Universidad de los Andes

Facultad de Ciencias

Departamento de Matemática

Existence and Stability of Bounded Solutions for a System of Parabolic Equations

Hugo Leiva - I. Sequera

Notas de Matemática

Serie: Pre-Print

No. 216

Mérida - Venezuela

2001

Notas de Matemática, No. 216
Mérida, 2001.

Existence and Stability of Bounded Solutions for a System of Parabolic Equations

Hugo Leiva - I. Sequera

Abstract

In this paper we study the existence and the stability of bounded solutions of the following non-linear system of parabolic equations with homogeneous Dirichlet boundary conditions

$$\begin{aligned} u_t &= D\Delta u + f(t,u), \quad t \geq 0, \quad u \in I\!R^n, \\ u &= 0 \quad , \quad \text{on } \partial\Omega \end{aligned}$$

where $f \in C^1(I\!R \times I\!R^n)$, $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, \quad i = 1, 2, \ldots, n$ and Ω is a bounded domain in $I\!R^N(N = 1, 2, 3)$. Roughly speaking we shall prove the following result: if f is globally Lipschitz with constant L, $3/4 < \alpha < 1$ and $\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} > L$, then the system has a bounded solution which is stable, where $d = \min\{d_i : i = 1, 2, \ldots, n\}$, λ_1 is the first eigenvalue of $-\Delta$ and $\Gamma(\cdot)$ the well known gamma function. Also, we prove that for some big class of functions f this bounded solution is almost periodic.

Key words. system of parabolic equations, bounded solutions, stability.
AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.
Running Title: BOUNDED SOLUTIONS FOR PARABOLIC EQS.

1 Introduction

In this paper we shall study the existence and the stability of bounded solutions for the following system of parabolic equations with homogeneous Dirichlet boundary conditions

$$u_t = D\Delta u + f(t,u), \quad t \geq 0, \quad u \in I\!R^n, \tag{1.1}$$

$$u = 0 \quad , \quad \text{on } \partial\Omega \tag{1.2}$$

where $f \in C^1(I\!R \times I\!R^n)$, $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, \quad i = 1, 2, \ldots, n$ and Ω is a bounded domain in $I\!R^N(N = 1, 2, 3)$.

We shall assume the following hypothesis:
H) there exists $L_f > 0$ such that

$$\|f(t,0)\| \leq L_f, \quad \forall t \in I\!R. \tag{1.3}$$

Under this assumption, roughly speaking we prove the following statement:
If f is globally Lipschitz in the second variable with a Lipschitz constant L, $3/4 < \alpha < 1$ and $\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} > L$, then the system admits only one bounded solution which is uniformly stable, where

$$d = \min\{d_i : i = 1, 2, \dots, n\}, \tag{1.4}$$

λ_1 is the first eigenvalue of $-\Delta$ and $\Gamma(\cdot)$ de well known gamma function. Also, we prove that for some particular f this bounded solution is almost periodic.

Several mathematical models may be written as a system of reaction-diffusion of the form (1.1), like a models of vibration of plates(see [1]) and a Lotka-Volterra system with diffusion(see [2]). Some ideas for this work can be found in [3], [4], [5] and [6].

2 Notations and Preliminaries

In this section we shall choose the space where this problem will be set.

Let $X = L^2(\Omega) = L^2(\Omega, I\!R)$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = H^2(\Omega, I\!R) \cap H_0^1(\Omega, I\!R). \tag{2.1}$$

Since this operator is sectorial, then the fractional power space X^α associated with A can be defined. That is to say: for $\alpha \geq 0$, $X^\alpha = D(A^\alpha)$ endowed with the graph norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in X^\alpha. \tag{2.2}$$

(see D. Henry [7] pg 29).

Precisely we have the following situation: Let $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the eigenvalues of A each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore
a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x, \tag{2.3}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{2.4}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x. \tag{2.5}$$

d)

$$X^{\alpha} = D(A^{\alpha}) = \{x \in X : \sum_{j=1}^{\infty} (\lambda_j)^{2\alpha} \|E_j x\|^2 < \infty\},$$

and

$$A^{\alpha} x = \sum_{j=1}^{\infty} (\lambda_j)^{\alpha} E_j x. \tag{2.6}$$

Also, we shall use the following notation:

$$Z := L^2(\Omega, \mathbb{R}^n) = X^n = X \times \cdots \times X, \quad \text{and} \quad C_n = C(\Omega, \mathbb{R}^n) = [C(\Omega)]^n,$$

with the usual norms.

