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Notas de Matemática

Serie: Pre-Print

No. 208

Mérida - Venezuela

2000

Notas de Matemática, No. 208
Mérida, 2000.

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Abstract.

In this note we obtain representations of $H^\infty(\mathbb{D}^N)$ generated by a polynomially bounded N-tuple of commuting operators and present necessary and sufficient conditions for the existence of such representations.

1. Introduction

Let $\mathcal{H}$ be a separable infinite dimensional, complex Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$, and let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting operators on $\mathcal{H}$. We say that $T = (T_1, \dots, T_N)$ is *polynomially bounded* if there exists a constant $M \geq 1$ such that

$$\|p(T_1, \dots, T_N)\| \leq M \sup_{z \in \mathbb{D}^N} |p(z)|$$

for every polynomial $p \in \mathbb{C}[z_1, \dots, z_N]$. We write $PB^N(\mathcal{H})$ for the class of such N-tuples.

As usual (cf. [?]), we identify $\mathcal{L}(\mathcal{H})$ with the dual space of $\mathcal{C}_1(\mathcal{H})$, the Banach space of the trace class operators via the duality

$$\langle T, S \rangle = \operatorname{trace}(TS), \qquad T \in \mathcal{C}_1(\mathcal{H}),\ S \in \mathcal{L}(\mathcal{H}).$$

This identification provides $\mathcal{L}(\mathcal{H})$ with a w^*-topology. We use $\mathcal{A}_T$ to denote the smallest w^*-closed subalgebra containing $T_1, \dots, T_N$ and the identity operator I on $\mathcal{H}$. We identify $\mathcal{A}_T$ with the dual space of $\mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}_T = \mathcal{Q}_T$.

If X is a compact Hausdorff space, we denote by $C(X)$ the Banach algebra of all continuous, complex-valued function on X under the supremum norm, and denote by $M(X)$ the space of the complex regular Borel measures on X that we will identify with $C(X)^*$.

The open unit disc in the complex plane $\mathbb{C}$ is denoted by $\mathbb{D}$, its boundary is the unit circle $\mathbb{T}$, the unit polydisc $\mathbb{D}^N$ and the torus $\mathbb{T}^N$ are the subsets of $\mathbb{C}^N$ which are cartesian products of N copies of $\mathbb{D}$ and $\mathbb{T}$ respectively. Let $m_N = m$ be the normalized Lebesgue measure on $\mathbb{T}^N$. Let $L^p(\mathbb{T}^N)$ $(1 \leq p < \infty)$ be the spaces of Borel measurable functions f on $\mathbb{T}^N$ for which

$$\|f\|_p = \left(\int_{\mathbb{T}^N} |f|^p \, dm \right)^{1/p} < \infty$$

and let $L^\infty(\mathbb{T}^N)$ be the space of the Borel measurable functions f on $\mathbb{T}^N$ for which $\|f\|_\infty < \infty$, where $\|f\|_\infty$ denotes the essential supremum norm on $\mathbb{T}^N$ with respect to m.

The algebra $H^\infty(\mathbb{D}^N)$ of bounded analytic functions on $\mathbb{D}^N$ can be identified with a subspace of $L^\infty(\mathbb{T}^N)$ and then $H^\infty(\mathbb{D}^N)$ can be seen as the dual of the space $L^1(\mathbb{T}^N)/{}^\perp H^\infty(\mathbb{D}^N)$ where

$${}^\perp H^\infty(\mathbb{D}^N) = \left\{ g \in L^1(\mathbb{T}^N) : \int_{\mathbb{T}^N} gf \, dm = 0 \text{ for all } f \in H^\infty(\mathbb{D}^N) \right\}.$$

The polydisc algebra $A(\mathbb{D}^N)$ is the algebra of the continuous complex functions on $\overline{\mathbb{D}}^N$ whose restrictions to $\mathbb{D}^N$ are continuous there (cf. [?]).

We will be concerned with *representations* of $H^\infty(\mathbb{D}^N)$, this are algebra homomorphisms

$$\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$$

such that $\Phi(1) = I$ which are continuous when $H^\infty(\mathbb{D}^N)$ and $\mathcal{L}(\mathcal{H})$ are provided by their w^*-topologies. Note that such a representation is necessarily bounded. Indeed, the image under Φ of the unit ball in $H^\infty(\mathbb{D}^N)$ must be w^*-compact, and hence bounded.

