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Abstract

In this paper we study the existence and the stability of bounded solutions of the following non-linear Strongly Damped Wave Equations with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt} + \eta(-\Delta)^{1/2}u_t + \gamma(-\Delta)u = f(t,u), & t \geq 0, \quad x \in \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & x \in \Omega \\ u(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega, \end{cases}$$

where $f : I\!R \times I\!R \to I\!R$ is a continuous and globally Lipschitz function with a Lipschitz constant $L > 0$. Ω is a bounded domain in $I\!R^N (N \geq 1)$. Roughly speaking we shall prove the following result: If

$$\lambda_1^{1/2} \max\left\{ \mathrm{Re}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right) \right\} > L,$$

where λ_1 is the first eigenvalue of $-\Delta$, then the equation admits only one bounded solution which is exponentially stable. Also, we prove that for some big class of functions f this bounded solution is almost periodic

Key words. wave equations, bounded solutions, exponential stability.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

1 Introduction

In this paper we shall study the existence and the stability of bounded solutions for the following nonlinear strongly damped equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} u_{tt} + \eta(-\Delta)^{1/2}u_t + \gamma(-\Delta)u = f(t,u), & t \geq 0, \quad x\Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), & x \in \Omega \\ u(t,x) = 0, \quad t \geq 0, \quad x \in \partial\Omega. \end{cases} \tag{1.1}$$

where $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous and globally Lipschitz function with a Lipschitz constant $L > 0$. i.e.,

$$|f(t,u) - f(t,v)| \leq L|u - v|, \quad t, u, v \in \mathbb{R}. \tag{1.2}$$

Ω is a bounded domain in $\mathbb{R}^N (N \geq 1)$. We shall assume the following hypothesis:

H) there exists $L_f > 0$ such that

$$\|f(t,0)\| \leq L_f, \quad \forall t \in \mathbb{R}. \tag{1.3}$$

Under this assumption, roughly speaking we prove the following statements: If

$$\beta = \lambda_1^{1/2} \max \left\{ \mathrm{Re} \left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right) \right\} > L.$$

where λ_1 is the first eigenvalue of $-\Delta$. then the equation admits only one bounded solution which is exponentially stable. Also. we prove that for some big class of functions f this bounded solution is almost periodic. Some ideas for this work can be found in [1], [2], [3] and [4].

In [5] they prove that the linear part of (1.1) generates an analytic semigroup of contractions $\{T(t)\}_{t \geq 0}$ ($\|T(t)\| \leq 1$). Here we prove easily that this semigroup is analytic and decay exponentilly to zero. Moreover.

$$\|T(t)\| \leq e^{-\beta t}, \quad t \geq 0.$$

2 Abstract Formulation of the Problem

In this section we shall choose the space where this problem will be set as an abstract second order ordinary differential equation.

Let $X = L^2(\Omega) = L^2(\Omega, \mathbb{R})$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, \mathbb{R}) \cap H_0^1(\Omega, \mathbb{R}). \tag{2.1}$$

The operator A has the following very well known properties: the spectrum of A consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty.$$

each one with finite multiplicity γ_n equal to the dimension of the corresponding eigenspace. Therefore.

a) there exists a complete orthonormal set $\{\phi_{n,k}\}$ of eigenvector of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x. \tag{2.2}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n x = \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k}. \tag{2.3}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and $x = \sum_{n=1}^{\infty} E_n x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At} x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{2.4}$$

d)

$$X^{\alpha} = D(A^{\alpha}) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2\alpha} \|E_n x\|^2 < \infty\}.$$

and

$$A^{\alpha} x = \sum_{n=1}^{\infty} (\lambda_n)^{\alpha} E_n x. \tag{2.5}$$

Hence, the equation (1.1) can be written as an abstract second order ordinary differential equation in X as follow

$$\begin{cases} u'' + \eta A^{1/2} u' + \gamma A u = & f^e(t, u), \quad t \geq 0, \\ u(0) = u_0, \quad u'(0) = & v_0. \end{cases} \tag{2.6}$$

where $f^e : I\!R_+ \times X \to X$ is given by:

$$f^e(t, u)(x) = f(t, u(x)), \quad x \in \Omega, \quad u \in X.$$

So,

$$\|f^e(t, u) - f^e(t, v)\| \leq L\|u - v\|, \quad t \in I\!R, u, v \in X. \tag{2.7}$$

