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On the dynamics of a nerve equation with delay

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1. Introduction

Stein, Leung, Mangeron and Oğuztöreli 4 proposed in 1974 the following integro-differential equation to describe the essential features of general types of neurons:

$$(1.1)\quad \frac{1}{a}\dot{x}(t) + x(t) = \frac{1}{1 + \exp\left[-f(t) - b\int_0^t x(t-u)\left[e^{-pu} - e^{-qu}\right]du\right]} \ ,$$

where $\dot{}$ is the derivative with respect to t, the constants $a > 0$, $b \in \mathbb{R}$, $0 < p < q$ are specific of certain groups of neurons. The function $x(t)$, $0 \leq x(t) \leq 1$, represents the normalized impulse frequency of the axon as a reaction to the input $f(t)$ of the nerve cell. The assumption $b < 0$ means that the integral term in Equation (1.1) reflects inhibition of new generation of impulses, due to exhaustion of the cell's resources by previous firing.

According to an der Heiden 4, introducing the following quantities:

$$\begin{aligned} y(t) &= \int_0^t x(t-u)e^{-pu}du \ , \\ z(t) &= \frac{1}{q-p}\int_0^t x(t-u)[e^{-pu} - e^{-qu}]du \ , \end{aligned}$$

and assuming that $f(t)$ is a constant function, the equation (1.1) becomes equivalent to the third order nonlinear autonomous system

$$(1.2)\quad \begin{aligned} \dot{x} &= \frac{a}{1 + \exp\{-f - b(q-p)z\}} - ax \ , \\ \dot{y} &= x - py \ , \\ \dot{z} &= y - qz \ , \end{aligned}$$

where the initial conditions must satisfy

$$0 \leq x_0 = x(0) \leq 1 \ , \quad y_0 = y(0) = 0 \ , \quad z_0 = z(0) = 0 \ .$$

These initial conditions reflect that equation (1.1) takes into account only the history of the cell back to the time $t = 0$. To avoid this inhomogeneity of the time an der Heiden 4 assume $x(t)$ to be determined by the impulse frequency in the time between $t - r$ and t. Then equation (1.1) has to be modified to

$$(1.3)\quad \frac{1}{a}\dot{x}(t) + x(t) = \frac{1}{1 + \exp\left[-f - b\int_0^r x(t-u)\left[e^{-pu} - e^{-qu}\right]du\right]} \ .$$

With the quantities:

$$\begin{aligned} y(t) &= \int_0^r x(t-u)e^{-pu}du \,, \\ z(t) &= \frac{1}{q-p}\int_0^r x(t-u)[e^{-pu}-e^{-qu}]du \,, \end{aligned}$$

we obtain from (1.3) the time lag differential system

$$\begin{aligned} \dot{x}(t) &= \frac{a}{1+\exp\{-f-b(q-p)z\}} - ax \,, \\ \dot{y}(t) &= x(t) - py(t) - x(t-r)e^{-pr} \,, \\ \dot{z}(t) &= y(t) - qz(t) + \frac{1}{q-p}x(t-r)(e^{-qr}-e^{-pr}) \,, \end{aligned} \tag{1.4}$$

where the initial conditions must satisfy

$$0 \le x_0 = x(0) \le 1 \,, \quad 0 \le y_0 \le \int_0^r e^{-pr}du \,,$$

$$0 \le z_0 \le \frac{1}{q-p}\int_0^r (e^{-pu}-e^{-qu})du \,.$$

Then by making $r \to \infty$ system (1.4) converges to system (1.2). In this paper we restrict our attention to Equation (1.3) and think preferable to treat the model as a retarded functional differential equation with finite delay, with no restriction in the delay's size. Moreover, we will neglect the term e^{-qu} by making $q \to \infty$. So, the main subject of this paper is the integro-differential equation

$$\frac{1}{a}\dot{x}(t) + x(t) = \frac{1}{1+\exp\left[-f-b\int_0^r x(t-u)e^{-pu}du\right]} \,. \tag{1.5}$$