Now, we define the following operator

$$\mathcal{A}_D : D(\mathcal{A}_D) \subset Z \to Z, \quad \mathcal{A}_D \psi = -D\Delta\psi = DA\psi, \tag{2.7}$$

where

$$D(\mathcal{A}_D) = H^2(\Omega, \mathbb{R}^n) \cap H_0^1(\Omega, \mathbb{R}^n).$$

Therefore, $\mathcal{A}_D$ is a sectorial operator and the fractional power space Z^{α} associated with $\mathcal{A}_D$ is given by

$$Z^{\alpha} = D(\mathcal{A}_D^{\alpha}) = X^{\alpha} \times \cdots \times X^{\alpha} = [X^{\alpha}]^n. \tag{2.8}$$

endowed with the graph norm

$$\|z\|_\alpha = \|\mathcal{A}_D^\alpha z\|, \quad z \in Z^\alpha, \tag{2.9}$$

where

$$\mathcal{A}_D^\alpha z = \sum_{j=1}^{\infty} D^\alpha (\lambda_j)^\alpha P_j z, \qquad D^\alpha = \mathrm{diag}(d_1^\alpha, d_2^\alpha, \cdots, d_n^\alpha), \tag{2.10}$$

and $P_j = \mathrm{diag}(E_j, E_j, \cdots, E_j)$ is an $n \times n$ matrix.

The C_o-semigroup $\{e^{-\mathcal{A}_D t}\}_{t\geq 0}$ generated by $-\mathcal{A}_D$ is given as follow

$$e^{-\mathcal{A}_D t} z = \sum_{j=1}^{\infty} e^{-\lambda_j D t} P_j z, \quad z \in Z. \tag{2.11}$$

Clearly, $\{P_j\}$ is a family of orthogonal projections in Z which is complete. So,

$$z = \sum_{j=1}^{\infty} P_j z, \quad \|z\|^2 = \sum_{j=1}^{\infty} \|P_j z\|^2 \ \text{ and } \ \|z\|_\alpha^2 = \sum_{j=1}^{\infty} \|P_j z\|_\alpha^2. \tag{2.12}$$

From (2.11) it follows that there exists a constant $M > 0$ such that for all $z \in Z^\alpha$

$$\|e^{-\mathcal{A}_D t} z\|_\alpha \leq M \|z\|_\alpha e^{-d\lambda_1 t}, \quad t \geq 0, \tag{2.13}$$

$$\|e^{-\mathcal{A}_D t} z\|_\alpha \leq M t^{-\alpha} \|z\| e^{-d\lambda_1 t}, \quad t > 0. \tag{2.14}$$

From Theorem 1.6.1 in D. Henry [7] it follows that for $\frac{3}{4} < \alpha \leq 1$ the following inclusions

$$Z^\alpha \subset C(\Omega, \mathbb{R}^n) \ \text{ and } \ Z^\alpha \subset L^p(\Omega, \mathbb{R}^n), \quad p \geq 2, \tag{2.15}$$

are continuous.

Now, the systems (1.1)-(1.2) can be written in an abstract way on Z as follow:

$$z' = -\mathcal{A}_D z + f^e(t, z), \quad z(t_0) = z_0 \quad t \geq t_0 > 0. \tag{2.16}$$

Where $f^e : \mathbb{R} \times Z^\alpha \to Z$ is given by:

$$f^e(t, z)(x) = f(t, z(x)), \quad x \in \Omega. \tag{2.17}$$

To show that equation (2.16) is well posed in Z^α we have to prove the following lemma.

