UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMÁTICAS

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Carlos Uzcátegui

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Abstract

Two topologies τ and ρ over X are said to be complementary if $\tau \wedge \rho$ is the indiscrete topology and $\tau \vee \rho$ the discrete topology. The lattice of topologies is complemented, i.e, every topology has a complement. We will show that every AT topology (i.e. a topology such that the intersection of arbitrary many open sets is open) over a countable set has a maximal complement in the lattice of topologies. This answer a question of S. Watson. This theorem is a corollary of an analogous result for the lattice of pre-order. We show that every pre-order P on a countable set X admits a maximal complement in the lattice of preorders over X. Moreover, if every connected component of P is neither discrete nor indiscrete, then such maximal complement has all its chains of size at most two.

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1 Introduction

The collection of topologies $TOP(X)$ over a set X is a lattice under inclusion $\subseteq$. The greatest element is the discrete topology (where every set is open) and the smallest element is the indiscrete topology (whose open sets are just $\emptyset$ and X). The lattice operations are defined by letting the meet $\tau \wedge \rho$ of two topologies be $\tau \cap \rho$ and the join $\tau \vee \rho$ be the least topology which contains both τ and ρ (i.e., the topology having $\tau \cup \rho$ as a subbasis). Moreover, $TOP(X)$ is a complete lattice. Two topologies τ and ρ over X are said to be complementary if $\tau \wedge \rho$ is the indiscrete topology and $\tau \vee \rho$ the discrete topology. Steiner [1] has shown long time ago that the lattice of topologies is complemented, i.e, every topology has a complement. We refer the reader to Watson's paper [3] where many new results about the complementation in $TOP(X)$ and an extensive bibliography can be found. This paper is motivated by a question from [3]. It was asked whether there are topologies with a maximal complement. It is known that a non discrete T_1 topology can not have a maximal complement [2]. A topology is said to be an Alexandroff-Tucker (AT) topology if the intersection of arbitrary many open sets is open. AT topologies are non T_1 (except for the discrete topology). We will show the following

Theorem 1: Every AT topology over a countable set has a maximal complement in the lattice of topologies.

This theorem is a corollary of an analogous result for the lattice of pre-order. This lattice is tight connected to the lattice of topologies and specially to the complementation of topologies. Let us recall some known facts about them. The collection $PO(X)$ of pre-orders over a set X (i.e. transitive and reflexive binary relations but not necessarily antisymmetric) forms also a lattice under reversed inclusion as the lattice order. The join of two pre-orders P and Q is then $P \cap Q$ and their meet is the transitive closure of $P \cup Q$. Moreover $PO(X)$ is also a complete complemented lattice. Complementation in $PO(X)$ is quite natural. Two pre-orders P and Q are complementary if their intersection, as binary relations, is the identity relation (denoted by Δ) and the transitive closure of their union is the largest binary relation $X \times X$. The reader should keep in mind that a maximal complement in $PO(X)$ is in fact $\subseteq$-minimal, since in $PO(X)$ the lattice order is given by reversed inclusion. Our main result is the following

Theorem 2: Every pre-order P on a countable set X admits a maximal complement in $PO(X)$.

Moreover, if every connected component of P is neither discrete nor indiscrete, then a maximal complement can be found with all its chains of size at most two. The following diagram shows a maximal complement for $\mathbb{Z}$ with its usual order. The pairs labelled with $\swarrow$ form the complementary relation.

$$\begin{array}{ccccccccccccccccc}
 & & -5 & & -3 & & -1 & & 1 & & 3 & & 5 & & 7 & & \\
\cdots & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \cdots \\
 & & -6 & & -4 & & -2 & & 0 & & 2 & & 4 & & 6 & &
\end{array}$$

Observe that all pairs labelled with $\uparrow$ belong to the usual order in $\mathbb{Z}$. Notice that if we erase one single $\swarrow$, then the resulting relation Q is not a complement of $\mathbb{Z}$, because the pair removed can not be recovered by the transitive closure of Q together with the order of $\mathbb{Z}$.