Given $\sigma \in \mathbb{R}$, $A > 0$ and any function $x : [\sigma - r, \sigma + A) \to \mathbb{R}^n$, we adopt the Hale's notation (see 4, for instance): for every $t \in [\sigma, \sigma + A)$, the function $x_t : [-r, 0] \to \mathbb{R}^n$ is defined by $x_t(\theta) := x(t+\theta)$, for any $\theta \in [-r, 0]$. In this setting Equation (1.5) can be written in the form:

$$\dot{x}(t) = -ax(t) + aM\left(\int_{-r}^0 e^{p\theta}x_t(\theta)\,d\theta\right), \tag{1.6}$$

where the function $M : \mathbb{R} \to \mathbb{R}$ is defined by $M(s) = \left(1 + e^{-f}e^{-bs}\right)^{-1}$, for all $s \in \mathbb{R}$.

Given $\phi \in C := C([-r, 0], \mathbb{R})$, we denote by $x(\cdot;\phi)$ the solution of Equation (1.6) defined for $-r \le t < A$, $A > 0$, with $x_0(\cdot;\phi) = \phi$. We tacitly assume the general theory of functional differential equations and most of the notations set in 4.

In this paper we show that for any $a > 0$, equation (1.6) admits a global attractor A_a^* . We prove that (1.6) has a unique equilibrium x^* which is asymptotically stable for a significant set of the parameters a, b, p and f. If some estimates are imposed on b, p and f, we show that there are some critical values of a where x^* looses the stability creating room for raising periodic solutions of Equation (1.6) through a Hopf bifurcation. Finally, we prove that when the unique equilibrium is local asymptotically stable and $b < 0$, the equilibrium is global asymptotically stable.

2. Asymptotic properties

In some instances is convenient to think of Equation (1.6) in its shorthand form:

$$\dot{x}(t) = F(x_t), \tag{2.1}$$

with the functional $\phi \in C \mapsto F(\phi) \in \mathbb{R}$ being defined by $F(\phi) := -a\phi(0) + aM\left(\int_{-r}^{0} e^{p\theta}\phi(\theta)\, d\theta\right)$, $\phi \in C$. Since the image $F(B_\delta)$ of any closed ball $B_\delta = \{\phi \in C \,|\, ||\phi|| \leq \delta\}$ is contained in the bounded interval $[-a\delta, a\delta + a]$, it follows that F is completely continuous. This implies the solutions $x(t) = x(t;\phi)$ of Equation (1.1) such that $x_0(\cdot;\phi) = \phi \in C$ exist for $-r \leq t < \infty$. In fact, for any t such that $x(t) \geq 1$, Equation (1.6) gives $\dot{x}(t) < 0$ and, similarly, if $x(t) \leq 0$ we have $\dot{x}(t) > 0$. The existence of the solution $x(t)$ in the large is now a straightforward consequence of Theorem 3.2, in Chapter 2, of 4.

Given $a > 0$, we say that a function is *a-lipschitzian* if it is uniformly Lipschitz-continuous with a Lipschitz constant a.
The following result holds

Lemma 2.1. *For any $a > 0$, the equation* (1.6) *has a global attractor $A_a^* \subset \Omega_a$, where*

$$\Omega_a := \{\phi \in C \,|\, \phi([-r,0]) \subset [0,1],\ \ \phi \textit{ is a-lipschitzian}\} \quad .$$

Proof. A direct application of Arzela-Ascoli's Theorem gives us that Ω_a is compact. Now, let us show that Ω_a is positively invariant. Indeed, let us pick $\phi \in \Omega_a$ and $\tau \geq 0$ such that either $x(\tau;\phi) = 1$ or $x(\tau;\phi) = 0$. Then, using Equation (1.6), we obtain that $x(\tau;\phi) = 1$ implies $\dot{x}(\tau;\phi) < 0$ and $x(\tau;\phi) = 0$ implies $\dot{x}(\tau;\phi) > 0$. Therefore, $0 \leq x(t;\phi) \leq 1$ for $0 \leq t \leq \infty$. Moreover, from (1.6) , it follows that $\dot{x}(t;\phi) \leq a$ for every $t \geq 0$, which implies that $x_t(\cdot;\phi)$ is a-lipschitzian.

Now having in mind that the function $x(t;\phi)$ is decreasing (increasing) with respect to t as long as $x(t;\phi)$ remains greater than 1 (less than 0), we certainly obtain that (1.6) is pointwise dissipative and the absorbing set is given by Ω_a .