Now, making the following change of variable $u' = v$ we can write the second order equation (2.6) as first order system of ordinary differential equations in the Hilbert space $Z = X \times X$ as follow:

$$z' = \mathcal{A}z + F(t, z), \quad z \in Z, \quad t \geq 0. \tag{2.8}$$

where

$$z = \begin{pmatrix} u \\ v \end{pmatrix}, \quad F(t,z) = \begin{pmatrix} 0 \\ f^e(t,u) \end{pmatrix}, \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & I_X \\ -\gamma A & -\eta A^{1/2} \end{pmatrix}, \tag{2.9}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times D(A^{1/2})$.

The proof of the following Theorem follows from Lemma 2.1 and 22 of [6].

Theorem 2.1 *The operator $\mathcal{A}$ given by (2.8), is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t\geq 0}$ given by*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z, \quad t \geq 0 \tag{2.10}$$

where $\{P_n\}_{n\geq 0}$ is a complete orthogonal projections in the Hilbert space Z given by

$$P_n = diag\,(E_n, E_n)\ , \ n \geq 1\ , \tag{2.11}$$

and

$$A_n = \begin{pmatrix} 0 & 1 \\ -\gamma\lambda_n & -\eta\lambda_n^{1/2} \end{pmatrix}, \ n \geq 1 \tag{2.12}$$

Moreover, this semigroup decay exponentially to zero. In fact, we have that

$$\|T(t)\| \leq e^{-\beta t}, \quad t \geq 0, \tag{2.13}$$

where

$$\beta = \lambda_1^{1/2} max\left\{ Re\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right) \right\}$$

3 Existence of the Bounded Solution

In this section we shall prove the existence and the stability of unique bounded Mild solutions of the system (2.8).

Definition 3.1 (Mild Solution) For mild solution $z(t)$ of (2.8) with initial condition $z(t_0) = z_0 \in Z$, we understand a function given by

$$z(t) = T(t - t_0)z_0 + \int_{t_0}^{t} T(t-s)F(s, z(s))ds, \quad t \in \mathbb{R}. \tag{3.1}$$

Remark 3.1 *It is easy to prove that any solution of (2.8) is a solution of (3.1). It may be thought that a solution of (3.1) is always a solution of (2.8) but this is not true in general. However, we shall prove in Theorem 4.1 that bounded solutions of (3.1) are solutions of (2.8).*

We shall consider $Z_b = C_b(\mathbb{R}, Z)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in Z. Z_b is a Banach space with suprem norm

$$\|z\|_b = \sup\{\|z(t)\|_Z \quad t \in \mathbb{R}\}, \quad z \in Z_b.$$

A ball of radio $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{z \in Z_b : \|z(t)\|_b \leq \rho, \quad t \in \mathbb{R}\}.$$

The proof of the following Lemma is similar to Lemma 3.1 of [4].

Lemma 3.1 *Let z be in Z_b. If z is a mild solution of (2.8), then z is a solution of the following integral equation*

$$z(t) = \int_{-\infty}^{t} T(t-s)F(s, z(s))ds, \quad t \in \mathbb{R}. \tag{3.2}$$

If z is a solution of (3.2), then z is a mild solution of (2.8) for $t \geq 0$.

The following Theorem refers to bounded Mild solutions of system (2.8). Even though, the proof is similar to Theorem 3.2 of [2], we will give the proof.

Theorem 3.1 *If for some η and γ with $\eta^2 \neq 4\gamma$ we have that*

$$\lambda_1^{1/2} max\left\{ Re\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right)\right\} > L, \tag{3.3}$$

then the equation (2.8) has one and only one bounded mild solution $z_b(t)$.

Moreover, this bounded solution is the only bounded solution of the equation (3.1) and is exponentially stable.