Let $\preceq$ be a pre-order over a set X, its associate AT topology $\tau_{\preceq}$ is defined as the topology generated by the collection of sets of the form $\{x \in X : a \preceq x\}$ with $a \in X$. This is in fact a characterization of AT topologies. Namely, given an AT topology τ, define $\preceq_\tau$ by $x \preceq_\tau y$ iff $x \in \overline{\{y\}}$. Then $\preceq_\tau$ is a pre-order and τ is its associate AT topology. Moreover, τ is T_0 iff $\preceq_\tau$ is a partial order (i.e it is antisymmetric). The lattice order in $PO(X)$ is taken as reversed inclusion so that it coincides with the induced order when we view $PO(X)$ (via $\tau_{\preceq}$) as a subset of $TOP(X)$. However $PO(X)$ is not a sublattice of $TOP(X)$.

Thus the question about the existence of maximal complements for an AT topology has two natural variants according to where we look at it: either inside $PO(X)$ or inside $TOP(X)$. However, if ρ is an AT topology, then the topology generated by ρ together with finitely many sets is again AT. Therefore if a complement of an AT topology is maximal in $PO(X)$, then it is also maximal in $TOP(X)$. In other words, Theorem 1 follows from Theorem 2. On the other hand, we have some partial results suggesting that some subtle conditions must be imposed in order to answer the general question about the existence of a maximal complement for an arbitrary topology over a countable set.

Our result can not be extended to arbitrary pre-orders over an uncountable set. By a Fordor's lemma type argument it can be easily shown that ω_1 with its usual order does not have a maximal complement in $PO(\omega_1)$. On the other hand, the main result can be easily extended to any pre-order P such that both P and P^{-1} are separable. In fact, the construction can be carried out inside a countable dense and co-dense subset of X. Therefore these type of partial orders do have maximal complements.

We will make next some comments about the analogous question for minimal complements. A standard Zorn's lemma argument shows that given a pre-order P and a complement Q of P there is a $\subseteq$-maximal partial order R extending Q and such that $R \cap P = \Delta$. It is clear that such R is a minimal complement of P in $PO(X)$. Thus the existence of minimal complement in $PO(X)$ is quite easy to establish. However, it is not clear if such minimal complements are also minimal in

$TOP(X)$. On the other hand, nothing similar happens for maximal complements, that is to say, there is a partial order P and a complement Q of P such that there is no maximal complement of P above Q in $PO(X)$. In fact, consider the following example. Let P be $\mathbb{N}$ with its usual order and $Q_A = \{(n,0) : n \in A\} \cup \Delta$ for $A \subseteq \mathbb{N}$. Notice Q_A is a complement of P iff A is infinite. Therefore for every infinite A there is no $R \subseteq Q_A$ such that R is a maximal complement for P.

We end the introduction by fixing the notation and terminology. X will always denote a countable set. We will denote a pre-order over X either by $\preceq$ or just P as a binary relation. In case we have more than one pre-order we will write $\preceq_P$ to avoid any possible confusion. The strict relation is denoted by $\prec$, that is to say, $x \prec y$ if $x \preceq y$ and $y \not\preceq x$. Since a pre-order is not necessarily antisymmetric it is convenient to use the following equivalence relation. Let $x \sim y$ if $x \preceq y$ and $y \preceq x$. A subset Y of a pre-ordered set $(X, \preceq)$ is said to be *open* (in the associated AT topology of $(X, \preceq)$) if whenever $x \in Y$ and $x \preceq y$, then $y \in Y$. An element x of a pre-ordered set $(X, \preceq)$ will be called *clopen* if whenever $y \preceq x \preceq z$ for some $y, z \in X$, then $x \sim y \sim z$ (i.e. the $\sim$-equivalence class of x is clopen). A non clopen element x of X will be called *maximal* if there is no $y \in X$ such that $x \prec y$. The notion of a *minimal* element is defined analogously. The collections of maximal and minimal elements of X will be respectively denoted by Max and Min. A subset $A \subseteq X$ is said to be *up-dense* if for every $x \in X$ there is $y \in X$ such that $x \preceq y$. Analogously we define the notion of a down-dense set. A (connected) component of a pre-order P is a non-empty subset D of X such that for every $x, y \in D$, there is a path in $P \cup P^{-1}$ from x to y. A component D of P is said to be trivial, if either D has only one element or the restriction of P to D is equal to $D \times D$.

