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Abstract

Let (X,τ) be a countable topological space. We say that τ is an analytic (Borel) topology if τ as a subset of the Cantor set 2^X (via characteristic functions) is an analytic (Borel) set. For example, the topology of the Arhangel'skii-Franklin space S_ω is $F_{\sigma\delta}$. In this paper we study the complexity, in the sense of the Borel hierarchy, of the subspaces of S_ω. We show that S_ω has subspaces with topologies of arbitrarily high Borel rank and it also has subspaces with a non Borel topology. Moreover, a closed subset of S_ω has this property iff it contains a copy of S_ω.

Keywords: Countable topological spaces, sequential spaces, Borel and analytic sets.
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1 Introduction

Let (X,τ) be a countable topological space. We say that τ is an analytic (Borel) topology if τ as a subset of the Cantor set 2^X (identifying a subset of X with its characteristic function) is an analytic (Borel) set. Most of the examples of countable topological spaces found in the literature are analytic. For example, every second countable topology is $F_{\sigma\delta}$, in particular, the topology of the rational is (in fact a complete) $F_{\sigma\delta}$ subset of $2^{\mathbb{Q}}$. Another examples of $F_{\sigma\delta}$ topologies are the Arens space [1] or its more general version, the Arhangel'skii-Franklin space S_ω [2]. A systematic study of analytic topologies was initiated in [11] where it was shown explicitly the connection between descriptive set theoretic properties and pure topological properties of a given space. For example, analytic topologies are tight related to spaces of continuous functions: a T_2 regular countable space is analytic iff it is homeomorphic to a countable subspace of $C_p(\mathbb{N}^{\mathbb{N}})$ (the space of real valued continuous functions over the Baire space $\mathbb{N}^{\mathbb{N}}$ with the pointwise topology) [11, theorem 6.1].

In this note we are interested in studying the complexity of the subspace topologies of a given countable space. It is clear that any subspace Y of a space X with an analytic topology also has an analytic topology. However, the complexity of the subspace topology of Y (measured in terms of the Borel hierarchy) might vary considerably depending on X and Y. On the one hand, if X is second countable or more generally with a F_σ basis (see section 3.2 for the definition), then the topology of every subspace of X is $F_{\sigma\delta}$. On the other hand, we will show in this paper that the Arhangel'skii-Franklin space S_ω (which has a $F_{\sigma\delta}$ topology) has subspaces with arbitrarily high Borel rank and also has non Borel subspaces (see §2 for the definition of S_ω and some general information about it).

Our main result is the following

Theorem 1.1. *Let X be a closed subset of S_ω. The following are equivalent:*

(i) X has a subspace whose topology is not Borel.

(ii) X has subspaces with Borel topology of arbitrarily high Borel rank.

(iii) X contains a copy of S_ω.

The proof uses the fact that S_ω is a sequential space, thus every closed subspace X has associated an ordinal $\rho(X)$ called the sequential rank (see the definition in §2). We will show the following

Theorem 1.2. *Let X be a closed subset of S_ω.*

(i) If $\rho(X) < \omega_1$, then the subspace topology of every subset of X is Borel.

(ii) If $\rho(X) = \omega_1$, then X has a closed copy of S_ω and a subspace whose topology is not Borel.

Examples of subspaces of S_ω with Borel topology of arbitrarily high rank are presented in §4 and are given essentially by the terminal nodes of wellfounded trees. We will construct Borel filters of arbitrarily high rank which in fact are the nbhd filter of the unique non isolated point of a certain subspace of S_ω. Part (i) and (ii) of theorem 1.2 are shown in §5 and §6 respectively.

A very natural question is to determine which countable spaces satisfy the conclusion of theorem 1.1. In particular, we would like to know when a countable space contains a copy of S_ω (a similar question was asked in [2]). In section §3 we show that if a countable space X with Borel topology satisfies that the nbdh filter of every point is Borel, then every subspace of X also has a Borel topology. Thus, by theorem 1.1, there must be a point s in S_ω such that the nbhd filter of s is not Borel. Since S_ω is homogeneous, then the nbhd filter of every point is not Borel (we will show that in fact, they are complete analytic sets (see proposition 6.4)). This stands in contrast with the fact that S_ω has a $F_{\sigma\delta}$ topology.

We end this introduction making some comments about the connection between analytic topologies and the descriptive complexity of $C_p(X)$, the space of real valued continuous function on a non discrete completely regular countable topological space X with the topology of pointwise convergence. There have been a lot of work on the classification of $C_p(X)$ (see [6, 4, 5] and the references therein). One of their main results is that if $C_p(X)$ is $F_{\sigma\delta}$ as a subset of $\mathbb{R}^X$, then $C_p(X)$ is homeomorphic to σ^ω (the countable product of the space of sequences eventually equal to zero) . It can be shown that a regular topology on X is analytic iff $C_p(X)$ is analytic. However the exact relationship between the complexity of the topology on X and that of $C_p(X)$ has not been fully investigated. For the case of spaces with only one non isolated point, i.e. spaces associated to filters, this has been done ([6, Lemma 4.2] and references therein). We have not pursued this issue here but we think it is worth and it will be treated elsewhere.

In [?, 5] was studied the problem of classifying $C_D(X)$ the set of continuous functions on X with the topology of pointwise convergence on D, where D is a countable dense subset of X, i.e, $C_D(X) = \{f|D : f \in C(X)\} \subseteq \mathbb{R}^D$. They have shown that the Borel complexity of $C_D(X)$ might vary considerably depending on D and X. For instance, for every countable ordinal α there is a space X_α and a dense subset D_α of X_α such that $C_p(X_\alpha)$ is $F_{\sigma\delta}$ and $C_{D_\alpha}(X_\alpha)$ has Borel rank larger than α (see [5, Prop. 2.6]. Our results go in the same line and show that a similar phenomenon happens within the single space S_ω (see 5.4).

2 Preliminaries

We will use the standard notions and terminology of descriptive set theory (see for instance [10]). $\omega^{<\omega}$ denotes the collection of finite sequences of natural numbers. If $s \in \omega^{<\omega}$, $|s|$ denote its length.