Lemma 2.1 *The function* f^e *given in (2.17) is locally Hölder continuous in* t *and locally Lipschitz in* z. *i.e., given an interval* $[a,b]$ *and a ball* $B_r^\alpha(0)$ *in* Z^α *there exist* $\theta > 0$ *and* $K > 0$ *such that*

$$\|f^e(t,z_1)-f^e(s,z_2)\| \le K(|t-s|^\theta+\|z_1-z_2\|_\alpha), \;\; \|z_1\|_\alpha, \|z_2\|_\alpha \le r, \;\; t,s \in [a,b].$$

Proof Since $f \in C^1(\mathbb{R}\times\mathbb{R}^n)$, then for each interval $[a,b]$ and a ball $B_\rho(0) \subset \mathbb{R}^n$ there exist constans $k > 0$ and $M(\rho) > 0$ such that

$$\|f(t,x) - f(s,y)\| \le k|t-s| + M(\rho)\|x-y\| \;\text{ if }\; \|x\|, \|y\| \le \rho, \quad t,s \in [a,b].$$

From the continuous inclusion $Z^\alpha \subset C_n$ there exists $l > 1$ such that

$$\sup_{x\in\Omega} \|z(x)\|_{\mathbb{R}^n} \le l\|z\|_\alpha, \quad z \in Z^\alpha.$$

Now, let $B_r^\alpha(0)$ be a ball in Z^α. Then putting $\rho = lr$ we get that

$$\|f(t,z_1(x)) - f(s,z_2(x))\| \le k|t-s| + M(lr)\|z_1(x) - z_2(x)\|, \quad x \in \Omega,$$

if $\|z_1\|_\alpha, \|z_2\|_\alpha \le r$ and $t,s \in [a,b]$.

Therefore, if $\|z_1\|_\alpha, \|z_2\|_\alpha \in B_r^\alpha(0)$ and $t,s \in [a,b]$, then

$$\|f^e(t,z_1) - f^e(s,z_2)\| \le k\mu(\Omega)^{1/2}|t-s| + M(lr)\|z_1 - z_2\|,$$

where $\mu(\Omega)$ denote the Lebesgue measure of Ω.

Now, from the continuous inclusion $Z^\alpha \subset L^2(\Omega, \mathbb{R}^n)$ there exists a constant $R > 0$ such that

$$\|z\|_{L^2} \le R\|z\|_\alpha, \quad z \in Z^\alpha.$$

Hence, if $\|z_1\|_\alpha, \|z_2\|_\alpha \in B_r^\alpha(0)$ and $t,s \in [a,b]$, then

$$\|f^e(t,z_1) - f^e(s,z_2)\| \le k\mu(\Omega)^{1/2}|t-s| + RM(lr)\|z_1 - z_2\|_\alpha.$$

We complete the proof by putting $\theta = 1$ and $K = \max\{k\mu(\Omega)^{1/2}, RM\}$. □

The following proposition can be proved in the same way as the foregoing lemma.

Proposition 2.1 *Supoose* f *is globally Lipschitz with a constant* L. *i.e.,*

$$\|f(t,u) - f(t,v)\| \le L\|u-v\|, \quad \forall t \in \mathbb{R}, \quad u,v \in \mathbb{R}^n.$$

Then

$$\|f^e(t,z_1) - f^e(t,z_2)\| \le LR\|z_1 - z_2\|_\alpha, \quad z_1, z_2 \in Z^\alpha, \quad t \in \mathbb{R}, \tag{2.18}$$

where $\|z\| \le R\|z\|_\alpha$, *for* $z \in Z^\alpha$. *Also, from the hypothesis H) we get that*

$$\|f^e(t,0)\| \le \mu(\Omega)L_f, \quad t \ge 0. \tag{2.19}$$

From Theorem 7.1.4 in [7], for all $T > t_0$ we have the following:

A continuous function $z(\cdot) : (t_0, T) \to Z^\alpha$ is solution of the integral equation

$$z(t) = e^{-A_D(t-t_0)}z_0 + \int_{t_0}^{t} e^{-A_D(t-s)}f^e(s, z(s))ds, \quad t \in (t_0, T] \qquad (2.20)$$

if and only if $z(\cdot)$ is a solution of (2.16).

From now on, we will suppose that $\frac{3}{4} < \alpha < 1$ and that $R = M = 1$..

3 Main Theorems

Now, we are ready to formulate the main results of this paper. Under the above conditions we can prove the following Theorems.