We say that a representation Φ of $H^\infty(\mathbb{D}^N)$ is *generated* by the N-tuple T if $\Phi(\pi_i) = T_i$, $i = 1, \ldots, N$, where $\pi_i(z) = \pi_i(z_1, \ldots, z_n) = z_i$ for $z \in \mathbb{D}^N$.

The classical Sz. Nagy-Foias functional calculus is a contractive, w^*-continuous representation of $H^\infty(\mathbb{D})$ generated by an absolutely continuous contraction $T \in \mathcal{L}(\mathcal{H})$ (cf. [?] and [?]). In [?] it is proved that a pair of completely nonunitary commuting contractions generates a contractive, w^*-continuous representation of $H^\infty(\mathbb{D}^2)$. In [?] (see also [?] and [?]) contractive, w^*-continuous representations of $H^\infty(\mathbb{D}^N)$ generated by N-tuples $(T_1, \ldots, T_N)$ of commuting contractions are constructed provided that the N-tuples satisfy the von Neumann Inequality and that they are absolutely continuous.

In [?] a w^*-continuous representation generated by one absolutely continuous polynomially bounded operator is constructed, and recently (see [?]) w^*-continuous representations generated by absolutely continuous polynomially bounded pairs of commuting operators were introduced.

Previously, in 1980, C. Apostol (cf. [?]) had constructed a continuous, w^*-continuous representation of $H^\infty(\mathbb{D}^N)$ generated by an N-tuple $T \in PB^N(\mathcal{H})$ satisfying the following condition, called in [?] the *Apostol condition*: for any $j = 1, \ldots, N$ and any pair of vectors $x, y \in \mathcal{H}$

$$\lim_{n\to\infty} \sup \left\{ |\langle p(T_1, \ldots, T_N)T_j^n x, y\rangle| : p \in \mathbb{C}[z_1, \ldots, z_N],\ \sup_{z\in\mathbb{D}^N} |p(z)| \le 1 \right\} = 0.$$

In [?], A. Octavio and M. Kosiek proved that an N-tuple of commuting contractions, for which von Neumann's Inequality holds, satisfies the Apostol condition if and only if it is absolutely continuous.

In this paper we obtain representations generated for polynomially bounded N-tuples of commuting operators, using ideas of [?], [?] and [?]. In addition, following [?] and [?], we present necessary and sufficient conditions for the existence of such representations.

In Section 2 we present preliminary concepts and results. In Section 3 we present our results.

2. Preliminaries

We will consider Borel measures on $\mathbb{T}^N$. The terminology and notation are the same as in [?, Ch.9] and [?, Ch.5]. Let X be a compact Hausdorff space, A be a function algebra on X and ρ a multiplicative linear functional on A, then a *representing measure* for ρ is a probability measure μ on X such that

$$\rho(f) = \int f\, d\mu$$

for every $f \in A$. The set M_ρ of the representing measures for ρ is a convex and w^*-compact subset of $M(X)$. We will write $A^\perp$ for the space

$$A^\perp = \left\{ \nu \in M(X) : \int f\, d\nu = 0,\ f \in A \right\}.$$

In particular if $A = A(\mathbb{D}^N)$, we write M_z ($z \in \mathbb{D}^N$) for the set of measures in $M(\mathbb{T}^N)$ that are representing measures for the multiplicative functional $f \to f(z)$, $f \in A(\mathbb{D}^N)$.

We say that a measure $\mu \in M(\mathbb{T}^N)$ is *absolutely continuous* (respect to M_0) if $\mu \ll \rho$ for some measure $\rho \in M_0$. A measure $\mu \in M(\mathbb{T}^N)$ is *singular* (respect to M_0) if $\mu \perp \rho$ for every $\rho \in M_0$.

A sequence $\{f_n\}$ in $A(\mathbb{D}^N)$ is said to be a *Montel sequence* if it is uniformly bounded and $f_n(z) \to 0$ as $n \to \infty$, for every $z \in \mathbb{D}^N$. A measure $\mu \in M(\mathbb{T}^N)$ is a *Henkin measure* if

$$\lim_{n\to\infty} \int_{\mathbb{T}^N} f_n \, d\mu = 0$$

for every Montel sequence $\{f_n\}$.