For $n \in \mathbb{N}$, $s^\frown n$ is the concatenation of s with n. For $\alpha \in \mathbb{N}^{\mathbb{N}}$, we denote $\alpha|n$ the restriction of α to $\{0,1,\cdots,n-1\}$. The Borel sets of rank α will be denoted by $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$, where for instance $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ are respectively the open and closed sets, $\mathbf{\Sigma}^0_2$ and $\mathbf{\Pi}^0_2$ are respectively the F_σ and G_δ sets and so on. A subset of a Polish space is analytic (or $\mathbf{\Sigma}^1_1$) if it is the continuous image of the Baire space $\mathbb{N}^{\mathbb{N}}$. A well known result of Souslin says that a subset of a Polish space is Borel iff it is analytic and co-analytic (see for instance [10, theorem 14.11]). Let X, Y be Polish spaces and $A \subseteq X$, $B \subseteq Y$. The set A is said to be *Wadge reducible* to B, denoted by $A \leq_w B$, if there is a continuous function $f : X \to Y$ such that $x \in A$ iff $f(x) \in B$ (see [10, §21.E]). Notice if $A \leq_w B$ and A is Borel (projective), then the Borel (projective) type of B is at least that of A. Let Γ be a class of sets in Polish spaces. If Y is a Polish space, a set $A \subseteq Y$ is called Γ-complete if $A \in \Gamma(Y)$ and $B \leq_w A$ for all $B \in \Gamma$ (see [10, §22.B]). The archetypical $\mathbf{\Sigma}^1_1$-complete set is the collection of ill founded trees on $\mathbb{N}$, i.e. trees with at least one infinite branch (see [10, 27.1]). Any $\mathbf{\Sigma}^1_1$-complete set is not Borel. Thus to show that an analytic subset A of a Polish space Z is not Borel it suffices to show that the set of ill founded trees is Wadge reducible to A.

Let A be a subset of a topological space X, the sequential closure of A is defined by transfinite recursion as follows [2]. Let $A^{(0)} = A$ and $A^{(1)}$ be the set of all limits of convergent sequences in A, $A^{(\alpha+1)} = [A^{(\alpha)}]^{(1)}$ and $A^{(\beta)} = \cup_{\alpha<\beta} A^{(\alpha)}$ for β a limit ordinal. The sequential closure of A, denoted $[A]_{\text{seq}}$, is the set $A^{(\omega_1)}$. The space X is called *sequential* if for every $A \subseteq X$ the closure of A is equal to its sequential closure, i.e. $\overline{A} = [A]_{\text{seq}}$. A subset $O \subseteq X$ is said to be *sequentially open* iff for all $x \in O$ and a sequence x_n converging to x there is N such that $x_n \in O$ for all $n > N$. A space is sequential iff every sequentially open set is in fact open. A closed subspace of a sequential space is sequential.

Definition 2.1. *Let X be a sequential space and $A \subseteq X$. The* sequential rank *of A in X, denoted $\sigma(A, X)$ is defined by*

$$\sigma(A,X) = min\{\alpha : \ A^{(\alpha)} = A^{(\alpha+1)}\}$$

The sequential rank of X *is defined by*

$$\rho(X) = \sup\{\sigma(A,X) : \ A \subseteq X\}$$

The local versions of these ordinals are defined as follows. Given $A \subseteq X$ and $s \in X$ define

$$\begin{aligned} \sigma(s,A) &= min\{\alpha : \ s \in A^{(\alpha)}\} \ \ for\ s \in \overline{A} \\ \rho(s,X) &= \sup\{\sigma(s,A) : \ s \in \overline{A} \ \& \ A \subseteq X\} \end{aligned}$$

□

The following elementary facts about these ordinals are stated for later reference.

Proposition 2.2. *Let X be a sequential space, $A \subseteq X$ and $s \in X$.*

1. $A^{(\sigma(A,X))} = \overline{A}$.
2. $\rho(s,X) = 0$ *iff s is isolated in X.*
3. $\sigma(A,X) = \sup_{s\in\overline{A}} \sigma(s,A)$.
4. $\rho(X) = \sup_{s\in X} \rho(s,X)$.

□

Now we recall the definition of S_ω and some basic facts about it. Define a topology τ over $\omega^{<\omega}$ by

$$U \in \tau \Leftrightarrow \{n \in \mathbb{N} : \ s\widehat{\ }n \notin U\} \text{ is finite for all } s \in U$$

Let S_ω be the space $(\omega^{<\omega}, \tau)$. It is clear that S_ω is T_2, zero dimensional and has no isolated points. Notice that a set U is τ_{FIN}-open iff there is $f : \omega^{<\omega} \to \mathbb{N}$ such that if $s \in U$, then $s\widehat{\ }n \in U$ for all $n \geq f(s)$. A sequence $\{x_i\}_i$ in S_ω converges to s iff $\{x_i\}_i$ is eventually of the form $s\widehat{\ }n_i$ for some increasing sequence of integers $\{n_i\}$. From this it follows that S_ω is sequential. For each $t \in \omega^{<\omega}$ we define

$$N_t = \{s \in \omega^{<\omega} : \ t \preceq s\}$$

Notice that N_t is a clopen set in S_ω. If we consider τ as a subset of $2^{\omega^{<\omega}}$ (which with the product topology is homeomorphic to the Cantor set), then it is clear that τ is $F_{\sigma\delta}$.

S_ω has showed up in many different contexts. The first occurrence was as an example of a sequential homogeneous space of sequential rank ω_1 [2]. A very interesting description of S_ω as a translation invariant topology over $\mathbb{Z}$ is given in [7]. S_ω has been implicitly used to study sequential convergence in $C_p(X)$ [8]. For instance, if Z is a topological space such that there is a continuous surjection from Z onto a non-meager subset of $\mathbb{R}$, then $C_p(Z)$ contains a copy of S_ω. Another occurrence of S_ω is in the following result: a linear normed space has the Schur property iff it has no copy of S_ω ([11, theorem 5.3] and [8, theorem 17]). Another interesting property of S_ω appears in [3, example 3.8].