Let us define $T_a(t) := x_t(\phi)$. Taking into account that F is a completely continuous it follows that $T_a(t)$ is completely continuous as well for any $t \geq r$. Finally, Theorem 3.4.8 p. 40 in 4 complete the proof of our claim. □

There exists a unique equilibrium x^* of Equation (1.6). Of course, the constant function $\phi^* \in C$, $\phi^*(\theta) \equiv x^*$, must belong to Ω_a. This can be seen by taking into account the properties of the function M and imposing that the function $\phi^*(\theta) \equiv x^*$, for $\theta \in [-r, 0]$, is a zero of the functional F defined by Equation (2.1). This gives

$$-x^* + M\left(x^* \int_{-r}^{0} e^{p\theta}\, d\theta\right) = 0, \tag{2.2}$$

that is,

$$x^* = M\left(\zeta x^*\right), \tag{2.3}$$

where $\zeta(p) := \left(1 - e^{-pr}\right)/p$. In the investigation on the effect of the parameters on the dynamics of Equation (1.6), the property

$$\lim_{p \to 0} \zeta(p) = r \tag{2.4}$$

will play an important role.

Remark 2.1. It will be useful later on that we can take the parameter b with $|b|$ arbitrarily large and the parameter p varying in any fixed interval $(0, p_0)$, keeping the equilibrium $x^* \in (0,1)$ uniformly away from 0 and 1, just controlling the input f. That is, we can do that independently of the parameter a and the delay r. This is easily seen noting that $\lim_{p\to 0} \zeta(p) = r$ and, according to the definition of x^*, it should satisfy

$$e^{-f} e^{-b\zeta(p)x^*} = \frac{1}{x^*} - 1.$$

Recalling the definition of the function M, one can see that the linear part of Equation (1.6) near the equilibrium x^* is given by

$$\dot{x}(t) = -ax(t) + a\xi \int_{-r}^{0} e^{p\theta} x_t(\theta)\, d\theta. \tag{2.5}$$

where

$$\xi := bx^*(1 - x^*). \tag{2.6}$$

We point out that x^* and therefore ξ are independent of a . This fact will be useful later.

The behavior of solutions of Equation (1.6) near x^* is determined by the characteristic equation of Equation (2.5):

$$\lambda = -a + a\xi \int_{-r}^{0} e^{(p+\lambda)\theta} d\theta. \tag{2.7}$$

If $\lambda = -p$ is a root of Equation (2.7) (note that this might occur if $p = a(1 - \xi r)$), it only contributes to provide an eigenfunction in the stable eigenspace, since it is a negative root. So, the asymptotic behavior of solutions of Equation (1.6) near the equilibrium x^* is determined by the location of the roots $\lambda \neq -p$ with respect to the imaginary axis. This remark leads to the study of the following equation equivalent to Equation (2.7) for $\lambda \neq -p$:

$$\left(\lambda^2 + (a+p)\lambda + a(p - \xi)\right) e^{\lambda r} + a\xi e^{-pr} = 0. \tag{2.8}$$

Let $z := \lambda r$ and define P, Q, R by

$$P = r(a+p) > 0, \quad Q = r^2 a(p-\xi), \quad R = r^2 a\xi e^{-pr}. \tag{2.9}$$

Substituting these new parameters and variable in Equation (2.8) one obtains

$$H(z) := \left(z^2 + Pz + Q\right) e^z + R = 0. \tag{2.10}$$

The next two results state all properties of the roots of Equation (2.16) that we will need along this work. Theorem 2.1 gives more information than Theorem 2.2 of Baptistini and Táboas in 4, but the proof is strongly inspired in that Theorem .

Lemma 2.2. i) *If $b < 0$, then all roots z of Equation* (2.10) *with $\Re(z) \geq 0$ are simple.*

Assume that $b > 0$.

ii) *All roots z of Equation* (2.10) *with $\Re(z) \geq 0$ are simple if and only if $R \leq P$.*

iii) *All complex roots z of Equation* (2.10) *with $\Re(z) \geq 0$ are simple.*

Proof. Suppose temporarily that there exists a number $z \in \mathbb{C}$ such that $H(z) = H'(z) = 0$, $z = \alpha + i\beta$, with $\alpha \geq 0$. Since $\overline{H(z)} = H(\overline{z})$, it suffices to analyze the upper semi-plane. So, let us suppose also $\beta \geq 0$.