2 Maximal complements in the lattice of pre-orders

Our main result was suggested by the following example. Let P be the partial order of all binary sequences $2^{<\omega}$ with the usual extension order $\preceq$. It is clear that there are elements a_n, b_n in $2^{<\omega}$ such that $a_n \prec b_n$ and moreover the a_n's form a dense set in $2^{<\omega}$. For instance, take the a_n's to be the collection of all sequences of even length ordered by size and let b_n be a_n followed by 0. Consider the following arrangement

$$
\begin{array}{ccccccccccccccccc}
 & & 000 & & 010 & & 100 & & 110 & & 00000 & & 00010 & & 00100 & & 00110 \\
 & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow \quad \cdots \\
\emptyset & & 00 & & 01 & & 10 & & 11 & & 0000 & & 0001 & & 0010 & & 0011
\end{array}
$$

Where $\emptyset$ is the empty sequence (which is the minimum of P). The top row corresponds to $b_1, b_2, \cdots$ and the bottom row to $\emptyset, a_1, a_2, \cdots$. Notice that all pairs labelled by $\uparrow$ belong to P. The collection Q of all pairs labelled by $\swarrow$ together with the identity Δ is a complementary relation for P. In fact, it is clear that $Q \cap P$ is equal to Δ. To see that the transitive closure of $Q \cup P$ contains every element of $2^{<\omega}$, just observe that we can travel from any sequence to some a_n (by the density of the a_n's) and then move towards $\emptyset$ following the diagram above by using alternatively P or Q. Moreover, Q is a maximal complement for P .

For the rest of this section, P will denote a pre-order over X. We will assume (unless stated otherwise) that P has no trivial components (see the introduction). The general case will be treated at the end of this section.

The main ingredient of the proof will be finding sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in X

carefully arranged as shown in the diagram below, as it was done for $\mathbb{Z}$ and $2^{<\omega}$.

$$\begin{array}{ccccccccccccccccc} & & d_3 & & d_2 & & d_1 & & b_1 & & b_2 & & b_3 & & b_4 & & \\ \cdots & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \uparrow & \swarrow & \cdots \\ & & c_3 & & c_2 & & c_1 & & a_1 & & a_2 & & a_3 & & a_4 & & \end{array}$$

where every pair labelled with $\uparrow$ belongs to P and every pair labelled with $\swarrow$ does not belong to P. Let Q be the collection of pairs labelled with $\swarrow$. If the sequence $\{a_n\}$ is up-dense in X and $\{d_n\}$ is down-dense, then Q is a complement of P. To make it maximal we will impose some extra conditions. The sequences $\{a_n\}$ and $\{b_n\}$ can indeed be found when X does not have maximal elements and analogously, when X does not have minimal elements, we can find the sequences $\{c_n\}$ and $\{d_n\}$. Thus the final step in the proof will be to partition X into four pieces such that in each piece we can define maximal complements and then glue them together to get a maximal complement for the whole space.

