{m}_c')$, $\mathcal{L}_p(\sigma(\mathcal{C}), \mathbf{m}_c')$, $L_p(\mathbf{m}_c')$ and $L_p(\sigma(\mathcal{C}), \mathbf{m}_c')$) are separable for $1 \le p < \infty$.

Proof. By the hypothesis on the range of $\mathbf{m}$ and by Theorem on Extension of [K] or by Corollary 2 of [P1] (where a self-contained short proof of the said theorem of [K] is given), $\mathbf{m}$ admits an X-valued σ-additive extension $\mathbf{m}_0$ on $\mathcal{B}_0(T)$, which is unique by the Hahn-Banach theorem. Then by Theorem 10 of [P2] or by Theorem 1 of [DP2] (where a simple proof of the said theorem of [P2] is given), $\mathbf{m}_0$ is $\mathcal{B}_0(T)$-regular and has a unique X-valued $\mathcal{B}_c(T)$-regular σ-additive extension $\mathbf{m}_c$ on $\mathcal{B}_c(T)$. Then the conclusions follow from Theorem 16.6 since $\mathcal{B}_c(T)$ (resp. $\mathcal{B}_0(T)$) is countably generated by the hypothesis on T and by Corollary to Proposition 2 (resp. and by Corollary to Proposition 16) of §14 of [Din].

Following [Ri2] we give the following

Definition 16.9. Let X be a quasicomplete (resp. sequentially complete) lcHs and $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $\mathbf{m}(\mathcal{P}) = \{\mathbf{m}(A) : A \in \mathcal{P}\}$. For $A, B \in \mathcal{P}$, we write $A \sim B$ if $\chi_A = \chi_B$ $\mathbf{m}$-a.e. in T. Then we write $\mathcal{P}(\mathbf{m}) = \mathbf{m}(\mathcal{P})/\sim$. (Note that $'\sim'$ is an equivalence relation.)

Remak 16.10. If $A, B \in \mathcal{P}$ and $A \sim B$, then $||\mathbf{m}||_q(A\Delta B) = 0$ for each $q \in \Gamma$ and hence $\rho(\mathbf{m})_q^{(p)}(A, B) = 0$ for $1 \leq p < \infty$. Consequently, $\mathcal{P}(\mathbf{m}) \subset L_p(\mathbf{m})$ for all $p \in [1, \infty)$ and by Theorem 16.2(i), $\tau_{\mathbf{m}}^{(p)}|_{\mathcal{P}(\mathbf{m})} = \tau_{\mathbf{m}}^{(r)}|_{\mathcal{P}}(\mathbf{m})$ for all $p, r \in [1, \infty)$. (Here by an abuse of notation we denote by $\tau_{\mathbf{m}}^{(p)}$ also the quotient topology induced on $L_p(\mathbf{m})$.)

Notation 16.11. In the light of Remark 16.10, when we write $\mathcal{P}(\mathbf{m})$, we consider it as a subset of some $L_r(\mathbf{m})$, $1 \leq r < \infty$, with the relative topology from $L_r(\mathbf{m})$.

Theorem 16.12. Let X and $\mathbf{m}$ be as in Definition 16.9. Then:

(i) If for some $p \in [1, \infty)$, $(\mathcal{P}(\mathbf{m}), \tau_{\mathbf{m}}^{(p)})$ is separable, then $L_r(\mathbf{m})$ (resp. $L_r(\sigma(\mathcal{P}), \mathbf{m})$) is separable for all $r \in [1, \infty)$. In that case, in the light of Notation 16.11 we simply say that $\mathcal{P}(\mathbf{m})$ is separable.

(ii) If X is further metrizable, then $(\mathcal{P}(\mathbf{m}), \tau_{\mathbf{m}}^{(r)})$ is separable for all $r \in [1, \infty)$ whenever $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable for some $p \in [1, \infty)$.

Proof. (i) By hypthesis there exists a countable $\tau_{\mathbf{m}}^{(p)}$-dense set D in $\mathcal{P}(\mathbf{m})$. Then by Remark 16.10, D is also $\tau_{\mathbf{m}}^{(r)}$-dense in $\mathcal{P}(\mathbf{m})$ for any $r \in [1, \infty)$. Now taking W as in the proof of Theorem 16.2(i) and using the fact that $\mathcal{I}_s/\sim$ is dense in $L_r(\mathbf{m})$ (resp. $L_r(\sigma(\mathcal{P}), \mathbf{m})$) by Theorem 15.6, we conclude that W is dense in $L_r(\mathbf{m})$ (resp. $L_r(\sigma(\mathcal{P}), \mathbf{m})$ and hence (i) holds.