These assumptions lead to the system

$$\begin{cases} \left(z^2 + Pz + Q\right) e^z + R = 0 \\ z^2 + (P+2)z + P + Q = 0. \end{cases} \tag{2.11}$$

The second equation of (2.11) is equivalent to

$$\begin{cases} \alpha^2 - \beta^2 + (P+2)\alpha + P + Q = 0 \\ 2\alpha\beta + (P+2)\beta = 0. \end{cases} \tag{2.12}$$

Let us suppose that $b < 0$. If $\beta \neq 0$, the second equation of (2.12) implies $2\alpha = -(P+2) < 0$, contradiction. Assuming that $\beta = 0$, the first equation of (2.12) gives $\alpha^2 + (P+2)\alpha + P + Q = 0$. Since in this case $Q > 0$, it follows that $\alpha < 0$. Contradiction.

Parts $ii) - iii)$ follow from the fact that (2.11) is equivalent to the equation $2\alpha + P = Re^{-\alpha}$. □

The following result not only gives us a necessary and sufficient condition for the roots having negative real parts. Indeed, it provides a complete description of the location of the roots of the characteristic equation (2.10)

Theorem 2.1. *Let $P, Q > 0$, $R < 0$ and define the 2-vectors $v(\beta) := (P\beta, Q - \beta^2)$, $w(\beta) := (-\sin\beta, \cos\beta)$, for $\beta \geq 0$. Consider the sequence $0 = \beta_0 < \beta_1 < \ldots \to \infty$ of all numbers such that $v(\beta_k)$ is a positive multiple of $w(\beta_k)$, $k = 1, 2, \ldots$. Then, the number of roots of Equation* (2.10) *with positive real parts is twice the number of β_k's such that $|v(\beta_k)| < -R$. To* *each such β_k there corresponds a pair of conjugate roots, z_k and $\bar{z}_k$, with $\Re(z_k) > 0$ and $2(k-1)\pi < |\Im(z_k)| < 2k\pi$.*

Proof. Since $\overline{H(z)} = H(\bar{z})$, where the bar indicates complex conjugation, it suffices to consider the solutions z of Equation (2.10) with $\Im(z) \geq 0$. Thus, denoting $z = \alpha + i\beta$, with $\beta \geq 0$, and splitting $H(z)$ in its real and imaginary parts, Equation (2.10) becomes equivalent to the system:

$$\begin{cases} \left[(\alpha^2 + P\alpha + Q - \beta^2)\cos\beta - (2\alpha + P)\beta \sin\beta\right] e^{\alpha} + R = 0 \\ (\alpha^2 + P\alpha + Q - \beta^2)\sin\beta + (2\alpha + P)\beta\cos\beta = 0 \end{cases} \tag{2.13}$$

that has the vector form:

$$\begin{cases} e^{\alpha}\left[\alpha(2\beta, \alpha + P) + v(\beta)\right] \cdot (-\sin\beta, \cos\beta) = -R \\ \left[\alpha(2\beta, \alpha + P) + v(\beta)\right] \cdot (\cos\beta, \sin\beta) = 0. \end{cases} \tag{2.14}$$

Considering $u(\alpha, \beta) := \alpha(2\beta, \alpha + P)$, noticing that $(\cos\beta, \sin\beta)$ and $(-\sin\beta, \cos\beta)$ are orthogonal and that the second equation in (2.14) is an orthogonality condition, one sees that System (2.14) is equivalent to the single vector equation:

$$\Lambda(\alpha, \beta) := e^{\alpha}\left[u(\alpha, \beta) + v(\beta)\right] = -Rw(\beta). \tag{2.15}$$