Proof Condition (3.3) implies that for $\rho > 0$ big enough we have the following estimate:

$$0 < L_f < (\beta(\eta, \gamma) - L)\rho. \tag{3.4}$$

where

$$\beta = \lambda_1^{1/2} \max \left\{ \operatorname{Re}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right) \right\}.$$

For the existence of such solution, we shall prove that the following operator has a unique fixed point in the ball B^b_ρ, $T : B^b_\rho \to B^b_\rho$

$$(Tz)(t) = \int_{-\infty}^{t} T(t-s)F(s, z(s))ds, \quad t \in I\!R.$$

In fact, for $z \in B^b_\rho$ we have

$$\|Tz(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} \{L\|z(s)\| + L_f\} \leq \frac{(L)\rho + L_f}{\beta}.$$

The condition (3.4) implies that

$$L\rho + L_f < \beta\rho \iff \frac{L\rho + L_f}{\beta} < \rho.$$

Therefore, $Tz \in B^b_\rho$ for all $z \in B^b_\rho$.

Now, we shall see that T is a contraction mapping. In fact, for all $z_1, z_2 \in B^b_\rho$ we have that

$$\|Tz_1(t) - Tz_2(t)\| \leq \int_{-\infty}^{t} e^{-\beta(t-s)} L\|z_1(s) - z_2(s)\|ds \leq \frac{L}{\beta}\|z_1 - z_2\|_b, \quad t \in I\!R.$$

Hence,

$$\|z_1 - Tz_2\|_b \leq \frac{L}{\beta}\|Tz_1 - z_2\|_b, \quad z_1, z_2 \in B^b_\rho.$$

The condition (3.4) implies that

$$0 < \beta - L \iff L < \beta \iff \frac{L}{\beta} < 1.$$

Therefore, T has a unique fixed point z_b in B^b_ρ

$$z_b(t) = (Tz_b)(t) = \int_{-\infty}^{t} T(t-s)F(s, z_b(s))dsds, \quad t \in I\!R,$$

From Lemma 3.1, z_b is a bounded solution of the equation (3.1). Since condition (3.4) holds for any $\rho > 0$ big enough independent of $L < \beta(\eta, \gamma)$, then z_b is the unique bounded solution of the equation (3.1).

To prove that $z_b(t)$ is exponentially stable in the large, we shall consider any other solution $z(t)$ of (3.1) and consider the following estimate for $t_0 \geq 0$

$$\begin{aligned}\|z(t) - z_b(t)\| &\leq \|T(t)(z(t_0) - z_b(t_0)) + \int_{t_0}^{t} T(t-s)\{F(s, z(s)) - F(s, z_b(s))\}\, ds\| \\ &\leq e^{-\beta t}\|(z(t_0) - z_b(t_0))\| + \int_{t_0}^{t} e^{-\beta(t-s)} L\|z(s) - z_b(s)\| ds.\end{aligned}$$

Then,

$$e^{\beta t}\|z(t) - z_b(t)\| \leq \|(z(t_0) - z_b(t_0))\| + \int_{t_0}^{t} e^{\beta s} L\|z(s) - z_b(s)\| ds.$$

Hence, applying the Gronwall's inequality we obtain

$$\|z(t) - z_b(t)\| \leq e^{(L-\beta)t}\|(z(t_0) - z_b(t_0))\|, \quad t \geq t_0.$$

From (3.4) we get that $L - \beta < 0$ and therefore $z_b(t)$ is exponentially stable in the large. □

Corolary 3.1 *If f is periodic in t of period τ ($f(t+\tau, \xi) = f(t, \xi)$), then the unique bounded solution given by Theorem 3.1 is also periodic of period τ.*