Theorem 3.1 *Consider B_ρ^α the ball of center zero and radio $\rho > 0$ in Z^α, and L_ρ the Lipschitz constant of f^e in $B_{2\rho}^\alpha$ and $\mu(\Omega)$ the Lebesgue measure of Ω. If the following estimate holds*

$$\left(\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} - 4L_\rho\right)\rho > \mu(\Omega)L_f, \qquad (3.1)$$

then the equation (2.16) admits one and only one bounded solution z_b, with $\|z_b(t)\| \le \rho$, $t \in \mathbb{R}_+$. Moreover, this bounded solution is locally stable

Theorem 3.2 *Suppose f satisfies condition (2.18) and*

$$\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} > L. \qquad (3.2)$$

Then equation (2.16) admits one and only one bounded solution $z_b(t)$ for $t \in \mathbb{R}_+$.

Moreover, this bounded solution is globally uniformly stable.

Before the proof of the main results we shall prove the following key lemma. Consider $Z_b^\alpha = C_b(\mathbb{R}, Z^\alpha)$ the space of bounded and continuous functions defined in $\mathbb{R}$ taking values in Z^α. Then Z_b^α is a Banach space with suprem norm

$$\|z\|_b = \sup\{\|z(t)\|_\alpha : t \in \mathbb{R}\}, \quad z \in Z_b^\alpha.$$

A ball of radio $\rho > 0$ and center zero in this space is given by

$$B_\rho^b = \{z \in Z_b^\alpha : \|z(t)\|_b \le \rho, \quad t \in \mathbb{R}\}.$$

Lemma 3.1 *Let z be in Z_b^α. If z is a solution of (2.16), then z is a solution of the following integral differential equation*

$$z(t) = \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s, z(s))ds, \quad t \in I\!R. \tag{3.3}$$

If z is a solution of (3.3), then z is a solution of (2.16) for $t \geq 0$.

Proof Suppose that z is a solution of (2.16). Then, from the variation constant formula (2.20) and the uniqueness of the solution of (2.16) we get that

$$z(t) = e^{-\mathcal{A}_D(t-t_0)}z(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s, z(s))ds, \quad t \geq t_0. \tag{3.4}$$

On the other hand, from (2.13) we obtain that

$$\|e^{-\mathcal{A}_D(t-t_0)}z(t_0)\|_\alpha \leq e^{-d\lambda_1(t-t_0)}\|z(t_0)\|_\alpha, \quad t \geq t_0,$$

and since $\|z(t)\|_\alpha \leq m, \quad t \in I\!R$, we get the following estimate

$$\|e^{-\mathcal{A}_D(t-t_0)}z(t_0)\|_\alpha \leq me^{-d\lambda_1(t-t_0)}, \quad t \geq t_0,$$

which implies that

$$\lim_{t_0 \to -\infty} \|e^{-\mathcal{A}(t-t_0)}z(t_0)\|_\alpha = 0.$$

Let $\rho > 0$ such that $\|z\|_b \leq \rho$ and L_ρ the Lipschitz constant of f^e in $B_{2\rho}^\alpha$. Then from the inequalities (2.13)-(2.14) we get the following estimates

$$\begin{aligned}
\int_{-\infty}^{t} \|e^{-\mathcal{A}_D(t-s)} f^e(s, z(s))\|_\alpha ds &\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} \|f^e(s, z(s))\|_\alpha ds \\
&\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} \left\{L_\rho \|z(s)\|_\alpha + \|f^e(s,0)\|\right\} ds \\
&\leq \left\{L_\rho \|z\|_b + \mu(\Omega) L_f\right\} \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} ds \\
&= \left\{L_\rho \|z\|_b + \mu(\Omega) L_f\right\} \frac{\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}.
\end{aligned}$$

Therefore, passing to the limit in (3.4) when t_0 goes to $-\infty$ we conclude that

$$z(t) = \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s, z(s))ds, \quad t \in I\!R.$$

Suppose that z is a solution of the integral equation (3.3). Then

$$\begin{aligned} z(t) &= \int_{-\infty}^{0} e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds \\ &+ \int_{0}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds. \end{aligned}$$

Hence, for $t \geq 0$ we get that

$$\begin{aligned} z(t) &= e^{-\mathcal{A}_D t} \int_{-\infty}^{0} e^{\mathcal{A}_D s} f^e(s,z(s))ds \\ &+ \int_{0}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds \\ &= e^{-\mathcal{A}_D t} z(0) + \int_{0}^{t} e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds, \end{aligned}$$

where

$$z(0) = \int_{-\infty}^{0} e^{\mathcal{A}_D s} f^e(s,z(s))ds.$$

Therefore, $z(t)$ is solution of the equation (2.16). □

Proof of Theorem 3.1.