The following result is a version of Valskii's decomposition adapted to the case of the unit polydisc. The proof on the ball case is given in detail in [?, Th. 9.2.1] and carries over word by word to the case of the polydisc (cf. [?]).

Theorem 2.1 (Valskii's Decomposition). *If μ is a Henkin measure then there exists $\nu \in A(\mathbb{T}^N)^{\perp}$ and $g \in L^1(\mathbb{T}^N)$ such that $\mu = \nu + gm$.*

We recall, following [?, V.17], that if X is a compact Hausdorff space, a *band of measures* is a norm closed subspace $\mathcal{B}$ of $M(X)$ such that if $\mu \in \mathcal{B}$ and ν is a measure on X that is absolutely continuous with respect to μ, then $\nu \in \mathcal{B}$. For any subset $\mathcal{S}$ of $M(X)$, the *band generated by $\mathcal{S}$* is the smallest band containing $\mathcal{S}$.

In particular (see [?, V.17.11]), if $\mathcal{S}$ is a closed convex set of measures and $\mathcal{B}$ is the band generated by $\mathcal{S}$, then $\nu \in \mathcal{B}$ if and only if there is a η in $\mathcal{S}$ such that $\nu \ll \eta$. So, in the case $\mathcal{S} = M_0$, we obtain that μ is absolutely continuous if and only if μ is in the band generated by M_0. We will denote this band by $\mathcal{B}_0$.

It is easy to see that $\mathcal{B}_0$ is just the band generated by the set of the representing measures of points of $\mathbb{D}^N$. So our terminology coincides with [?], [?], [?], and [?].

Additionally, we will need the following characterization of $\mathcal{B}_0$. For $N = 2$ it was proved in [?] and in the general case in [?] (see also [?, Section 7.1]).

Proposition 2.2. *Let $\mu \in M(\mathbb{T}^N)$, $\mu \in \mathcal{B}_0$ if and only if $f_n \to 0$ in the w^*-topology of $L^\infty(|\mu|)$ for every Montel sequence $\{f_n\}$ in $A(\mathbb{D}^N)$.*

We will request also the following well known results.

Theorem 2.3. *(cf. for example [?, V.17.4]) If $\mathcal{B}$ is a band of measures on a compact space X and $\nu \in M(X)$, then $\nu = \nu_a + \nu_s$, where $\nu_a \in \mathcal{B}$ and $\nu_s \perp \mu$ for every $\mu \in \mathcal{B}$. The measures ν_a and ν_s are unique.*

The above decomposition of ν is called the *Lebesgue decomposition* of ν respect to $\mathcal{B}$.

Recall that if A is an function algebra on the compact space X and $\mathcal{B}$ is a band of measures on X, then $\mathcal{B}$ is a *reducing band* (for A) if for every $\mu \in A^{\perp}$ with Lebesgue decomposition $\mu = \mu_a + \mu_s$, μ_a in $\mathcal{B}$ and $\mu_s \perp \eta$ for every $\eta \in \mathcal{B}$; it follows that μ_a and μ_s both belong to $A^{\perp}$.

With this notation, we state the Abstract F. and M. Riesz Theorem used in the following section.

Theorem 2.4. *(cf. for example [?, V.18.2]) If A is a function algebra and ρ is multiplicative linear functional on A, then the band generated by the representing measures for ρ is a reducing band.*

3. Representations generated by N-tuples

Let $T = (T_1, \ldots, T_N) \in PB^N(\mathcal{H})$. The map $p \to p(T)$, defined in a natural way for polynomials $p \in \mathbb{C}[z_1, \ldots, z_N]$, extends to a norm continuous algebra homomorphism

$$\Psi : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H}).$$

We will say that such a homomorphism Ψ is *generated* by T. We see, using the Hahn-Banach theorem, that for every pair $x, y \in \mathcal{H}$ there exists a measure $\mu(x, y) \in M(\mathbb{T}^N)$ such that

$$\langle \Psi(f)x, y\rangle = \int_{\mathbb{T}^N} f \, d\mu(x,y), \qquad f \in A(\mathbb{D}^N),\ x, y \in \mathcal{H}.$$

We can assume that $\|\mu(x,y)\| \leq \|\Psi\| \, \|x\| \|y\|$. Such family $\{\mu(x,y) : x, y \in \mathcal{H}\}$ will be called a family of *representing measures.*

If the norm continuous homomorphism $\Psi : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ generated by T has a family of representing measures $\{\mu(x,y) : x, y \in \mathcal{H}\}$ that are absolutely continuous we will say that T is *absolutely continuous*. If Ψ has a family of representing measures that are singular, then we will say that T is *singular*.