Proof Let z_b be the unique solution of (3.1) in the ball B^b_ρ. Then, $z(t) = z_b(t+\tau)$ is also a solution of the equation (3.1) lying in the ball B^b_ρ. In fact, consider $z_0 = z_b(0)$ and

$$\begin{aligned} z_b(t+\tau) &= T(t+\tau)z_0 + \int_0^{t+\tau} T(t+\tau-s)F(s, z_b(s))ds \\ &= T(t)T(\tau)z_0 + \int_0^{\tau} T(t+\tau-s)F(s, z_b(s))ds \\ &+ \int_{\tau}^{t+\tau} T(t+\tau-s)F(s, z_b(s))ds \\ &= T(t)\left\{T(\tau)z_0 + \int_0^{\tau} T(\tau-s)F(s, z_b(s))ds\right\} \\ &+ \int_0^{t} T(t-s)F(s, z_b(s+\tau))ds \\ &= T(t)z_b(\tau) + \int_0^{t} T(t-s)F(s, z_b(s+\tau))ds \end{aligned}$$

Therefore,

$$z(t) = T(t)z_b(\tau) + \int_0^t T(t-s)F(s, z(s))ds,$$

and by the uniqueness of the fixed point of the contraction mapping T in this ball, we conclude that $z_b(t) = z_b(t+\tau)$, $t \in \mathbb{R}$. □

Remark 3.2 *Under some condition the bounded solution given by Theorem 3.1 is almost periodic: for example we can study the case when the function f has the following form:*

$$f(t,\xi) = g(\xi) + P(t), \quad t, \xi \in \mathbb{R}, \tag{3.5}$$

where $P \in C_b(\mathbb{R}, \mathbb{R})$, the space of continuous and bounded functions.

Corolary 3.2 *Suppose f has the form (3.5). Then bounded solution $z_b(\cdot, P)$ given by Theorem 3.1 depends continuously on $P \in C_b(\mathbb{R}, \mathbb{R})$.*

Proof Let $P_1, P_2 \in C_b(\mathbb{R}, \mathbb{R})$ and $z_b(\cdot, P_1), z_b(\cdot, P_2)$ be the bounded functions given by Theorem 3.1. Then

$$\begin{aligned} z_b(t, \cdot, P_1) - z_b(t, \cdot, P_2) &= \int_{-\infty}^t T(t-s)[g(z_b(s, P_2)) - g(z_b(s, P_2))]ds \\ &+ \int_{-\infty}^t T(t-s)[P_1(s) - P_2(s)]ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_b(\cdot, P_1) - z_b(\cdot, P_2)\|_b &\leq \frac{L}{\lambda_1^{1/2}\max\left\{\mathrm{Re}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right)\right\}}\|z_b(\cdot, P_1) - z_b(\cdot, P_2)\|_b \\ &+ \frac{1}{\lambda_1^{1/2}\max\left\{\mathrm{Re}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right)\right\}}\|P_1 - P_2\|_b. \end{aligned}$$

Therefore,

$$\|z_b(\cdot, P_1) - z_b(\cdot, P_2)\|_b \leq \frac{1}{\lambda_1^{1/2}\max\left\{\mathrm{Re}\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}\right)\right\} - 1}\|P_1 - P_2\|_b.$$

□

Lemma 3.2 *Suppose f is as (3.5). Then, if $P(t)$ is almost periodic, then the unique bounded solution of the system (3.1) given by Theorem 3.1 is also almost periodic.*

Proof To prove this lemma, we shall use the following well known fact, due to S. Bohr. A function $f \in C(\mathbb{R}; Z)$ is almost periodic (a.p) if and only if the Hull $H(h)$ of h is compact in the topology of uniform convergence.

Where $H(h)$ is the closure of the set of translates of h under the topology of uniform convergence

$$H(h) = \overline{\{h_\tau : \tau \in \mathbb{R}\}}, \qquad h_\tau(t) = h(t+\tau), t \in \mathbb{R}.$$

Since the limit of a uniformly convergent sequence of a.p. functions is a.p., then the set A_ρ of a.p. functions in the ball B_ρ^b is closed, where ρ is given by Theorem 3.1

Claim. The contraction mapping T given in Theorems 3.1 leaves A_ρ invariant. In fact: if $z \in A_\rho$, then $h(t) = g(z(t)) + P(t)$ is also an a.p. function. Now, consider the function

$$\begin{aligned} \mathcal{F}(t) = (Tz)(t) &= \int_{-\infty}^{t} T(t-s)\{g(z(s)) + P(s)\}\,ds \\ &= \int_{-\infty}^{t} T(t-s)h(s)ds, \quad t \in \mathbb{R}. \end{aligned}$$