3 On the complexity of the neighborhood filters

In this section we will make some comments about the problem of determining when every subspace topology of a Borel topology is also Borel. Let us start by analyzing the case of a (Hausdorff) space with only one non isolated point. Let $\mathcal{F}$ be a filter over $\mathbb{N}$ and X be $\omega + 1$ with the topology where every $n \in \mathbb{N}$ is isolated and the nbhds of ω are the elements of $\mathcal{F}$. Let $Y \subseteq X$, then the restriction of $\mathcal{F}$ to Y, denoted by $\mathcal{F}_Y$, is easily seen to satisfied that $A \in \mathcal{F}_Y$ iff $A \subseteq Y$ and $A \cup (X \setminus Y) \in \mathcal{F}$. This shows that $\mathcal{F}_Y \leq_w \mathcal{F}$. Therefore, if $\mathcal{F}$ is Borel, then $\mathcal{F}_Y$ is also Borel and thus the subspace topology of Y is Borel for every $Y \subseteq X$.

Recall that the nbhd filter $\mathcal{F}_x$ of a point $x \in X$ is the filter over $X \setminus \{x\}$ defined by $A \in \mathcal{F}_x$ if there is an open set V such that $x \in V \subseteq A \cup \{x\}$. Notice that if τ is analytic, then every $\mathcal{F}_x$ is also analytic. It is elementary to show that V is open iff $V \setminus \{x\} \in \mathcal{F}_x$ for all $x \in V$. In particular, this says that if every $\mathcal{F}_x$ is Borel, then τ is also Borel. The converse is not true, as we will see in section §6, S_ω has a Borel topology but in fact all its nbhd filters are non Borel.

A basis $\mathcal{B}$ for a countable topological space X is said to be F_σ if $\mathcal{B}$ as a subset of 2^X is a F_σ set. Every space with a F_σ basis has a $F_{\sigma\delta}$ topology [11, proposition 3.2]. The converse is not true, since S_ω has a $F_{\sigma\delta}$ topology but it does not admit a F_σ basis [11, proposition 5.2] (this will be deduced also from one of the results presented in this paper). A countable T_2 regular space X has an F_σ basis iff X has a closed subbasis iff X is homeomorphic to a countable subspace of $C_p(2^{\mathbb{N}})$ ([11, theorems 3.2, 3.4 and 6.1]). It is easy to check that having a F_σ basis is a hereditary property. Moreover, in this case, every nbhd filter is F_σ. In fact, let $\{F_n\}_n$ be closed subsets of 2^X such that $\mathcal{B} = \bigcup_n F_n$ is a basis for τ. Then

$$A \in \mathcal{F}_x \ \Leftrightarrow \ \exists n \in \mathbb{N} \, \exists V \, [\, V \in F_n \ \& \ x \in V \subseteq A \cup \{x\} \,]$$

The set of all $(V,A)\in 2^X\times 2^X$ such that $V\in F_n$ & $x\in V\subseteq A\cup\{x\}$ is compact for any $x\in X$ and $n\in\mathbb{N}$. So $\mathcal{F}_x$ is a countable union of projections of compact sets, therefore it is F_σ for all x. We state this result in the following

Proposition 3.1. *If τ has a F_σ basis, then $\mathcal{F}_x$ is F_σ for all x.* □

The converse of the previous result does not hold. A counterexample is Arens space S_2 which can be defined as $\omega^{\leq 2}$ with the topology it inherits from S_ω (see [?, Example 1.6.19]). This topology does not admit a F_σ basis [11] but in this space every nbhd filter is Borel (in fact, it is $\mathbf{\Sigma}^0_4$, see lemma 4.5).

We will denote the closure operator of a topological space (X,τ) by cl_X or cl_τ. The following result characterizes when every $\mathcal{F}_x$ is Borel for an analytic (and therefore Borel) topology.

Theorem 3.2. *Let τ be an analytic topology over a countable set X. The following are equivalent*

1. *$\mathcal{F}_x$ is Borel for every $x\in X$.*
2. *For each $x\in X$, the set $C_x=\{A\subseteq X:\ x\in\overline{A}\}$ is Borel.*
3. *cl_τ is a Borel function from 2^X into 2^X.*
4. *The relation $R(A,Y)$ given by "A is closed in Y" is Borel (in $2^X\times 2^X$).*

Proof: Since τ is analytic, then R is analytic and each C_x is coanalytic. The following equivalences are straightforward:

$$\begin{array}{rcl} A\in\mathcal{F}_x & \Leftrightarrow & x\notin A\ \&\ X\setminus(A\cup\{x\})\notin C_x\\ A\in C_x & \Leftrightarrow & X\setminus(A\cup\{x\})\notin\mathcal{F}_x\\ R(A,Y) & \Leftrightarrow & \forall B\,[\mathrm{cl}_\tau(A)=B\to B\cap Y\subseteq A]\\ A\in C_x & \Leftrightarrow & x\in A\ \text{ or }\ [x\notin A\ \&\ \neg R(A,A\cup\{x\})]\\ \mathrm{cl}_\tau(A)=B & \Leftrightarrow & A\subseteq B\ \ \&\ \ \forall x(x\in B\to A\in C_x)\ \ \&\ \ X\setminus B\in\tau. \end{array}$$

Notice that the complementation mapping is a homeomorphism of the Cantor set and thus the function $A\mapsto X\setminus(A\cup\{x\})$ is continuous for every $x\in X$. To finish the proof we notice that the first two equivalences above show that $\mathcal{F}_x$ is Borel iff C_x is Borel. The third one shows that if cl_τ is Borel, then R is co-analytic and, being analytic, it is then Borel by Souslin's theorem [10, theorem 14.11]. The forth one shows that if R is Borel, then C_x is Borel for all x. And the last equivalence shows that if C_x is Borel for all x, then cl_τ has an analytic graph and thus it is a Borel function [10, theorem 14.12]. □

In view of the previous result it is natural to introduce the following notion. Let us say that a topology on a countable set X is *hereditarily Borel* if the subspace topology of every $Y\subseteq X$ is Borel. Thus by theorem 3.2 we have the following result.