If $\alpha \geq 0$ is fixed and β increases from 0 to $+\infty$, the pair $(\xi, \eta) = \Lambda(\alpha, \beta)$ describes clockwise an unbounded arc of parabola, starting at the vertex $(0, \alpha^2 + P\alpha + Q)$ while $(\xi, \eta) = w(\beta)$ starts at $(0, 1)$ to turn counterclockwise an infinite number of laps around the unit circle, with center at the origin. Therefore, for each $\alpha \geq 0$ and $k = 1, 2, \ldots$, there exists precisely one $\beta_k \in ((2k-1)\pi, 2k\pi)$ such that the 2-vectors $\Lambda(\alpha, \beta_k)$ and $w(\beta_k)$ are aligned and have the same orientation. Moreover, for each k, $k = 1, 2, \ldots$, the number $\beta_k = \beta_k(\alpha)$ depends continuously on α. So, there exists a root z of Equation (2.10) with $\Re(z) = \alpha$ if, and only if, for some $k \in \mathbb{Z}_{++}$, $|\Lambda(\alpha, \beta_k)| = -R$.

Notice that, for each positive integer k, $v(\beta_k(0)) = \Lambda(0, \beta_k(0))$, $\Lambda(\alpha, \beta_k(\alpha))$ depends continuously on α and $\lim_{\alpha\to\infty} |\Lambda(\alpha, \beta_k(\alpha))| = \infty$. Therefore, to distinct β_k's can be associated distinct positive α's such that $|\Lambda(\alpha, \beta_k(\alpha))| = -R$. It remains only to prove that there is at most one positive number α with this property.

Notice that for any $\alpha > 0$ we have

$$\begin{aligned} \frac{\partial}{\partial\alpha}|\Lambda(\alpha, \beta)|^2 &= 2e^{2\alpha}\left[(2\alpha + P)^2\beta^2 + (\alpha^2 + P\alpha + Q - \beta^2)^2\right] \\ &\quad + e^{2\alpha}\left[4(2\alpha + P)\beta^2 + 2(2\alpha + P)(\alpha^2 + P\alpha + Q - \beta^2)\right] \\ &= 2\left[|\Lambda(\alpha, \beta)|^2 + e^{2\alpha}\left[(2\alpha + P)\beta^2 \right.\right. \\ &\qquad \left.\left. + (2\alpha + P)(\alpha^2 + P\alpha + Q)\right]\right] > 0. \end{aligned} \tag{2.16}$$

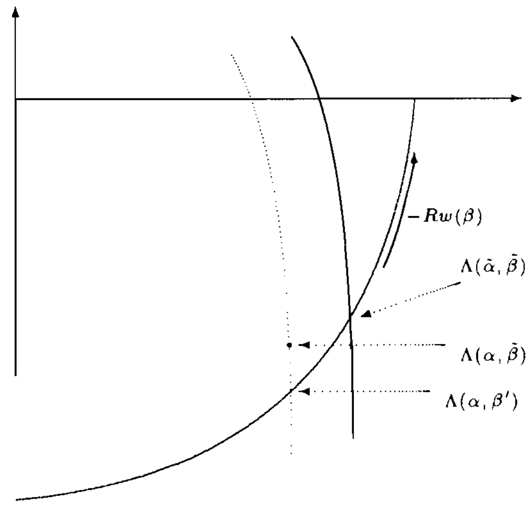

FIGURE 1

The point $\Lambda(\tilde{\alpha},\tilde{\beta})$ is one of the intersections of the branch of parabola $(\xi,\eta)=\Lambda(\tilde{\alpha},\beta), \beta\geq 0$, with the circle $|(\xi,\eta)|=-R$, that is, $|\Lambda(\tilde{\alpha},\tilde{\beta})|=-R$. We are going to consider only the case where $|\Lambda(\tilde{\alpha},\beta)|>-R$, for $\beta>\tilde{\beta}$. The proof in the case where $|\Lambda(\tilde{\alpha},\beta)|>-R$, for $\beta<\tilde{\beta}$, is carried out similarly.

Let $\tilde{\alpha}-\epsilon<\alpha<\tilde{\alpha}$ for a small $\epsilon>0$ and consider $\beta_k(\alpha)$ defined as above. According to (2.16) we have $|\Lambda(\alpha,\tilde{\beta})|<-R$. Since $w(\beta)$ varies counterclockwise and $\Lambda(\alpha,\beta)$ varies clockwise as β increases, we can assure that $|\Lambda(\alpha,\beta_k(\alpha))|<-R$. In fact, if β' is close to $\tilde{\beta}$ and $|\Lambda(\alpha,\beta')|=-R$, it follows that $\beta'>\tilde{\beta}$. As Figure 2 shows, according to the mentioned orientation of $w(\cdot)$ and $\Lambda(\alpha,\cdot)$, the point $\beta=\beta_k(\alpha)$ (where the alignment of $\Lambda(\alpha,\beta)$ with $w(\beta)$ occurs) satisfies $\beta_k(\alpha)<\beta'$. Therefore, $\Lambda(\alpha,\beta_k(\alpha))<-R$.