The following result is a version of a decomposition theorem of [?] and it is analogous to [?, Th. 3.4.] and [?, Th. 1.5.]. We give the proof, following [?, Th. 2.3] and the version in [?, Th. 1.5.], for the sake of completeness.

Theorem 3.1 (Mlak's Decomposition). *Let $T \in PB^N(\mathcal{H})$. There exist subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{H}$ in* Lat(T) *such that $T|\mathcal{M}$ is absolutely continuous, $T|\mathcal{N}$ is singular, and $T = T|\mathcal{M} \dotplus T|\mathcal{N}$.*

Proof. Let $\Psi : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the norm continuous homomorphism generated by T and let $\{\mu(x,y) : x, y \in \mathcal{H}\}$ be a family of representing measures for Ψ such that

$$\|\mu(x,y)\| \leq \|\Phi\| \, \|x\| \, \|y\|, \qquad x, y \in \mathcal{H}$$

and let

$$\mu(x,y) = \mu_a(x,y) + \mu_s(x,y)$$

be the Lebesgue decomposition of $\mu(x,y)$.

Let $\Psi_a, \Psi_s : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the maps defined by

$$\langle \Psi_a(f)x, y\rangle = \int_{\mathbb{T}^N} f \, d\mu_a(x,y), \qquad \langle \Psi_s(f)x, y\rangle = \int_{\mathbb{T}^N} f \, d\mu_s(x,y)$$

for every $f \in A(\mathbb{D}^N)$ y $x, y \in \mathcal{H}$.

For to see that Ψ_a and Ψ_s are linear, we observe that for every $x_1, x_2, y \in \mathcal{H}$,

$$\begin{aligned}\mu(x_1+x_2,y) - (\mu(x_1,y) + \mu(x_2,y)) &= [\mu_a(x_1+x_2,y) - (\mu_a(x_1,y) + \mu_a(x_2,y))] \\ &\quad + [\mu_s(x_1+x_2,y) - (\mu_s(x_1,y) + \mu_s(x_2,y))],\end{aligned}$$

is the Lebesgue decomposition of a measure in $A(\mathbb{D}^N)^{\perp}$, and then using the generalized abstract F. and M. Riesz theorem we see that

$$\mu_a(x_1+x_2,y) - (\mu_a(x_1,y) + \mu_a(x_2,y)) \in A(\mathbb{D}^N)^{\perp},$$

and

$$\mu_s(x_1+x_2,y) - (\mu_s(x_1,y) + \mu_s(x_2,y)) \in A(\mathbb{D}^N)^{\perp}.$$

Hence

$$\int_{\mathbb{T}^N} f \, d\mu_a(x_1+x_2,y) = \int_{\mathbb{T}^N} f \, d\mu_a(x_1,y) + \int_{\mathbb{T}^N} f \, d\mu_a(x_2,y),$$

and

$$\int_{\mathbb{T}^N} f \, d\mu_s(x_1+x_2,y) = \int_{\mathbb{T}^N} f \, d\mu_s(x_1,y) + \int_{\mathbb{T}^N} f \, d\mu_s(x_2,y),$$

for every $f \in A(\mathbb{D}^N)$. In a similar way we can conclude the proof that Ψ_a and Ψ_s are both linear.

Now we see that Ψ_a and Ψ_s are multiplicative.