From Lemma 3.1 it is enough to prove that the following operator $T : Z_b^\alpha \to Z_b^\alpha$ define by:

$$Tz(t) = \int_{-\infty}^{0} e^{-\mathcal{A}_D(t-s)} f^e(s,z(s))ds,$$

has a unique fixed point in B_ρ^b.

For $z \in B_\rho^b$ we get

$$\begin{aligned} \|Tz(t)\|_\alpha &\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} \left\{ L_\rho \|z(s)\|_\alpha + \|f^e(s,0)\| \right\} ds \\ &\leq \left\{ L_\rho \|z\|_b + \mu(\Omega) L_f \right\} \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} ds \\ &\leq \left\{ L_\rho \rho + \mu(\Omega) L_f \right\} \frac{\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} < \rho. \end{aligned}$$

Hence, $Tz \in B_\rho^b, \quad z \in B_\rho^b$.

Now, we prove that T is a contraction mapping. In fact, for $z_1, z_2 \in B^b_\rho$ we have that

$$\begin{aligned}\|Tz_1(t) - Tz_2(t)\|_\alpha &\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} \|f^e(s, z_1(s)) - f^e(s, z_2(s))\| ds \\ &\leq \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} L_\rho \|z_1(s) - z_2(s)\|_\alpha ds \\ &\leq \frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \|z_1 - z_2\|_b.\end{aligned}$$

So, from (3.1) we get that

$$\frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} < 1.$$

Then, T is a contraction mapping. Therefore, T has a unique fixed point z_b in B^b_ρ. i.e.,

$$z_b(t) = \int_{-\infty}^{0} e^{-\mathcal{A}_D(t-s)} f^e(s, z(s)) ds,$$

and from Lemma 3.1 $z_b(t)$ is solution of (2.16) for $t \geq 0$.

To prove that $z_b(t)$ is locally stable, we consider any othe solution $z(t)$ of (2.16) such that $\|z(t_0) - z(t_0)_b\|_\alpha < \frac{\rho}{2}$ with $t_0 \geq 0$. Then , $\|z(t_0)\|_\alpha < 2\rho$. As long as $\|z(t)\|_\alpha$ remainds less than 2ρ we get the following estimates:

$$\begin{aligned}\|z(t) - z(t)_b\|_\alpha &\leq e^{-\lambda_1 dt} \|z(t_0) - z(t_0)_b\|_\alpha \\ &+ \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-d\lambda_1(t-s)} L_\rho \|z(s) - z_b(s)\|_\alpha ds, \quad t \in [t_0, t_1] \\ &\leq \|z(t_0) - z(t_0)_b\|_\alpha + \frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \sup_{s\in[t_0,t_1]} \|z(s) - z_b(s)\|_\alpha.\end{aligned}$$

If $t_1 = \inf\{t > t_0 : \|z(t)\|_\alpha < 2\rho\}$, then either $t_1 = \infty$ or $\|z(t_1)\|_\alpha = 2\rho$. Suppose that $\|z(t_1)\|_\alpha = 2\rho$. Then from the above estimate we get that

$$\rho < \frac{\rho}{2} + 2\frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \rho = \left(\frac{1}{2} + 2\frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}\right) \rho.$$

From condition (3.1) we get that

$$\frac{1}{2} + 2\frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} < 1$$

which is a contradiction. Therefore,$t_1 = \infty$ and $z(t) \in B^b_{2\rho}$ for $t \geq t_0$.