Assume now $\tilde{\alpha}+\epsilon>\alpha>\tilde{\alpha}$ for a small $\epsilon>0$. By (2.16) we have $|\Lambda(\alpha,\tilde{\beta})|>-R$. Taking in account once more the directions in which $w(\beta)$ and $\Lambda(\alpha,\beta)$ vary with β increasing, a figure analogous to Figure 2 can be drawn, now with $\Lambda(\alpha,\tilde{\beta})$ outside the circle of radius $-R$, and very similar arguments lead to $|\Lambda(\alpha,\beta_k(\alpha))|>-R$.

If $\tilde{\alpha}>0$ satisfies $|\Lambda(\tilde{\alpha},\beta_k(\tilde{\alpha}))|=-R$, for some positive integer k, we have proved that in some neighborhood of $\tilde{\alpha}$, $\Lambda(\alpha,\beta_k(\alpha))$ is inside the circle of radius $-R$ for $\alpha<\tilde{\alpha}$ and it is outside for $\alpha>\tilde{\alpha}$. It follows now by the continuity of $\Lambda(\alpha,\beta_k(\alpha))$ that, for each fixed positive integer k, there is at most one $\tilde{\alpha}>0$ such that $|\Lambda(\tilde{\alpha},\beta_k(\tilde{\alpha}))|=-R$. □

It is an obvious consequence of the proof of Lemma 2.1 the following

Corollary 2.1. *To each term β_k (at most two) of the sequence $\beta_0 < \beta_1 < \ldots$, defined above, such that $|v(\beta_k)| = -R$, there corresponds a pair of imaginary roots $\pm i\beta_k$ of Equation* (2.10).

Remark 2.2. A geometric interpretation of Lemma 2.1 is the following: If β varies from 0 to $+\infty$, the point $v(\beta)$ describes clockwise the arc of parabola in the plane $\xi\eta$:

$$\begin{cases} \xi = P\beta \\ \eta = Q - \beta^2, \quad \beta \geq 0, \end{cases}$$

starting at the vertex $(0, Q)$. The point $w(\beta)$ describes, counterclockwise, infinitely many successive laps over the unit circle. Both, $v(\beta)$ and $w(\beta)$ are in the semi-plane $\xi > 0$ if and only if β belongs to the range $(\pi, 2\pi) \cup (3\pi, 4\pi) \cup \ldots \cup ((2k-1)\pi, 2k\pi) \cup \ldots$. At the k-th lap of $w(\beta)$ there exists precisely one value β_k in $(2k-1)\pi, 2k\pi)$, $k = 1, 2, \ldots$, for which $v(\beta_k)$ and $w(\beta_k)$ are aligned. Lemma 2.1 states that all roots of Equation (2.8) have negative real part if, and only if, $|v(\beta_k)| > R$, $k = 1, 2, \ldots$.

Remark 2.3. Lemma 2.1 provides a method to compute the dimension of the unstable manifold of the equilibrium x^*.

We are now in a position to state a sufficient condition for the local stability of the equilibrium x^*.

Theorem 2.2. *If*

$$0 < -b \leq 4\sqrt{p^2 + (\pi/r)^2}, \tag{2.17}$$

then the equilibrium x^* of Equation (1.6) *is asymptotically stable, for every* $a > 0$.

Proof. Keeping the notations of Lemma 2.1, it will be shown that $|v(\beta)| > -R$, for $\beta \in (\pi, 2\pi) \cup (3\pi, 4\pi) \cup \ldots$.

We claim that $|v(\pi)|^2 = \pi^4 + (P^2 - 2Q)\pi^2 + Q^2 > R^2$.