Corollary 3.3. *Let τ be an analytic topology over X. If every nbhd filter $\mathcal{F}_x$ of X is Borel, then the topology of X is hereditarily Borel. Moreover, the Borel rank of the subspace topologies is uniformly bounded.* □

Remark 3.4. *We do not know whether the converse of 3.3 holds. That is to say, if X has an analytic topology such that the nbhd filter of some point is not Borel, then X has a subspace with a non Borel topology.*

We end this section by showing a general fact about the Baire measurability of cl_τ.

Proposition 3.5. *Let τ be a meager (as a subset of 2^X) T_1 topology with infinite many limit points. Then cl_τ is not of Baire class 1.*

Proof: Since τ is T_1, then it is a dense subset of 2^X. Thus the collection of τ-closed sets is also dense and meager. Given any non τ-closed set B, there is a sequence of finite F_n sets such that $B = \lim_n F_n$ (in the product topology of 2^X). Since τ is T_1, then $\overline{F_n} = F_n$ and therefore cl_τ is not continuous at B. This shows that the collection of non continuity points of cl_τ is a comeager set, therefore cl_τ can not be of Baire class 1. □

Remark 3.6. *In particular, by [11, corollary 2.6]), cl_τ is not of Baire class 1 when τ is an analytic T_1 topology with infinite many limit points.*

4 Subspaces of S_ω with topology of arbitrarily high Borel rank

In this section we will show the following

Theorem 4.1. *For any countable ordinal α there is $X \subseteq S_\omega$ such that the subspace topology of X is a Borel set of rank $\geq \alpha$.*

The idea for the proof of 4.1 is to associate to a well founded tree T on $\mathbb{N}$ a subspace X_T of S_ω is such way that the Borel rank of the topology of X_T will be, roughly speaking, equal to the rank of T. Let $E(T)$ be the terminal nodes of T. The subspaces we will construct are of the form $\{\emptyset\} \cup E(T)$. Let us observe that any antichain D (i.e. there are no two elements in D one extending the other) is discrete as a subset of S_ω, so in particular $E(T)$ is a discrete set. Therefore, we will actually construct filters of arbitrarily high Borel rank. Our filters are similar to those constructed in [4]. It is interesting to realize that these filters correspond to nbhd filters of points in a subspace of S_ω.

Definition 4.2. *For any well founded tree T on $\mathbb{N}$, let $\mathcal{F}_T$ be the nbhd filter of $\emptyset$ in the subspace $\{\emptyset\} \cup E(T)$ of S_ω.*

We will construct by recursion a ω_1-sequence of trees T_α such that $\mathcal{F}_{T_\alpha}$ is $\mathbf{\Sigma}^0_\alpha$-complete, that is to say, they will satisfy the following two conditions:

(i) $\mathcal{F}_{T_\alpha}$ is $\mathbf{\Sigma}^0_\alpha$.

(ii) For every A in $\mathbf{\Sigma}^0_\alpha$ there is a continuous function $V : 2^{\mathbb{N}} \to 2^{E(T_\alpha)}$ such that $x \in A$ iff $V(x) \in \mathcal{F}_{T_\alpha}$.

Recall the exact Borel rank of a $\mathbf{\Sigma}^0_\alpha$-complete is precisely α.

Before stating the preliminary lemmas needed for the proof of theorem 4.1 we will make a general observation which shows that the subspaces we will construct can not be sequential.

Proposition 4.3. *Let $X \subseteq S_\omega$. If X is a sequential subspace of S_ω, then the topology of X is $\mathbf{\Pi}^0_3$.*

Proof: Since X is sequential, then $V \subseteq X$ is open in X iff V is sequentially open. Therefore V is open in X iff for all $s \in V$ the following holds

$$\text{If } \{n : s\widehat{\ }n \in X\} \text{ is infinite, then } \exists N \forall m \geq N[s\widehat{\ }m \in X \to s\widehat{\ }m \in V]$$

and from this it follows that the topology of X is $\mathbf{\Pi}^0_3$. □

We will use the following result [4, Lemma 8.2] (see also [10, 23.5]).

Lemma 4.4. *Let $A \subseteq 2^{\mathbb{N}}$ and $\alpha > 1$ a countable ordinal.*

1. *A belongs to $\mathbf{\Pi}^0_{\alpha+1}$ iff there are sets A_m in $\mathbf{\Pi}^0_{\beta_m}$ for some $\beta_m < \alpha$ such that*

$$x \in A \Leftrightarrow \forall n \exists m \geq n \; x \in A_m$$

2. *A is in $\mathbf{\Sigma}^0_{\alpha+1}$ iff there are sets A_m in $\mathbf{\Sigma}^0_{\beta_m}$ for some $\beta_m < \alpha$ such that*

$$x \in A \Leftrightarrow \exists n \forall m \geq n \; x \in A_m$$

For $\alpha = 1$, the sets A_m can be chosen to be clopen. □

The base for the induction is given in the following

Lemma 4.5. *Let $T = \omega^{\leq 2}$. Then $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_4$-complete.*

Proof: Notice that $E(T) = \omega^2$. Let $V \subseteq E(T)$. It is easy to check that

$$V \in \mathcal{F}_T \text{ iff } \exists N \, \forall n \geq N \, \exists M \, \forall m \geq M \; \langle n, m \rangle \in V \tag{1}$$

From this it follows that $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_4$. To see $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_4$-complete fix $A \subseteq 2^{\mathbb{N}}$ a $\mathbf{\Sigma}^0_4$ set. By lemma 4.4 there are clopen sets $F(n,m)$ such that

$$x \in A \Leftrightarrow \exists N \, \forall n \geq N \, \exists M \, \forall m \geq M \; x \in F(n,m) \tag{2}$$

Let $V : 2^{\mathbb{N}} \to 2^{E(T)}$ given by $V(x) = \{\langle n, m \rangle : \; x \in F(n,m)\}$. Since the $F(n,m)$'s are clopen, then V is continuous. From (1) and (2) we conclude that $x \in A$ iff $V(x) \in \mathcal{F}_T$. □

Remark 4.6. *Recall that Arens space S_2 is the subspace $\omega^{\leq 2}$ of S_ω. Thus the previous lemma might be known (see [4, Remark 8.11]) but we have included the proof for the sake of completeness.*

Lemma 4.7. *Let T_n be well founded trees such that $\mathcal{F}_{T_n}$ is $\mathbf{\Sigma}^0_{\alpha_n}$-complete. Let T be the following tree*

$$T = \bigcup_n \{\langle n \rangle^\frown s : \; s \in T_n\} \cup \{\emptyset\}$$

Then T is well founded and $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_{\alpha+1}$-complete for $\alpha = \sup\{\alpha_n + 1 : \; n \in \mathbb{N}\}$.