(ii) If X is further metrizable, then $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is metrizable. If $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable, then by Theorem 11, Ch. 4 of [Ke], $\mathcal{P}(\mathbf{m})$ is separable for the relative topology from $L_p(\mathbf{m})$ (resp. from $L_p(\sigma(\mathcal{P}), \mathbf{m})$). Then by (i), $L_r(\mathbf{m})$ (resp. $L_r(\sigma(\mathcal{P}), \mathbf{m})$) is separable for all $r \in [1, \infty)$.

To give a characterization of the separability of $\mathcal{S}(\mathbf{n})$ similar to Propositions 2 and 3(ii) of [Ri2] when $\mathcal{S}$ is a σ-ring of sets and $\mathbf{n} : \mathcal{S} \to X$ is σ-additive, we give the following concept. See also ([KK], pp.32-33).

Definition 16.13. Let X be a quasicomplete (resp. sequentially complete) lcHs and let $\mathcal{S}$ be a σ-ring of subsets of T. Let $\mathbf{m} : \mathcal{P} \to X$ (resp. $\mathbf{n} : \mathcal{S} \to X$) be σ-additive. If there exists a countably generated δ-ring $\mathcal{P}_0 \subset \mathcal{P}$ (resp. σ-ring $\mathcal{S}_0 \subset \mathcal{S}$) such that $\mathcal{P}_0(\mathbf{m}) = \mathcal{P}(\mathbf{m})$ (resp. $\mathcal{S}_0(\mathbf{n}) = \mathcal{S}(\mathbf{n})$)(see Definition 16.9), then $\mathcal{P}$ (resp. $\mathcal{S}$) is said to be $\mathbf{m}$-essentially countably generated.

Theorem 16.14. Let X ba a quasicomplete (resp. sequentially complete) lcHs and $\mathcal{S}$ be σ-ring of subsets of T. Let $\mathbf{m} : \mathcal{P} \to X$ (resp. $\mathbf{n} : \mathcal{S} \to X$) be σ-additive. Then:

(i) If $\mathcal{P}$ is $\mathbf{m}$-essentially (resp. $\mathcal{S}$ is $\mathbf{n}$-essentially) countably generated, then $\mathcal{P}(\mathbf{m})$ (resp. $\mathcal{S}(\mathbf{n})$) is separable (see Notation 16.11) and hence, for $1 \leq p < \infty$, $\mathcal{L}_p(\mathbf{m})$ and $L_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$) are separable; (resp. $\mathcal{L}_p(\mathbf{n})$ and $L_p(\mathbf{n})$ (resp. $\mathcal{L}_p(\mathcal{S}, \mathbf{n})$ and $L_p\mathcal{S}, \mathbf{n})$) ar separable. (Note that $\sigma(\mathcal{S}) = \mathcal{S}$.)

(ii) If $\mathcal{S}(\mathbf{n})$ is separable and if the range $\mathbf{n}(\mathcal{S})$ is metrizable for the relative topology from X, then $\mathcal{S}$ is $\mathbf{n}$-essentially countably generated. (Consequently, by (i), $\mathcal{L}_p(\mathbf{n})$ and $L_p(\mathbf{n})$ (resp. $\mathcal{L}_p(\mathcal{S}, \mathbf{n})$ and $L_p(\mathcal{S}, \mathbf{n})$) are separable for all $p \in [1, \infty)$).

Proof. (i) We shall prove for the case of $\mathcal{P}(\mathbf{m})$. The proof for $\mathcal{S}(\mathbf{n})$ is similar. Let $\mathcal{P}_0$ be a countably generated δ-ring such that $\mathcal{P}_0(\mathbf{m}) = \mathcal{P}(\mathbf{m})$. Then by Theorem 16.3, for all $p \in [1, \infty)$, $\mathcal{P}_0$ is separable for the topology $\tau_{\mathbf{m}}^{(p)}$. Let $1 \leq p < \infty$, p fixed. Then in the light of Theorem 5C of [H], there exists a $\tau_{\mathbf{m}}^{(p)}$-dense countable subring $\mathcal{R}_0$ of $\mathcal{P}_0$. Let $\mathcal{I}_{\mathcal{R}_0}$ be the set of all $\mathcal{R}_0$-simple functions and let $W = \{\sum_1^r \alpha_j \chi_{F_j} : \alpha_j = a_j + ib_j, a_j, b_j \text{ real rational}, F_j \in \mathcal{R}_0, j = 1, 2, ..., r, r \in \mathbb{N}\}$.