Define

$$\|z - z_b\|_b^+ = \sup_{t \geq t_0} \|z(t) - z_b(t)\|_\alpha.$$

Then,

$$\|z - z_b\|_b^+ \leq \|z(t_0) - z_b(t_0)\|_\alpha + \frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \|z - z_b\|_b^+.$$

Hence,

$$\left(1 - \frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}\right) \|z - z_b\|_b^+ \leq \|z(t_0) - z_b(t_0)\|_\alpha.$$

Let us put $\Lambda = \frac{L_\rho \Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}$. Then

$$\|z - z_b\|_b^+ \leq \frac{1}{1-\Lambda} \|z(t_0) - z_b(t_0)\|_\alpha.$$

From here we get the stability of $z_b(t)$. □

Proof of Theorem 3.2.

Since $\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} - L > 0$, then there exist $\rho_1 > 0$ such that

$$\left(\frac{(\lambda_1 d)^{1-\alpha}}{\Gamma(1-\alpha)} - L\right) \rho_1 > \mu(\Omega) L_f.$$

Then, from Theorem 3.1 we get for each $\rho > \rho_1$ the existence of an unique bounded solution of the equation (2.16) in the ball B_ρ^b; therefore the system (2.16) has one and only one bounded solution $z_b(t)$.

To prove that $z_b(t)$ is stable, we consider any other solution of (2.16) and the following estimate

$$\|z(t) - z(t)_b\|_\alpha \leq \frac{1}{1-\Lambda} \|z(t_0) - z_b(t_0)\|_\alpha.$$

and $\Lambda = \frac{L\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}$. Since in this case Λ does not depent on the bounded function z_b and t_0, the stability is uniform. □

Corolary 3.1 *If f is periodic in t of period τ ($f(t+\tau,\xi) = f(t,\xi)$), then the unique bounded solution given by Theorems 3.1 and 3.2 is also periodic of period τ.*

Proof Let z_b be the unique solution of (2.16) in the ball B_ρ^b. Then, $z(t) = z_b(t+\tau)$ is also a solution of the equation (2.16) lying in the ball B_ρ^b, and by the uniqueness of the fixed point of the contraction mapping T in this ball, we conclude that $z_b(t) = z_b(t+\tau), \quad t \in I\!R$. □

Remark 3.1 *Under some condition the bounded solution given by Theorems 3.1 and 3.2 is almost periodic; for example we can study the case when the function f has the following form:*

$$f(t,\xi) = g(\xi) + P(t), \quad t \in \mathbb{R}, \xi \in \mathbb{R}^n, \tag{3.5}$$

where $P \in C_b(\mathbb{R}, \mathbb{R}^n)$, the space of continuous and bounded functions.

Corolary 3.2 *Suppose f has the form (3.5). Then the bounded solution $z_b(\cdot, P)$ given by Theorem 3.2 depends continuously on $P \in C_b(\mathbb{R}, \mathbb{R}^n)$.*

Proof Let $P_1, P_2 \in C_b(\mathbb{R}, \mathbb{R}^n)$ and $z_b(\cdot, P_1), z_b(\cdot, P_2)$ be the bounded functions given by Theorem 3.2. Then

$$\begin{aligned} z_b(t,\cdot,P_1) - z_b(t,\cdot,P_2) &= \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)}[g(z_b(s,P_2)) - g(z_b(s,P_2))]ds \\ &+ \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)}[P_1(s) - P_2(s)]ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b &\leq \frac{L\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b \\ &+ \frac{\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \|P_1 - P_2\|_b. \end{aligned}$$

Therefore,

$$\|z_b(\cdot,P_1) - z_b(\cdot,P_2)\|_b \leq \frac{\frac{\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}}{1 - \frac{L\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}}} \|P_1 - P_2\|_b.$$

□

We conclude this work with the following lemma about almost periodicity of the bounded solutions of the equation (2.16).

Lemma 3.2 *Suppose f is as (3.5). Then, if $P(t)$ is almost periodic, then the unique bounded solution of the system (2.16) given by Theorems 3.1 and 3.2 is also almost periodic.*

Proof To prove this lemma, we shall use the following well known fact, due to S. Bohr. A function $f \in C(\mathbb{R}; Z^\alpha)$ is almost periodic (a.p) if and only if the Hull $H(h)$ of h is compact in the topology of uniform convergence.