Proof: It is clear that T is well founded. Notice that $E(T)$ is the union of $\{\langle n \rangle^\frown s : \; s \in E(T_n)\}$. Let $\Phi_n : S_\omega \to N_{\langle n \rangle}$ be defined by $\Phi_n(s) = \langle n \rangle^\frown s$. It is clear that Φ_n is an homeomorphism. Let $\mathcal{G}_n = \Phi_n[\mathcal{F}_{T_n}]$. Then $\mathcal{G}_n$ is the nbhd filter of $\langle n \rangle$ in the subspace $\Phi_n[E(T_n) \cup \{\emptyset\}] = \{\langle n \rangle^\frown s : \; s \in E(T_n)\} \cup \{\langle n \rangle\}$ and moreover $\mathcal{G}_n$ is $\mathbf{\Sigma}^0_{\alpha_n}$.

Let $A \subseteq E(T)$, we claim

$$A \in \mathcal{F}_T \text{ iff } \exists N \, \forall n \geq N \; A \cap N_{\langle n \rangle} \in \mathcal{G}_n \tag{3}$$

From this and lemma 4.4 it follows that $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_{\alpha+1}$.

To show (3), suppose $A \in \mathcal{F}_T$ and let W be an open set in S_ω such that $\emptyset \in W$ and $W \cap E(T) = A$. There is N such that $\langle n \rangle \in W$ for all $n \geq N$. Notice that $W_n = W \cap N_{\langle n \rangle}$ is an open set in $N_{\langle n \rangle}$, $\langle n \rangle \in W_n$ and $W_n \cap \Phi_n[E(T_n)] \subseteq A \cap N_{\langle n \rangle}$. Conversely, suppose the right hand side of (3) holds and let W_n be an open set in $N_{\langle n \rangle}$ such that $\langle n \rangle \in W_n$ and $W_n \cap \Phi_n[E(T_n)] = A \cap N_{\langle n \rangle}$ for all $n \geq N$. Let W be the union of the W_n's together with $\emptyset$. Then W is an open nbhd of $\emptyset$. It is routine to check that $W \cap E(T) \subseteq A$.

Now we will show that $\mathcal{F}_T$ is $\mathbf{\Sigma}^0_{\alpha+1}$-complete. Let $A \subseteq 2^{\mathbb{N}}$ be a $\mathbf{\Sigma}^0_{\alpha+1}$ set. By lemma 4.4 there are $\mathbf{\Sigma}^0_{\beta_n}$ sets A_n with $\beta_n < \alpha$ such that

$$x \in A \Leftrightarrow \exists N \forall n \geq N \; x \in A_n$$

We can assume that $\beta_n \leq \alpha_n$ (in fact, suppose $\alpha_0 < \beta_0$. Find the least n such that $\beta_0 \leq \alpha_n$. Replace the original sequence $\{A_k\}$ by $\{A'_k\}$ which now starts with n copies of $2^{\mathbb{N}}$ and then the original sequence $\{A_k\}$. Now $\beta'_0 \leq \alpha_0$. Repeat this procedure as many times as necessary).

Since $\mathcal{F}_{T_n}$ is $\mathbf{\Sigma}^0_{\alpha_n}$-complete, there are continuous functions $V_n : 2^{\mathbb{N}} \to 2^{E(T_n)}$ such that

$$x \in A_n \Leftrightarrow V_n(x) \in \mathcal{F}_{T_n} \tag{4}$$

Let $V(x) = \bigcup_n \Phi_n[V_n(x)]$. Notice that $V : 2^{\mathbb{N}} \to 2^{E(T)}$ is continuous and $V(x) \cap N_{\langle n \rangle} = \Phi_n[V_n(x)]$. From this, (3), (4) and the definition of $\mathcal{G}_n$ we have $V(x) \in \mathcal{F}_T$ iff $\exists N \; \forall n \geq N \; \Phi_n[V_n(x)] \in \mathcal{G}_n$ iff $\exists N \; \forall n \geq N \; V_n(x) \in \mathcal{F}_{T_n}$ iff $x \in A$. □

Remark 4.8. *The filter $\mathcal{F}_T$ occurring in the proof of the previous result could be stated in terms of the Hausdorff operation (see [10, Exercies 23.5]) and the Frechet product (see [4, Section 8]). Thus the $\mathbf{\Sigma}^0_{\alpha+1}$-completeness of $\mathcal{F}_T$ can be proved based on some general results about these operations. Our filters are similar to the filters F_α's constructed in [4, Section 8]. For instances, $\mathcal{F}_T$ with $T = \omega^{\leq 2}$ corresponds to F_2.*

Proof of 4.1: We will define by recursion a sequence U_α of well founded trees such that $\mathcal{F}_{U_\alpha}$ is $\mathbf{\Sigma}^0_\alpha$-complete for α an even integer greater than 2 or an odd infinite ordinal.

We start with $U_4 = \omega^{\leq 2}$ which works by lemma 4.5. Now taking T_n equal to U_{2k} for all n and applying lemma 4.7 we obtain U_{2k+2}. For infinite ordinals we start by taking $T_n = U_{2n}$ in lemma 4.7 and obtain $U_{\omega+1}$. Now for the inductive step the pattern should be clear. □

It is quite easy to define topologies on the $\mathbf{\Pi}$ side of the Borel hierarchy once we have available topologies on the $\mathbf{\Sigma}$ side.