The proof of theorem 1 and 2 will be given in a sequence of lemmas. The definition of $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ will be by recursion. The basic fact used in the inductive step is the following

Lemma 2.1. *Let $F \subseteq X$ be a finite set and $x \in X$. Suppose P does not have maximal elements. Then*

$$\bigcap_{z\in F}\{y \in X : y \not\preceq z\} \cap \{y \in X : x \preceq y\} \neq \emptyset$$

And dually (by reversing $\preceq$), if X does not have minimal elements, then

$$\bigcap_{z\in F}\{y \in X : z \not\preceq y\} \cap \{y \in X : y \preceq x\} \neq \emptyset$$

Proof: Just notice that, when there are no maximal elements, the set $\{y \in X : y \preceq z\}$ is nowhere dense in X with the associate AT topology (see the introduction). □

Lemma 2.2. *Suppose P does not have neither maximal nor minimal. Then there are sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ in X such that:*

(i) For all $x \in X$ there is n such that $d_n \preceq x \preceq a_n$

(ii) $c_n \prec d_n \prec a_n \prec b_n$, for all n.

(iii) $a_n \not\preceq b_m$ for all $m < n$.

(iv) $c_m \not\preceq d_n$ for all $m < n$.

(v) $a_n \not\preceq d_m$ for all m and n.

Proof: Let x_n be an enumeration of X. The sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ and $\{d_n\}$ will be defined recursively. Since X has neither minimal nor maximal elements, pick a_1, b_1, c_1 and d_1 such that $c_1 \prec d_1 \prec x_1 \prec a_1 \prec b_1$.

Suppose we have defined a_i, b_i, c_i and d_i for $i \leq k$ such that (ii)-(v) holds and $d_i \prec x_i \prec a_i$ for all $i \leq k$. By lemma 2.1 there are a_{k+1} and d_{k+1} such that $a_{k+1} \not\preceq b_i$ for all $i \leq k$, $x_{k+1} \prec a_{k+1}$, $c_i \not\preceq d_{k+1}$ for all $i \leq k$ and $d_{k+1} \prec x_{k+1}$. Next, pick b_{k+1} and c_{k+1} such that $a_{k+1} \prec b_{k+1}$ and $c_{k+1} \prec d_{k+1}$.

It remains to be checked that condition (v) holds. In fact, it suffices to show that $a_{k+1} \not\preceq d_i$ for $i \leq k$ and $a_i \not\preceq d_{k+1}$ for $i \leq k+1$. But this follows from the fact that $a_{k+1} \not\preceq b_i$, $c_i \not\preceq d_{k+1}$ and $c_i \prec d_i \prec a_i \prec b_i$ for all $i \leq k$. □

As we said before, if P has minimal elements but no maximal elements, then the previous construction can be obviously done to get only the sequences $\{a_n\}$ and $\{b_n\}$. By duality we have an analogous result when P has no minimal elements. We state these facts in the next lemma.

Lemma 2.3. *(1) Suppose P does not have maximal elements. Then there are sequences $\{a_n\}$ and $\{b_n\}$ such that: (i) For all $x \in X$ there is n such that $x \preceq a_n$; (ii) $a_n \prec b_n$, for all n and (iii) $a_n \not\preceq b_m$ for all $m < n$.*

(2) Suppose P does not have minimal elements. Then there are sequences $\{c_n\}$ and $\{d_n\}$ such that: (i) For all $x \in X$ there is n such that $d_n \preceq x$; (ii) $c_n \prec d_n$, for all n and (iii) $c_m \not\preceq d_n$ for all $m < n$.

□

Lemma 2.4. *Suppose P does not have neither maximal nor minimal elements. Then P has a maximal complement in $PO(X)$ that moreover has chains of size at most two.*

Proof: Let a_n, b_n, c_n and d_n be as in lemma 2.2 and define an order Q (viewed as a binary relation) as follows

$$Q = \{(b_{n+1}, a_n) : n \geq 1\} \cup \{(d_n, c_{n+1}) : n \geq 1\} \cup \{(b_1, c_1)\} \cup \Delta$$

where Δ is the diagonal.