Let $f, g \in A(\mathbb{D}^N)$ and $x, y \in \mathcal{H}$,

$$\int_{\mathbb{T}^N} fg \, d\mu(x,y) = \langle \Psi(f)x, \Psi(g)^* y\rangle = \int_{\mathbb{T}^N} f \, d\mu(x, \Psi(g)^* y),$$

then,

$$\begin{aligned}g\mu(x,y) - \mu(x, \Psi(g)^* y) &= [g\mu_a(x,y) - \mu_a(x, \Psi(g)^* y)] \\ &\quad + [g\mu_s(x,y) - \mu_s(x, \Psi(g)^* y)],\end{aligned}$$

is the Lebesgue decomposition of a measure in in $A(\mathbb{D}^N)$. Thus,

$$\int_{\mathbb{T}^N} fg\,d\mu_a(x,y) = \langle \Psi_a(f)x, \Psi(g)^*y\rangle = \int_{\mathbb{T}^N} g\,d\mu(\Psi_a(f)x,y),$$

for $f,g \in A(\mathbb{D}^N)$ and $x,y \in \mathcal{H}$. The same is true when we replace $\mu_a(x,y)$ and $\Psi_a(f)$ by $\mu_s(x,y)$ and $\Psi_s(f)$ respectively. So we see that for every $f \in H^\infty(\mathbb{D}^N)$ and every $x,y \in \mathcal{H}$,

$$f\mu_a(x,y) - \mu(\Psi_a(f)x,y) = [f\mu_a(x,y) - \mu_a(\Psi_a(f)x,y)] - \mu_s(\Psi_a(f)x,y),$$

and

$$f\mu_s(x,y) - \mu(\Psi_s(f)x,y) = -\mu_a(\Psi_s(f)x,y) + [f\mu_s(x,y) - \mu_s(\Psi_s(f)x,y)],$$

are the Lebesgue decomposition of measures in $A(\mathbb{D}^N)^\perp$. So we have

$$\langle \Psi_a(gf)x,y\rangle = \int_{\mathbb{T}^N} g\,d\mu_a(\Psi_a(f)x,y) = \langle \Psi_a(g)\Psi_a(f)x,y\rangle,$$

for all $f,g \in A(\mathbb{D}^N)$ and $x,y \in \mathcal{H}$. As before, the result holds if we replace Ψ_a and μ_a by Ψ_s and μ_s respectively, and, so we see that, Ψ_a and Ψ_s are multiplicative.

A similar argument show that

$$\langle \Psi_s(g)\Psi_a(f)x,y\rangle = \int_{\mathbb{T}^N} g\,d\mu_s(\Psi_a(f)x,y) = 0,$$

and

$$\langle \Psi_a(g)\Psi_s(f)x,y\rangle = \int_{\mathbb{T}^N} g\,d\mu_a(\Psi_s(f)x,y) = 0,$$

for every $f,g \in A(\mathbb{D}^N)$ and $x,y \in \mathcal{H}$.

Clearly $\|\Psi_a\| \le \|\Psi\|$ and $\|\Psi_s\| \le \|\Psi\|$, Ψ_a is absolutely continuous, Ψ_s is singular and $\Psi = \Psi_a + \Psi_s$. Let $\mathcal{M} = \Psi_a(1)\mathcal{H}$ and $\mathcal{N} = \Psi_s(1)\mathcal{H}$. We are going to see that $\mathcal{M}$ and $\mathcal{N}$ satisfy the conclusions of our theorem.

Certainly, $\Psi_a(1)$ y $\Psi_s(1)$ are projections so that $\mathcal{M}$ and $\mathcal{N}$ are closed subspaces. The identity

$$x = \Psi(1)x = \Psi_a(1)x + \Psi_s(1)x,$$

shows that $\mathcal{H} = \mathcal{M} + \mathcal{N}$. Moreover, if $y = \Psi_a(1)x_1$ and $y = \Psi_s(1)x_2$ for $x_1, x_2 \in \mathcal{H}$ then $\Psi_s(1)y = \Psi_a(1)y = 0$ and $y = \Psi(1)y = \Psi_a(1)y + \Psi_s(1)y = 0$; so that $\mathcal{H} = \mathcal{M} \dotplus \mathcal{N}$ as we wanted.