Then W is countable and by an argument similar to that in the proof of Theorem 16.2(i), W is $\tau_{\mathbf{m}}^{(p)}$-dense in $\mathcal{I}_{\mathcal{R}_0}$. Let $f \in \mathcal{L}_p(\mathbf{m})$ (resp. $f \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) and let V be a neighborhood of f in $\mathcal{L}_p(\mathbf{m})$ (resp. in $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$). Then there exist an $\epsilon > 0$ and $q_1, q_2, ..., q_n$ in Γ such that $V \supset \{g \in \mathcal{L}_p(T, \mathcal{P}, \mathbf{m})\ (\text{resp}\ g \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})) : (\mathbf{m}_{q_i})_p^\bullet(f - g, T) < \epsilon, i = 1, 2, ..., n\}$. By Theorem 15.6, there exists $s \in \mathcal{I}_s$ such that $(\mathbf{m}_{q_i})_p^\bullet(f - s, T) < \frac{\epsilon}{2}$ for $i = 1, 2, ..., n$ and as $\mathcal{P}_0(\mathbf{m}) = \mathcal{P}(\mathbf{m})$, there exists $s' \in \mathcal{I}_{\mathcal{R}_0}$ such that $s = s'$ $\mathbf{m}$-a.e. in T. Moreover, as W is dense in $\mathcal{I}_{\mathcal{R}_0}$, there exists $\omega \in W$ such that $(\mathbf{m}_{q_i})_p^\bullet(s' - \omega, T) < \frac{\epsilon}{2}$ for $i = 1, 2, ..., n$. Then it follows that $\omega \in V$ and hence $\mathcal{L}_p(\mathbf{m})$ (resp. $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable. Then by Problem H, Ch. 4 of [Ke], $L_p(\mathbf{m})$ (resp. $L_p(\sigma(\mathcal{P}), \mathbf{m})$) is separable. Hence (i) holds.

(ii) Let $\Omega = \mathbf{n}(\mathcal{S})$. By hypothesis, Ω and hence $\Omega - \Omega$ is metrizable for the relative topology from X. Hence there exists a sequence $(V_n)_1^\infty$ of τ-closed absolutely convex τ-neighborhoods of 0 in X such that

(i) $V_{n+1} + V_{n+1} \subset V_n$ for $n \geq 1$; and
(ii) $V_n \bigcap (\Omega - \Omega), n \in \mathbb{N}$, form a neighborhood base of 0 in $(\Omega - \Omega, \tau|(\Omega - \Omega))$.

Let τ_1 be the locally convex topology on X for which $(V_n)_1^\infty$ is a neighborhood base of 0 in X. Then, arguing as in the proof of Theorem 2.1 of [S], we observe that $\tau_1|_\Omega = \tau|_\Omega$. If q_n is the Minkowski functional of V_n, then q_n is a τ_1-continuous seminorm on X and hence $(q_n)_1^\infty \subset \Gamma$.

Claim 1. $\tau_{\mathbf{n}}^{(p)}|_{\mathbf{n}(\mathcal{S})}$ is generated by $(\rho(\mathbf{n})_{q_k}^{(p)})_{k=1}^\infty$ for $p \in [1, \infty)$.

In fact, let $1 \leq p < \infty$ be fixed and let τ_0 be the topology induced by $(\rho(\mathbf{n})_{q_k}^{(p)})_{k=1}^\infty$ on $\mathbf{n}(\mathcal{S})$. Clearly, $\tau_0 \leq \tau_{\mathbf{n}}^{(p)}|_{\mathbf{n}(\mathcal{S})}$. Conversely, given $q \in \Gamma$, there exists q_n such that $U_q \bigcap \mathbf{n}(\mathcal{S}) \subset U_{q_n} \bigcap \mathbf{n}(\mathcal{S})$ as $\tau_1|_\Omega = \tau|_\Omega$. Then for $x^* \in U_{q_n}^0$ we have $|x^*(x)| \leq 1$ whenver $x \in U_q \bigcap \mathbf{n}(\mathcal{S})$ and hence by Theorem 13.2, $(\mathbf{m}_{q_n})_p^\bullet(\chi_A - \chi_B, T) \leq (\mathbf{m}_q)_p^\bullet(\chi_A - \chi_B, T)$ for $A, B \in \mathcal{S}$. Therefore, $\tau_{\mathbf{n}}^{(p)}|_{\mathbf{n}(\mathcal{S})} \leq \tau_0$ and hence the claim holds.