First we show that Q is a complement of P. From conditions (ii), (iii) and (iv) it is clear that $P \cap Q = \Delta$. We will show that the transitive closure of $P \cup Q$ is $X \times X$. To avoid confusion we will denote the pre-orders P and Q respectively by $\preceq_P$ and $\preceq_Q$. Let $x, y \in X$ and let n, m be such that $c_m \preceq_P y$ and $x \preceq_P b_n$. Then a path from x to y in $P \cup Q$ is as follows

$$x \preceq_P b_n \preceq_Q a_{n-1} \preceq_P b_{n-1} \cdots \preceq_Q a_1 \preceq_P b_1 \preceq_Q c_1 \preceq_P d_1 \preceq_Q \cdots \preceq_Q c_m \preceq_P y \tag{1}$$

Next we show that Q is a maximal complement. It suffices to show that $Q \setminus \{(x, y)\}$ is not a complement of P for any $(x, y) \in Q$ with $x \neq y$. There are three cases for (x, y) to consider: (b_{n+1}, a_n), (d_n, c_{n+1}) or (b_1, c_1). All three cases are similar and thus we will only analyze the first one.

Let Q' be $Q \setminus \{(b_{n+1}, a_n)\}$. We will show by induction on the length of paths that (b_{n+1}, x) is not in the transitive closure of $P \cup Q'$ when x is either c_m, d_m, a_i or b_i with $i \leq n$.

It is clear that any path in $P \cup Q'$ starting at b_{n+1} and of length one is of the form $b_{n+1} \preceq_P x$. From properties (ii) and (v) in lemma 2.2 we get that x can not be equal to neither c_m nor d_m for any m. And from (ii) and (iii) it is clear that x can not be equal to neither a_i nor b_i for $i \leq n$.

For the inductive step, suppose that if x is either c_m, d_m, a_i or b_i with $i \leq n$, then there are no paths of length k in $P \cup Q'$ from b_{n+1} to x. Let us consider a path in $P \cup Q'$ of length $k+1$

$$b_{n+1} \preceq_{R_1} x_1 \preceq_{R_2} x_2 \cdots \preceq_{R_k} x_k \preceq_{R_{k+1}} x_{k+1}$$

where R_i is either P or Q'. By the inductive hypothesis we necessarily have that x_k can not be equal to neither c_m, d_m, a_i nor b_i with $i \leq n$. Therefore x_k has to be equal to either a_j or b_j for some $j > n$. We consider two cases: (i) $R_{k+1} = Q'$ and (ii) $R_{k+1} = P$. For case (i), we have that

x_k has to be equal to b_j for some $j > n$. Since $(b_{n+1}, a_n) \notin Q'$, then $j > n+1$. Thus x_{k+1} is equal to a_{j-1}, $j-1 > n$ and we are done. For case (ii), by the inductive hypothesis we can also suppose that $R_k = Q'$. Thus x_k is equal to a_j for some $j > n$. It follows from condition (ii) and (v) that x_{k+1} can not be equal to neither c_i nor d_i for any i. Also, from condition (ii) and (iii) we get that x_{k+1} can not be equal to neither a_i nor b_i for any $i < j$. Since $j > n$, we are done.

□

Lemma 2.5. *Assume P has not maximal elements and, in addition, suppose that every element of X has at least one minimal element below it. Then P has a maximal complement in PO(X) that moreover has all its chains of size at most two.*

Proof: Let a_n and b_n be as in lemma 2.3 and let Min be the collection of minimal elements of X. Let $\sim$ be the equivalence relation over X defined at the end of the introduction. Let Min^* be formed by only one representative of each equivalent class of elements of Min. Define a partial order Q as follows

$$Q = \{(b_{n+1}, a_n) : \ n \geq 1\} \cup \{(b_1, z) : \ z \in Min^*\} \cup \Delta$$

where Δ is the diagonal. Since $a_1 \prec b_1$, then $b_1 \notin Min$, therefore $P \cap Q = \Delta$. On the other hand, since any element x in X has an element of Min below it, then it is clear that there is a path in $P \cup Q$ from b_1 to x. A path from x to b_1 can be built as in (1). Thus Q is a complement of P. The proof that Q is in fact maximal can be done exactly as in the proof of lemma 2.4. But notice that the use of Min^* instead of Min is essential to guarantee maximality. □

A similar argument shows, *mutatis mutandis*, the following.