If $y \in \mathcal{M}$, $y = \Psi_a(1)x$ for one $x \in \mathcal{H}$, so that for every $j = 1,\dots,N$

$$\begin{aligned} T_j y = T_j\Psi_a(1)x &= \Psi(\chi_j)\Psi_a(1)x \\ &= \Psi_a(\chi_j)\Psi_a(1)x + \Psi_s(\chi_j)\Psi_a(1)x \\ &= \Psi_a(\chi_j)x = \Psi_a(1)\Psi_a(\chi_j)x \in \mathcal{M}, \end{aligned}$$

and, then, $T_j\mathcal{M} \subset \mathcal{M}$. Similarly $T_j\mathcal{N} \subset \mathcal{N}$ for every $j = 1,\dots,N$. □

We observe that the decomposition $\Psi = \Psi_a + \Psi_s$ is unique. Indeed, suppose that $\Psi = \hat{\Psi}_a + \hat{\Psi}_s$, with $\hat{\Psi}_a$ absolutely continuous and $\hat{\Psi}_s$ singular. Let $\{\nu_a(x,y) : x,y \in \mathcal{H}\}$ and $\{\nu_s(x,y) : x,y \in \mathcal{H}\}$ be the families of absolutely continuous representing measures and singular representing measures respectively. For every $x,y \in \mathcal{H}$ we see that

$$\mu(x,y) - (\nu_a(x,y) + \nu_s(x,y)) = (\mu_a(x,y) - \nu_a(x,y)) + (\mu_s(x,y) - \nu_s(x,y)),$$

is the Lebesgue decomposition of a measure in $A(\mathbb{D}^N)^\perp$ and so $\Psi_a = \hat{\Psi}_a$ and $\Psi_s = \hat{\Psi}_s$.

The precedent theorem leads to us to consider the class of polynomially bounded N-tuples of commuting operators what are absolutely continuous. We denote this class by $ACPB^N(\mathcal{H})$. We will see that the class $ACPB^N(\mathcal{H})$ is just the class what generates representations of $H^\infty(\mathbb{D}^N)$ w^*-continuous.

In [?] it is proved that an N-tuple of commuting contractions that satisfies the von Neumann Inequality is absolutely continuous if and only if it satisfies the Apostol condition. In the next result we extend this result to N-tuples in the class $ACPB^N(\mathcal{H})$ and give additional characterizations of these class of N-tuples.

Theorem 3.2. *Let $T \in PB^N(\mathcal{H})$ generating the norm continuous homomorphism $\Psi : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$. The following statements are equivalent:*

(1) *The N-tuple T generates a w^*-continuous representation of $H^\infty(\mathbb{D}^N)$.*
(2) *For every $\{f_n\}$ Montel sequence in $A(\mathbb{D}^N)$,*

$$\Psi(f_n) \xrightarrow{w^*} 0$$

(3) *The N-tuple T is absolutely continuous.*
(4) *The N-tuple T satisfies the Apostol condition.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) can be proved like in [?, Lemma 1.1]. Apostol's construction in [?] shows (4) $\Rightarrow$ (1). We will prove the equivalence (2) $\Leftrightarrow$ (3) and the implication (1) $\Rightarrow$ (4).
(2) $\Rightarrow$ (3) If (2) is true, Ψ has a family of Henkin representing measures. Let $\{\mu(x,y) : x, y \in \mathcal{H}\}$ be a such family of Henkin representing measures. We write $\mu(x,y) = \nu(x,y) + g_{x,y}m$ with $\nu(x,y) \in A(\mathbb{D}^N)$ and $g_{x,y} \in L^1(\mathbb{T}^N)$ given by the Valskii decomposition of $\mu(x,y)$. Since

$$\langle \Psi(f)x, y\rangle = \int_{\mathbb{T}^N} f\, d\mu(x,y) = \int_{\mathbb{T}^N} f g_{x,y}\, dm$$

for every $f \in A(\mathbb{D}^N)$, the family $\{g_{x,y}m : x, y \in \mathcal{H}\}$ is just a family of absolutely continuous representing measures for Ψ.
(3) $\Rightarrow$ (2) Let $\{\mu(x,y) : x, y \in \mathcal{H}\}$ be a family of absolutely continuous representing measures for Ψ and $\{f_n\}$ a Montel sequence, then

$$\langle \Psi(f_n)x, y\rangle = \int_{\mathbb{T}^N} f_n\, d\mu(x,y) \to 0, \qquad x, y \in \mathcal{H}$$

by the proposition 2.2.
(1) $\Rightarrow$ (4) If T generates a representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ w^*-continuous and the Apostol condition is false then for one $j \in \{1, \ldots, N\}$ and some $x, y \in \mathcal{H}$ there exist $\epsilon > 0$ and a sequence of polynomials $\{p_k\}$ with $\|p_k\|_\infty \leq 1$ and

$$|\langle p_k(T_1, \ldots, T_N)T_j^{n_k}x, y\rangle| > \epsilon$$

for $\{n_k\}$ an increasing sequence of positive integers.