Proposition 4.9. *Let (X, τ) be a countable topological space. Suppose $X = \bigcup_n U_n$ where U_n is a pairwise disjoint family of non empty open sets. Suppose that τ restricted to U_n is $\mathbf{\Sigma}^0_{\alpha_n}$-complete and α_n is an increasing sequence of countable ordinals. Then τ is $\mathbf{\Pi}^0_\lambda$-complete, where λ is $\sup_n(\alpha_n + 1)$.*

Proof: Let $V \subseteq X$, it is clear that $V \in \tau$ iff $V \cap U_n$ is open in U_n for all n. Thus τ is $\mathbf{\Pi}^0_\lambda$. Fix $A \subseteq Y$ be a $\mathbf{\Pi}^0_\lambda$ subset of a zero dimensional Polish space Y. Let B_n be $\mathbf{\Sigma}^0_{\beta_n}$ set with $\beta_n < \lambda$ such that $A = \bigcap_n B_n$. We can suppose w.l.o.g that $\beta_n \leq \alpha_n$. Then as τ restricted to U_n is $\mathbf{\Sigma}^0_{\alpha_n}$-hard there are continuous functions $f_n : Y \to 2^{U_n}$ such that $y \in B_n$ iff $f_n(y)$ is open in U_n. Define $f : Y \to 2^X$ by $f(y) = \bigcup_n f_n(y)$. Since the U_n's are pairwise disjoint, then f is easily seen to be continuous and $y \in A$ iff $f(y) \in \tau$. □

Remark 4.10. *The method of constructing subspaces used in the proof of 4.1 and 4.9 does not provide examples of topologies of any possible Borel type. For instance, it will be interesting to determine whether S_ω has subspaces with topology of type $\mathbf{\Pi}^0_{2n}$, $\mathbf{\Sigma}^0_{2n+1}$ ($n \geq 2$) and $\mathbf{\Sigma}^0_\omega$.*

Remark 4.11. *Notice also that from 4.1, 3.1, 3.2 and 3.3 it follows that S_ω does not have a F_σ basis. A fact that was proved in [11, proposition 5.2] by a different method.*

5 Subspaces of S_ω whose topology are hereditarily Borel

In this section we will show the following

Theorem 5.1. *Let $X \subseteq S_\omega$ be a closed subspace with $\rho(X) < \omega_1$. Then the closure operator cl_X for the subspace topology of X is Borel. In particular, every subspace of X has a Borel topology and, moreover, the Borel rank of the topologies of the subspaces of X is uniformly bounded.*

Let X be a closed subspace of S_ω with $\rho(X) < \omega_1$. In order to use 3.2 and 3.3 we need to show that the following sets C_s are Borel for all $s \in X$

$$C_s = \{A \subseteq X : s \in \overline{A}\}$$

We will also allow $s \notin X$, since in this case C_s would be empty.

Lemma 5.2. *Let X be a closed subspace of S_ω. For all N the following holds*

$$C_s = \{A \subseteq X : s \in A\} \cup \bigcap_{n \geq N} \bigcup_{m \geq n} C_{s\frown m}$$

Proof: Let $A \subseteq S_\omega$ and $s \in \overline{A} \setminus A$. A straightforward induction on $\sigma(s, A)$ shows that there is an increasing sequence of integers $\{n_i\}_i$ such that $s\frown n_i \in \overline{A}$ for all i. From this the inclusion $\subseteq$ follows. For the other one, just observe that $s\frown n_i$ converges to s. □

Lemma 5.3. *Let X be a closed subspace of S_ω and $s \in X$. If $\rho(s, X) < \omega_1$, then there is N such that $\rho(s\frown m, X) < \rho(s, X)$ for all $m \geq N$ such that $s\frown m \in X$.*

Proof: Let $\alpha = \rho(s, X)$ and B the set of all m such that $\rho(s\frown m, X) \geq \alpha$ and $s\frown m \in X$. Suppose, towards a contradiction, that B is infinite. Notice that $\alpha > 0$, otherwise s would be isolated in X and therefore there would be only finitely many m such that $s\frown m \in X$. We will only analyze the case when α is a limit ordinal, the case when α is a successor ordinal can be treated similarly.

Since $\alpha < \omega_1$, then we can fix an increasing sequence $\alpha_n < \omega_1$ of ordinals converging to α. For each $n \in B$, there is A_n such that $s\frown n \in \overline{A_n}$ and $\sigma(s\frown n, A_n) \geq \alpha_n$. We can assume w.l.o.g. that $A_n \subseteq N_{s\frown n}$. Let

$$A = \bigcup_{n \in B} A_n$$

Notice that $s \in \overline{A}$. We claim that $\sigma(s, A) > \alpha$, which is a contradiction. In fact, suppose $s \in A^{(\alpha)}$, then there is m such that $s \in A^{(\alpha_m+1)}$. Therefore there is an increasing sequence of integers n_i such that $s\frown n_i \in A^{(\alpha_m)}$ for all i. Since $A_n \subseteq N_{s\frown n}$ and the $N_{s\frown n}$'s are disjoint, then it follows that $s\frown n_i \in A_{n_i}^{(\alpha_m)}$ for all i. Thus $\alpha_{n_i} \leq \sigma(s\frown n_i, A_{n_i}) \leq \alpha_m$, which is impossible if $m < n_i$. □

Proof of 5.1: By propositions 3.2 and 3.3 it suffices to show that C_s is a Borel subset of 2^X for all $s \in X$. We will show it by induction on $\rho(s, X)$.