Lemma 2.6. *Assume P has no minimal elements and, in addition, suppose that every element has at least one maximal element above it. Then P has a maximal complement Q whose chains have at most two elements.* □

The next case we need to consider is when every element has at least one maximal element above it and also at least one minimal element below it. This case is handled in the following.

Lemma 2.7. *Suppose that every element of X has at least one maximal element above it and also at least one minimal element below it. Then P has a maximal complement that moreover has all chains of size at most two.*

Proof: Let Min and Max be respectively the collection of minimal and maximal elements of X. As in the proof of lemma 2.5, let Min^* and Max^* be formed by taking only one element of each equivalence class. Pick $a \in Min^*$ and $b \in Max^*$. Define Q as follows

$$\{(y, x) : \ x \in Min^* \ \& \ y \in Max^*\} \cup \Delta$$

It is straightforward to show that Q is a minimal complement for P. □

Now we will show how to glue together two maximal complements of a partition of the space and get a maximal complement of the whole space. This result is similar to proposition 2.10 of [3]. Recall that a subset Y of X is said to be open if for all x, y with $x \preceq y$ and $x \in Y$, then $y \in Y$.

Lemma 2.8. *Let X_1, X_2 be a partition of X such that X_1 is open. Let P_i be the restriction of P to X_i. Suppose that Q_i is a maximal complement for P_i with chains of at most two elements. Then P has a maximal complement whose chains have at most two elements.*

Proof: Since X_1 is open and X_2 is disjoint from X_1, then there is no $x \in X_1$ and $y \in X_2$ such that $x \preceq y$. We consider two cases.

(i) X_2 is also open. Since each chain in Q_i is of size at most two (and we are assuming that P is not trivial) then there are $a \prec_{Q_1} b$ and $c \prec_{Q_2} d$. Define Q as follows

$$Q = Q_1 \cup Q_2 \cup \{(d, a), (b, c)\}$$

Notice that since the X_i's are both open, then $(d, a), (b, c) \notin P$. Since each chain in Q_i has at most 2 elements, then it is easy to verify that Q is indeed transitive and moreover it is a pre-order whose chains has at most 2 elements. It is also routine to check that Q is a complement for P. To see that Q is a maximal complement, observe first that Q_i is equal to the restriction of Q to X_i. So it suffices to show that if we remove either (d, a) or (b, c) then the resulting pre-order is not a complement of P. But this is clear, since every path starting from a point of X_1 and ending in a point of X_2 necessarily uses (b, c) and analogously every path starting from a point in X_1 and ending in a point of X_2 necessarily uses (d, a). □

(ii) Suppose now that X_2 is not open in X. As before, there are $a \prec_{Q_1} b$ and $c \prec_{Q_2} d$. Define Q as follows

$$Q = Q_1 \cup Q_2 \cup \{(b, c)\}$$

Since X_2 is not open in X, then there are $x_i \in X_i$ such that $x_2 \prec x_1$. A completely similar argument as in case (i) but now replacing (d, a) by (x_2, x_1) shows that Q is a maximal complement for P. □

Now we have all we need to proof our main result. From this point on we will not assume that P has no trivial components.

Theorem 2.9. *Every pre-order P over a countable set has a maximal complement in $PO(X)$. Moreover, if P has no trivial components a maximal complement can be found such that its chains have size at most two.*

Proof: We first consider the case when P has no trivial components. Let Min and Max be respectively the collection of minimal and maximal elements of X. Consider the following subset of X.

$$\begin{aligned} X_1 &= \{x \in X : \nexists a \in Min\ \nexists b \in Max\ a \preceq x \preceq b\} \\ X_2 &= \{x \in X : \exists a \in Min\ \nexists b \in Max\ a \preceq x \preceq b\} \\ X_3 &= \{x \in X : \nexists a \in Min\ \exists b \in Max\ a \preceq x \preceq b\} \\ X_4 &= \{x \in X : \exists a \in Min\ \exists b \in Max\ a \preceq x \preceq b\} \end{aligned}$$