By hypothesis, $\mathcal{S}(\mathbf{n}) = \mathbf{n}(\mathcal{S})/\sim$ is separable in $\tau_{\mathbf{n}}^{(p)}|_{\mathcal{S}(\mathbf{n})}$ and hence in τ_0 by Claim 1 (by an abuse of notation (see Remark 16.10)). Hence there exists a countable τ_0-dense family $\mathcal{F} \subset \mathcal{S}(\mathbf{n})$. Let $\mathcal{S}_0 = \sigma(\mathcal{F})$. Then, obviously, $\mathcal{S}_0(\mathbf{n}) \subset \mathcal{S}(\mathbf{n})$. Now let $A \in \mathcal{S}(\mathbf{m})$. As τ_0 is psudo-metrizable, there exists a sequence $(F_n)_1^\infty \subset \mathcal{F}$ such that $\lim_n (\mathbf{m}_{q_k})_p^\bullet(\chi_A - \chi_{F_n}, T) = 0$ for each $k \in \mathbb{N}$. Then by Theorems 5.18(vi) and 5.19 for $\mathbf{m}_{q_k}$, there exists a subsequence $(F_{n,1})_{n=1}^\infty$ of $(F_n)_1^\infty$ such that $\chi_{F_{n,1}} \to \chi_A$ $\mathbf{m}_{q_1}$-a.e. in T. As $(\mathbf{m}_{q_2})_p^\bullet(\chi_A - \chi_{F_{n,1}}, T) \to 0$, by the said theorems there exists a subsequence $(F_{n,2})_{n=1}^\infty$ of $(F_{n,1})_{n=1}^\infty$ such that $\chi_{F_{n,2}} \to \chi_A$ $\mathbf{m}_{q_2}$-a.e. in T. Continuing this process indefinitely, we note that the diagonal sequence $(\chi_{F_{n,n}})_{n=1}^\infty$ is some subsequence $(\chi_{F_{n_k}})_{k=1}^\infty$ of $(\chi_{F_n})_1^\infty$ and $\chi_{F_{n_k}} \to \chi_A$ $\mathbf{m}$-a.e. in T as $\tau_{\mathbf{n}}^{(p)}|_{\mathbf{n}(\mathcal{S})} = \tau_0$. Let $N \in \mathcal{S}$ with $||\mathbf{m}||_q(N) = 0$ for all $q \in \Gamma$ such that $\chi_{F_{n_k}}(t) \to \chi_A(t)$ for $t \in T \backslash N$. Then, for each $t \in A \backslash N$, $\chi_{F_{n_k}}(t) \to 1$ and hence $\liminf_k F_{n_k} \supset A \backslash N$. Similarly, for each $t \in T \backslash A \backslash N$, $\chi_{F_{n_k}}(t) \to 0$ so that $\liminf_k (T \backslash F_{n_k}) \supset T \backslash A \backslash N$ or equivalently, $\limsup_k F_{n_k} \subset A \cup N$. Thus it follows that $\chi_A = \limsup_k F_{n_k}$ $\mathbf{m}$-a.e. in T. As $\limsup_k F_{n_k} \in \mathcal{S}_0$, $A \in \mathcal{S}_0(\mathbf{m})$ so that $\mathcal{S}(\mathbf{m}) = \mathcal{S}_0(\mathbf{m})$. Hence $\mathcal{S}$ is $\mathbf{m}$-essentially countably generated.

Remark 16.15. The above theorem for $p = 1$ subsumes Propositions 2 and 3(ii) of [Ri2] given for a σ-additive vector measure $\mathbf{m}$ defined on a σ-algebra Σ of subsets of T with values in a sequentially complete lcHs. There the proof is based on the facts that $\mathbf{m}$ is a closed measure as shown in [Ri1] and that the σ-algebra generated by an algebra of sets is the same as its sequential closure. But our proof is different, more general and covers all $p \in [1, \infty)$.

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Departamento de Matemáticas, Facultad de Ciencias
Universidad de los Andes, Mérida, Venezuela
e-mail address: panchapa@ciens.ula.ve