If $\rho(s, X) = 0$, then s is isolated in X, therefore C_s consists of all $A \subseteq X$ such that $s \in A$, and thus C_s is a closed subset of 2^X. Suppose that C_t is Borel for all $t \in X$ with $\rho(t, X) < \alpha$ and let $s \in X$ with $\rho(s, X) = \alpha$. By lemma 5.3 there is N such that $\rho(s\frown m, X) < \alpha$ for all $m \geq N$ such that $s\frown m \in X$. By the inductive hypothesis, $C_{s\frown m}$ is Borel for all $m \geq N$. Now from lemma 5.2 it follows that C_s is also Borel. □

Remark 5.4. *Let T be a tree on $\mathbb{N}$, then T as a subset of S_ω is closed and thus a sequential space. By 4.3 the topology of T is $F_{\sigma\delta}$. If T is well founded, it has associated a rank as a tree, which we will denote by $rk(T)$ (see [?, §2.E]). It is routine to check that $\rho(T) \leq rk(T)$. For the trees U_α constructed in the proof of 4.1, it can be easily verified by induction that $\rho(U_\alpha) = rk(U_\alpha) \leq \alpha$. It can also be verified that every subspace of U_α has a Borel topology of rank at most α and there is one (namely $E(U_\alpha) \cup \{\emptyset\}$) whose topology is Borel of rank exactly α. So for this examples, the sequential rank $\rho(X)$ gives a good bound for the Borel complexity of the topology of every subspace of X.*

6 Subspaces of S_ω with an analytic non Borel topology

In this section we will show the following

Theorem 6.1. *Let $X \subseteq S_\omega$ be a closed subspace with $\rho(X) = \omega_1$. Then there is $Y \subseteq X$ such that the subspace topology of Y is not Borel. Moreover, there is a closed copy of S_ω inside X.*

The key lemma is the following

Lemma 6.2. *Let $D \subseteq \omega^{<\omega}$ be an antichain and $s \in \overline{D}$ with $\rho(s, \overline{D}) = \omega_1$. Then the topology of $D \cup \{s\}$ is a complete analytic set, in particular, it is not Borel.*

Since any antichain is discrete in S_ω then s is the only non isolated point of $D \cup \{s\}$. So the topology of $D \cup \{s\}$ is given by the nbhd filter of s in $D \cup \{s\}$.

A particular and concrete example of an antichain D such that $\rho(\emptyset, D) = \omega_1$ is the following:

$$D = \{s\widehat{\ }2n : \ s(i) \text{ is odd for all } i < |s| \text{ and } n \in \mathbb{N}\} \tag{5}$$

Notice the collection of all finite sequences of odd integers is a subset of $\overline{D}$ and $\emptyset \in \overline{D}$. Thus $\overline{D}$ contains a closed copy of S_ω. Therefore $\rho(\emptyset, \overline{D}) = \omega_1$ and from lemma 6.2 we conclude that the subspace topology of $D \cup \{\emptyset\}$ is analytic and non Borel.

We will need another property of the ordinal ρ defined in §2.

Lemma 6.3. *Let $X \subseteq S_\omega$ be a closed subspace and $s \in \omega^{<\omega}$. If $\rho(s, X) = \omega_1$, then $\rho(s\widehat{\ }m, X) = \omega_1$ for infinite many m's.*

Proof: Suppose that $\rho(s\widehat{\ }m, X) < \omega_1$ for all $m \geq N$ with $s\widehat{\ }m \in X$. Let $\alpha = \sup\{\rho(s\widehat{\ }m, X) : s\widehat{\ }m \in X, m \geq N\}$. Let $A \subseteq X$ such that $s \in \overline{A}$. It suffices to show that $\sigma(s, A) \leq \alpha + 1$. We can assume that $s \notin A$. Then there is an increasing sequence of integers $\{n_i\}_i$ such that $s\widehat{\ }n_i \in \overline{A}$. By hypothesis $\rho(s\widehat{\ }n_i, X) \leq \alpha$. Therefore $\sigma(s\widehat{\ }n_i, A) \leq \alpha$, thus $\sigma(s, A) \leq \alpha + 1$. □

Using these two lemmas we give the proof of 6.1

Proof of 6.1: From part (4) of 2.2 we know that there is $s \in X$ such that $\rho(s, X) = \omega_1$. We will construct an antichain $D \subseteq X$ such that $s \in \overline{D}$ and $\rho(s, \overline{D}) = \omega_1$. Thus $Y = D \cup \{s\}$ will be the required subspace of X. By lemma 6.3 $\rho(s\widehat{\ }m, X) = \omega_1$ for infinitely many m. The idea is to put in D "half" of these sequences $s\widehat{\ }m$ and repeat this process with the other "half". More formally, for each sequence t such that $\rho(t, X) = \omega_1$ put

$$B_t = \{m : \ \rho(t\widehat{\ }m, X) = \omega_1\}$$

and let B_t^0, B_t^1 be a partition of B_t into two infinite pieces. We define by recursion two sequences of sets D_n and E_n as follows:

$$\begin{aligned} D_1 &= \{\, s\widehat{\ }m : \ m \in B_s^1 \,\} \\ E_1 &= \{\, s\widehat{\ }m : \ m \in B_s^0 \,\} \\ D_{n+1} &= \{\, t\widehat{\ }m : \ t \in E_n \ \& \ m \in B_t^1 \,\} \\ E_{n+1} &= \{\, t\widehat{\ }m : \ t \in E_n \ \& \ m \in B_t^0 \,\} \end{aligned}$$

Let

$$D = \bigcup_{n \geq 1} D_n \ , \ E = \bigcup_{n \geq 1} E_n$$

It is not hard to verify by induction on n that $E_n \subseteq \overline{D}$. It is clear that D is an antichain and $s \in \overline{D}$. To see that $\rho(s, \overline{D}) = \omega_1$ it suffices to verify that $E \cup \{s\}$ is a closed copy of S_ω. It is clear that $E \cup \{s\}$ is a copy of S_ω. To check that $E \cup \{s\}$ is closed, notice that if $t \in E$, $t' \prec t$ and $|s| < |t'|$, then $t' \in E$. □

Now we give the

Proof of 6.2: Since S_ω is an homogeneous space, we can assume w.l.o.g that $s = \emptyset$. Consider the following function F that maps a tree T on $\mathbb{N}$ to a subset of D:

$$F(T) = \{r \in D : \ \exists t \in T \ |t| = |r| \ \& \ t(i) \leq r(i) \text{ for all } i < |r|\}$$

For a given r there are only finitely many sequences t such that $|t| = |r|$ and $t(i) \leq r(i)$ for all $i < |r|$, thus F is continuous.