It is clear that they form a partition of X and moreover by lemmas 2.4, 2.5, 2.6 and 2.7 we know that P restricted to each X_i has a maximal complement with chains of size at most two. On the other hand, it is routine to verify the following

(i) $X_1 \cup X_2$ is open in X.
(ii) X_2 is open in $X_1 \cup X_2$.
(iii) X_4 is open in $X_3 \cup X_4$.

Therefore by lemma 2.8 and (ii) and (iii) above there are maximal complements for P restricted to $X_1 \cup X_2$ and $X_3 \cup X_4$ with chains of size at most two. Now, again by lemma 2.8 and (i) above, we get a maximal complement for P.

Finally, we handle the trivial components. Let Y be the union of all trivial components of P and assume Y is not empty. We consider two cases:

(1) Suppose $Z = X \setminus Y$ is not empty. Notice that Z and Y are open. From the result we just proved, there is a maximal complement Q for the restriction of P to Z. Let Y^* be a set containing one and only one element of each $\sim$-equivalence class of elements of Y. Pick $a, b \in Z$ such that $a \prec b$. Define a relation R as follows:

$$R = Q \cup \{(a, y), (y, b) : \; y \in Y^*\}$$

It is routine to check that R is a maximal complement for P. Notice that R has chain of size three.

(2) Suppose $X = Y$ is empty. This is equivalent to say that P is an equivalence relation. We can assume that P is neither equal to Δ nor to $X \times X$. Let W be the set of all elements of X whose equivalence class has size one and let Z be $X \setminus W$. We consider two cases.

(2a) Suppose Z has finitely many equivalence classes and let $\{a_n, b_n\}$ with $1 \leq n \leq m$ be a selection of two elements of each equivalence class in Z. If $m = 1$, the maximal complement is just the equivalence relation Q defined by letting all elements of W be equivalent to a representative of the unique equivalence class in Z. So we will assume that $m \geq 2$. Let

$$Q = \{(a_n, b_{n+1}) : 1 \leq n \leq m-1\} \cup \{(a_m, b_1)\} \cup \{(a_1, y), (y, b_2) : \; y \in W\} \cup \Delta$$

It is routine to check that Q is indeed a maximal complement for P (just draw a diagram similar to the one at the beginning of this section).

(2b) Suppose Z has infinitely many equivalence classes and let $\{a_n, b_n\}$ for $n \in \mathbb{Z}$ be a selection of two elements of each equivalence class in Z. Then a maximal complement Q for P is defined as follows:

$$Q = \{(a_n, b_{n+1}), (b_{n+1}, a_n) : n \in \mathbb{Z}\} \cup \{(a_1, y), (y, a_1), (y, b_2), (b_2, y) : \; y \in W\} \cup \Delta$$

□

As we have explained in the introduction, the following theorem is a consequence of theorem 2.9

Theorem 2.10. *Any AT topology over a countable set admits a maximal complement in the lattice of topologies.*

Proof: Let (X, τ) be an AT topology and let $\preceq$ be the associated pre-order (namely, $x \preceq y$ iff $x \in \overline{\{y\}}$) and denote by P the pre-order $(X, \preceq)$. Let Q be a maximal complement for P in $PO(X)$ given by 2.9. Then the associate AT topology ρ of Q is a complement for τ. To see that ρ is maximal in $TOP(X)$ just observe that if $\rho \subseteq \eta \subseteq \eta'$ and η' is a complement of τ, then η is also a complement of τ. So if there is $V \in \eta' \setminus \rho$, then add V to ρ to get an AT complement η of τ properly extending ρ. But this would contradict the maximality of Q in $PO(X)$. □

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Los Andes, Mérida 5101, Venezuela
e-mail: uzca@ciens.ula.ve