Then, it is enough to establish that $H(\mathcal{F})$ is compact in the topology of uniform convergence. Let $\{\mathcal{F}_{\tau_k}\}$ be any sequence in $H(\mathcal{F})$. Since h is a.p. we can select from $\{h_{\tau_k}\}$ a Cauchy subsequence $\{h_{\tau_{k_j}}\}$, and we have that

$$\begin{aligned} \mathcal{F}_{\tau_{k_j}}(t) = \mathcal{F}(t+\tau_{k_j}) &= \int_{-\infty}^{t+\tau_{k_j}} T(t+\tau_{k_j}-s)h(s)ds \\ &= \int_{-\infty}^{t} T(t-s)h(s+\tau_{k_j})ds. \end{aligned}$$

Hence

$$\begin{aligned} \|\mathcal{F}_{\tau_{k_j}}(t) - \mathcal{F}_{\tau_{k_i}}(t)\|_\alpha &\le \int_{-\infty}^{t} e^{-\beta(t-s)}\|h(s+\tau_{k_j}) - h(s+\tau_{k_i})\|ds \\ &\le \|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b \int_{-\infty}^{t} e^{-\beta(t-s)}ds = \frac{1}{\beta}\|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b. \end{aligned}$$

Therefore, $\{\mathcal{F}_{\tau_{k_j}}\}$ is a Cauchy sequence. So, $H(\mathcal{F})$ is compact in the topology of uniform convergence, $\mathcal{F}$ is a.p. and $TA_\rho \subset A_\rho$.

Now, the unique fixed point of T in the ball B^b_ρ lies in A_ρ. Hence, the unique bounded solution $z_b(t)$ of the equation (3.1) given in Theorem 3.1 is also almost periodic. □

4 Smoothness of the Bounded Solution

In this part, we shall prove that the bounded Mild solution $z_b(t)$ of the equation (2.8) is also a classic solution of this equation: that is to say, we shall proof the smoothness of this solution. With this we conclude this work.

Theorem 4.1 *The bounded Mild solution $z_b(t)$ of the equation (2.8) given by Theorem 3.1 is a classic solution of this equation on $\mathbb{R}$, i.e.,*

$$z_b'(t) = \mathcal{A}z_b(t) + F(t, z_b(t)), \quad t \in \mathbb{R}.$$

Proof Let $z_b(t)$ be the only bounded mild solution of (2.8) given by Theorem 3.1. Then

$$z_b(t) = \int_{-\infty}^{t} T(t-s)g(s)ds = \int_{0}^{\infty} T(s)g(t-s)ds, \quad t \in \mathbb{R}$$

where $g(s) = F(s, z_b(s))$. Therefore, $g \in C_b(\mathbb{R}, Z)$ and $\|g(s)\| \leq \|g\|_b$, $s \in (-\infty, t)$.

Let us put $x(s) = T(t-s)g(s)$, $s \in (-\infty, t)$. Then $x(s)$ is a continuous function, and since $\{T(t)\}_{t\geq 0}$ is analytic, then

$$x(s) \in D(\mathcal{A}), \quad \text{for} \quad s < t.$$

Claim. $\mathcal{A}x(s)$ is continuous on $(-\infty, t)$ and the improper integral

$$\int_{-\infty}^{t} \mathcal{A}x(s)ds, \quad t \in \mathbb{R}.$$

exists.

In fact, there exists a complete system of orthogonal projections $\{q_i(n)\}_{i=1}^{2}$ in $\mathbb{R}^2$ such that

$$\begin{cases} -A_n &= \rho_1(n)q_1(n) + \rho_2(n)q_2(n) \\ e^{A_n t} &= e^{-\rho_1(n)t}q_1(n) + e^{-\rho_2(n)t}q_2(n), \end{cases}$$

where

$$\rho(n) = \lambda_n \left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2} \right).$$

Hence,

$$-Az = \sum_{n=1}^{\infty} \{\rho_1(n) P_{n1} z + \rho_2(n) p_{n2} z\}$$

and

$$T(t)z = \sum_{n=1}^{\infty} \left\{ e^{-\rho_1(n)t} P_{n1} z + e^{-\rho_2(n)t} P_{n2} z \right\}.$$

where, $P_{ni} = q_i(n) P_n$ is a complete system of orthogonal projections in Z.