We claim that T is ill founded iff $F(T) \cup \{\emptyset\}$ is open in $D \cup \{\emptyset\}$. In fact, suppose first that T is ill founded. Let α be an infinite branch of T. Define

$$W = \{t \in \omega^{<\omega} : \ \alpha(i) \leq t(i) \text{ for all } i < |t|\}$$

It is clear that W is an open set of S_ω and $\emptyset \in W$. Let

$$O = \bigcup_{t \in F(T)} N_t \cup W$$

O is an open set of S_ω. We will show that

$$F(T) \cup \{\emptyset\} = (D \cup \{\emptyset\}) \cap O$$

It is clear that $F(T) \cup \{\emptyset\} \subseteq (D \cup \{\emptyset\}) \cap O$. On the other hand, let $r \in D \cap O$. There are two cases: (i) If $r \in N_t \cap O$ with $t \in F(T)$, then $t = r$ as D is an antichain. (ii) If $r \in W \cap D$, then $\alpha(i) \leq r(i)$ all $i < |r|$. Since α is a branch of T, then $r \in F(T)$ by the definition of $F(T)$.

Suppose now that $F(T) \cup \{\emptyset\}$ is open in $D \cup \{\emptyset\}$ and let O be an open subset of S_ω such that $F(T) \cup \{\emptyset\} = (D \cup \{\emptyset\}) \cap O$. By recursion we will define $\alpha \in \mathbb{N}^\mathbb{N}$ and a sequence $r_n \in \omega^{<\omega}$ such that:

(1) $r_n \in O \cap D$ for all n.

(2) $\alpha|j \in O \cap \overline{D}$ for all j.

(3) $r_n(i) \leq \alpha(i)$ for all $i < |r_n|$.

(4) $\rho(\alpha|j, \overline{D}) = \omega_1$ for all j.

(5) $|r_n| < |r_{n+1}|$ for all n.

Granting this has been done we finish the proof. To show that T is not well founded, let

$$T_0 = \{t \in T : \ t(i) \leq \alpha(i) \text{ for all } i < |t|\}$$

It is clear that T_0 is a finitely branching subtree of T. So it suffices to show that T_0 is infinite. In fact, by (1) $r_n \in O \cap D \subseteq F(T)$, thus there is $t_n \in T$ such that $t_n(i) \leq r_n(i)$ for all $i < |r_n| = |t_n|$. From (5) we conclude that the t_n's are all different and from (3) we have $t_n \in T_0$ for all n.

So it remains to show that such $\alpha \in \mathbb{N}^{\mathbb{N}}$ and $r_n \in \omega^{<\omega}$ exist. Since $\emptyset \in O \cap \overline{D}$, then there is $r_0 \in O \cap D$. By lemma 6.3 there are infinitely many n such that $\rho(\langle n \rangle, \overline{D}) = \omega_1$. Thus let $\alpha(0) \geq r_0(0)$ be such that $\langle \alpha(0) \rangle \in O$ and $\rho(\langle \alpha(0) \rangle, \overline{D}) = \omega_1$. We can continue this way and define $\alpha(i)$ for all $i < |r_0|$. Thus (1) and (3) are satisfied for $n = 0$ and (2) and (4) for $j < |r_0|$.

Suppose we have defined r_n and $\alpha(i)$ for all $i < |r_n| = k$. Let

$$s = \langle \alpha(0), \alpha(1), \cdots, \alpha(k-1) \rangle$$

By (2) $s \in \overline{D} \cap O$, thus there is $r_{n+1} \in D \cap O$ extending s. By (4) $\rho(s, \overline{D}) = \omega_1$, therefore r_{n+1} extends properly s. Hence $|r_n| < |r_{n+1}|$ and (5) holds. Now we repeat the same argument as for the case $n = 0$ and define α up to $|r_{n+1}|$ such that (2) and (4) holds for every $j < |r_{n+1}|$. □

From 6.1, 3.2 and 3.3 we know that there must be some $s \in S_\omega$ such that $\mathcal{F}_s$ is not Borel. Since S_ω is homogeneous, then every $\mathcal{F}_t$ is not Borel. Even more, $\mathcal{F}_t$ is a complete analytic set for every $t \in S_\omega$. We will show it for $t = \emptyset$. In fact, let D be an antichain such that $\emptyset \in \overline{D}$ and the topology of $D \cup \{\emptyset\}$ is a complete analytic set (for instance, the one given by (5)). Since D is discrete, the nbhd filter $\mathcal{G}$ of $\emptyset$ in $D \cup \{\emptyset\}$ is a complete analytic set. It is easy to check that $A \in \mathcal{G}$ iff $A \subseteq D$ and $A \cup (S_\omega \setminus D) \in \mathcal{F}_\emptyset$. Thus $\mathcal{G} \leq_w \mathcal{F}_\emptyset$. So we have shown the following

Proposition 6.4. *Let $\mathcal{F}_s$ be the nbdh filter of s in S_ω. Then $\mathcal{F}_s$ is a complete analytic set.*

Remark 6.5. *In [12] was defined the following filter and shown to be a complete analytic set. For every tree T let $F(T)$ be the set $\{r \in S_\omega : \ \exists t \in T \ |t| = |r| \ \ \& \ \ t(i) \leq r(i) \text{ for all } i < |r|\}$. For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, let T_α be the set of all initial segments of α. Then T_α is a tree. Let $\mathcal{F}$ be the filter given by $S \in \mathcal{F}$ iff there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $F(T_\alpha) \subseteq S$. Then it is clear that $\mathcal{F} \subseteq \mathcal{F}_\emptyset$. The proof of 6.2 follows closely the proof that $\mathcal{F}$ is a complete analytic set. In fact, it shows that if T is not well founded, then $F(T) \in \mathcal{F}$ and when T is well founded, then $F(T) \notin \mathcal{F}_\emptyset$.*

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