Where $H(h)$ is the closure of the set of translates of h under the topology of uniform convergence

$$H(h) = \overline{\{h_\tau : \tau \in I\!R\}}, \qquad h_\tau(t) = h(t+\tau), t \in I\!R.$$

Since the limit of a uniformly convergent sequence of a.p. functions is a.p., then the set A_ρ of a.p. functions in the ball B_ρ^b is closed, where ρ is given by Theorem 3.1 or 3.2.

Claim. The contraction mapping T given in Theorems 3.1 and 3.2 leaves A_ρ inva-
riant. In fact; if $z \in A_\rho$, then $h(t) = g(z(t)) + P(t)$ is also an a.p. function. Now, consider the function

$$\begin{aligned} F(t) = (Tz)(t) &= \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)} \{g(z(s)) + P(s)\}\, ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)} h(s) ds, \quad t \in I\!R. \end{aligned}$$

Then, it is enough to establish that $H(F)$ is compact in the topology of uniform convergence. Let $\{F_{\tau_k}\}$ be any sequence in $H(F)$. Since h is a.p. we can select from $\{h_{\tau_k}\}$ a Cauchy subsequence $\{h_{\tau_{k_j}}\}$, and we have that

$$\begin{aligned} F_{\tau_{k_j}}(t) = F(t+\tau_{k_j}) &= \int_{-\infty}^{t+\tau_{k_j}} e^{-\mathcal{A}_D(t+\tau_{k_j}-s)} h(s) ds \\ &= \int_{-\infty}^{t} e^{-\mathcal{A}_D(t-s)} h(s+\tau_{k_j}) ds. \end{aligned}$$

Hence,

$$\begin{aligned} \|F_{\tau_{k_j}}(t) - F_{\tau_{k_i}}(t)\|_\alpha &\le \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\lambda_1 d(t-s)} \|h(s+\tau_{k_j}) - h(s+\tau_{k_i})\|_\alpha ds \\ &\le \|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\lambda_1 d(t-s)} ds \\ &= \frac{\Gamma(1-\alpha)}{(\lambda_1 d)^{1-\alpha}} \|h_{\tau_{k_j}} - h_{\tau_{k_i}}\|_b. \end{aligned}$$

Therefore, $\{F_{\tau_{k_j}}\}$ is a Cauchy sequence. So, $H(F)$ is compact in the topology of uniform convergence, F is a.p. and $TA_\rho \subset A_\rho$.

Now, the unique fixed point of T in the ball B_ρ^b lies in A_ρ. Hence, the unique bounded solution $z_b(t)$ of the equation (2.16) given in Theorems 3.1 and 3.2 is also almost periodic. □

References

[1] LUIZ A. F. de OLIVEIRA "On Reaction-Diffusion Systems" E. Journal of Differential Equations, Vol. 1998(1998), N0. 24, pp. 1-10.

[2] J. LOPEZ G. and R. PARDO SAN GIL "Coexistence in a Simple Food Chain with Diffusion" J. Math. Biol. (1992) 30: 655-668.

[3] H. LEIVA "Stability of a Periodic Solution for a System of Parabolic Equations" J. Applicable Analysis, Vol. 60, pp. 277-300(1996).

[4] H. LEIVA, "Existence of Bounded Solutions of a Second Order System with Dissipation" J. Math. Analysis and Appl. **237**, 288-302(1999).

[5] L. GARCIA and H. LEIVA, "Center Manifold and Exponentially Bounded Solutions of a Forced Newtonian System with Dissipation" E. Journal Differential Equations. conf. 05, 2000, pp. 69-77.

[6] H. LEIVA, "Existence of Bounded Solutions of a Second Order Evolution Equation and Applications" Journal Math. Physis. Vol. 41, N0 11, 2000.

[7] D.Henry "Geometric theory of semilinear parabolic equations" , Springer, New York (1981).

[8] J.M. ALONSO, J. MAWHIN AND R. ORTEGA, "Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation", J.Math. Pures Appl., 78, 49-63 (1999).

[9] W. ARENDT AND C.J.K. BATTY, "Almost periodic solutions fo first and second-order cauchy problems", J.D. Equations **137**, 363-383 (1997).