Therefore,

$$Ax(s) = \sum_{n=1}^{\infty} \left\{ -\rho_1(n) e^{-\rho_1(n)(t-s)} P_{n1} g(s) - \rho_2(n) e^{-\rho_2(n)(t-s)} p_{n2} g(s) \right\}.$$

So,

$$\|Ax(s)\| \leq \max_{n \geq 1} \left\{ \lambda_n |\rho| e^{-\lambda_n \mathrm{Re}(\rho)(t-s)} \right\} \|g\|_b,$$

with

$$\rho = \frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2}.$$

Hence, $Ax(s)$ is a continuous function on $(-\infty, t)$. Now, consider the following improper integrals:

$$\begin{aligned}
\int_{-\infty}^{t} Ax(s) ds &= \int_0^{\infty} AT(s) g(t-s) ds \\
&= \int_0^{\infty} \sum_{n=1}^{\infty} \left\{ -\rho_1(n) e^{-\rho_1(n)s} P_{n1} g(t-s) - \rho_2(n) e^{-\rho_2(n)s} P_{n2} g(t-s) \right\} ds \\
&= \sum_{n=1}^{\infty} \left\{ \int_0^{\infty} -\rho_1(n) e^{-\rho_1(n)s} P_{n1} g(t-s) ds - \int_0^{\infty} \rho_2(n) e^{-\rho_2(n)s} P_{n2} g(t-s) ds \right\}.
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\left\| \int_0^{\infty} -\rho_i(n) e^{-\rho_i(n)s} P_{ni} g(t-s) ds \right\| &\leq \int_0^{\infty} \lambda_n |\rho| e^{-\lambda_n \mathrm{Re}(\rho)s} \|P_{ni} g(t-s)\| ds \\
&\leq \frac{|\rho|}{\mathrm{Re}(\rho)} \|g\|_b.
\end{aligned}$$

Therefore, the improper integral

$$\int_{-\infty}^{t} Ax(s) ds. \quad \text{exists}$$

Now, from Theorem 1.3.5 of [7] we have that

$$\int_{-\infty}^{t} x(s)ds \in D(\mathcal{A}), \quad \text{and} \quad \mathcal{A}\int_{-\infty}^{t} x(s)ds = \int_{-\infty}^{t} \mathcal{A}x(s)ds.$$

i.e.,

$$\int_{-\infty}^{t} T(t-s)g(s)ds \in D(\mathcal{A}), \quad \text{and} \quad \mathcal{A}\int_{-\infty}^{t} T(t-s)g(s)ds = \int_{-\infty}^{t} \mathcal{A}T(t-s)g(s)ds.$$

Now, we are ready to prove that $z_b(t)$ is a solution of (2.8). In fact, consider

$$\begin{aligned}
\frac{z_b(t+h) - z_b(t)}{h} &= \frac{1}{h}\int_{-\infty}^{t+h} T(t+h-s)g(s)ds - \frac{1}{h}\int_{-\infty}^{t} T(t-s)g(s)ds \\
&= \left(\frac{T(h)-I}{h}\right)\int_{-\infty}^{t} T(t-s)g(s)ds + \frac{1}{h}\int_{t}^{t+h} T(t+h-s)g(s)ds.
\end{aligned}$$

Using the definition of infinitesimal generator of a semigroup and passing to the limit as $h \to 0^+$ we get that

$$z_b'(t) = \mathcal{A}\int_{-\infty}^{t} T(t-s)g(s)ds + T(0)g(t).$$

So,

$$z_b'(t) = \mathcal{A}z_b(t) + F(t, z_b(t)), \quad t \in I\!R.$$

□

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