If $h_k(z) = z_j^{n_k}$ for $z \in \mathbb{D}^N$, since $h_k \xrightarrow{w^*} 0$, then too $p_k h_k \xrightarrow{w^*} 0$ and so

$$|\langle p_k(T_1, \ldots, T_N)T_j^{n_k}x, y\rangle| \to 0$$

which is a contradiction.

$\square$

Remark: If T generates a representation of $H^\infty(\mathbb{D}^N)$ w^*-continuous and $\{\mu(x,y) : x, y \in \mathcal{H}\}$ is a family of representing measures with $\mu(x,y) = \mu_a(x,y) + \mu_s(x,y)$ its Lebesgue decomposition for every $x, y \in \mathcal{H}$, the family $\{\mu_a(x,y) : x, y \in \mathcal{H}\}$ is also a family of representing measures. Indeed, if we write $\mu(x,y) = \nu(x,y) + g_{x,y}m$ with $\nu(x,y) \in A(\mathbb{D}^N)$ and $g_{x,y} \in L^1(\mathbb{T}^N)$ given by the Valskii decomposition of $\mu(x,y)$ then, $\mu(x,y) - g_{x,y}m \in A(\mathbb{D}^N)^\perp$ and

$$\mu(x,y) - g_{x,y}m = (\mu_a(x,y) - g_{x,y}m) + \mu_s(x,y),$$

is its Lebesgue decomposition and so $\mu_a(x,y) - g_{x,y}m \in A(\mathbb{D}^N)^\perp$. Therefore $\int f\, d\mu_a(x,y) = \int f g_{x,y}\, dm$ for every $f \in A(\mathbb{D}^N)$.

So, if $T \in ACPB^N(\mathcal{H})$ generates the norm continuous homomorphism Ψ, we can suppose that Ψ has a family of absolutely continuous representing measure $\{\mu(x,y) : x, y \in \mathcal{H}\}$ such that

$$\|\mu(x,y)\| \leq \|\Psi\| \, \|x\| \|y\|, \qquad x, y \in \mathcal{H}.$$

Finally, we summarize the properties of the representations generated by N-tuples in $ACPB^N(\mathcal{H})$. Our result generalizes the corresponding result for pairs of [?]. The proof is immediate from the Theorem 3.2 and well know results on dual algebras (cf. for example [?]).

Main Theorem 3.3. *Let $T \in ACPB^N(\mathcal{H})$. There exist a unique homomorphism of algebras*

$$\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{A}_T$$

with the following properties:

1. *For every $j = 1, \ldots, N$, $\Phi(\pi_j) = T_j$;*
2. *The homomorphism $\Phi | A(\mathbb{D}^N) = \Psi$, the norm continuous algebra homomorphism generated by T, and $\|\Psi\| = \|\Phi\|$;*
3. *The homomorphism Φ is w^*-continuous;*
4. *The range of Φ is w^*-dense in $\mathcal{A}_T$;*
5. *There exist a bounded, linear, one-one map*
$$\Phi_* : \mathcal{Q}_T \to L^1(\mathbb{T}^N)/{}^\perp H^\infty(\mathbb{D}^N)$$
such that $(\Phi_*)^* = \Phi$;
6. *If Φ is bounded below, then the range of Φ is $\mathcal{A}_T$ and Φ is an invertible algebra isomorphism of $H^\infty(\mathbb{D}^N)$ onto $\mathcal{A}_T$; in this case Φ_* is invertible linear isomorphism of the space $\mathcal{Q}_T$ onto $L^1(\mathbb{T}^N)/{}^\perp H^\infty(\mathbb{D}^N)$.*

Acknowledgement: This paper is part of the author's Ph.D. thesis, written at Instituto Venezolano de Investigaciones Científicas under the direction of Professor Alfredo Octavio.

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