In fact, since $Q^2 - R^2 > 0$, it suffices to show that $\pi^2 + (P^2 - 2Q) \geq 0$. Definitions (2.9) imply that the number $P^2 - 2Q = r^2[a^2 + 2\xi a + p^2]$, as a function of $a > 0$, takes its minimum at $a = -\xi$. Therefore, recalling that $0 < x^*(1 - x^*) < 1/4$, one sees that Hypothesis (2.17) implies

$$\begin{aligned} P^2 - 2Q \geq r^2(p^2 - \xi^2) = \\ r^2\left[p^2 - b^2\left(x^*(1 - x^*)\right)^2\right] \geq r^2[p^2 - (b^2/16)] \geq -\pi^2. \end{aligned} \tag{2.18}$$

Then, $\pi^2 + (P^2 - 2Q) \geq 0$.

Since

$$\begin{aligned} \frac{\left(|v(\beta)|^2\right)'}{2} = 2\beta^3 + (P^2 - 2Q)\beta = \\ \beta\left(2\beta^2 + (P^2 - 2Q)\right) \geq \beta\left(\pi^2 + (P^2 - 2Q)\right) \geq 0, \end{aligned} \tag{2.19}$$

for $\beta \geq \pi$, it follows that

$$\beta \geq \pi \Rightarrow |v(\beta)| > -R. \tag{2.20}$$

Recall that $v(\beta)$ and $w(\beta)$ can be aligned with the same orientation only if either $\beta = 0$ or $\beta \in (\pi, 2\pi) \cup (3\pi, 4\pi) \cup \ldots$. Since $v(0) = Q > -R$, Theorem 2.2 is now a straightforward consequence of the condition (2.20) combined with Lemma 2.1. □

Corollary 2.2. *Suppose $0 < -b < 4p$. Then there exists $r_0 > 0$ such that, for any $r \geq r_0$, the equilibrium x^* of Equation (1.6) is local asymptotically stable.*

Proof. This is evident because the inequality $0 < -b < 4p$ allows to choose $r_0 > 0$ in such a way that $r \geq r_0$ implies condition (2.17). □

Define $\tilde{x} \in (0,1)$ as the unique solution of the equation $x = M(rx)$ and the number $\tilde{\xi} = \tilde{\xi}(b) < 0$ by $\tilde{\xi}(b) := b\tilde{x}(1-\tilde{x})$. According to the definition of M, it is easily seen that $\tilde{x}$ may be kept uniformly away from 0 and 1 for b arbitrarily large, just controlling the input f. This implies that we can make $|\tilde{\xi}(b)|$ and $|b|$ arbitrarily large, controlling $\tilde{x} \in (0,1)$ by convenient choices of the input f. Note that such a procedure might be carried on independently of the remaining parameters.

Theorem 2.3. *If $-b$ is sufficiently large, there are possible choices of the parameter $p > 0$, restricted to a small interval $(0, p_0)$, and the input f, such that, for sufficiently large a, the equilibrium x^* of Equation (1.6) is unstable.*

Proof. The condition $|v(\beta)| < -R$ is equivalent to

$$F(\beta) := \beta^4 + (P^2 - 2Q)\beta^2 + Q^2 - R^2 < 0. \tag{2.21}$$

The definitions given in (2.9) imply $Q^2 - R^2 > 0$ and, therefore, a necessary condition for the existence of $\beta > 0$ satisfying the inequality (2.21) is that

$$P^2 - 2Q < 0 \tag{2.22}$$

holds. By using (2.9) again one sees that (2.22) is equivalent to the following relation between the original parameters:

$$a^2 + p^2 < -2a\xi. \tag{2.23}$$

According to the remarks made before the statement of Theorem 2.3, we can take $|\tilde{\xi}|$ so large that the inequality $a^2 < -2a\tilde{\xi}$ holds for arbitrarily large a. Since $\xi \to \tilde{\xi}$, as $p \to 0$, (2.23) is true by choosing $p > 0$ sufficiently small.

The discriminant $\Delta := (P^2 - 2Q)^2 - 4(Q^2 - R^2)$ of $F(\beta)$ in terms of the original parameters is given by

$$\Delta = r^4\big[(a+p)^4 - 4a(a+p)^2(p-\xi) + 4a^2\xi^2e^{-2pr}\big]. \tag{2.24}$$

Letting $p \to 0$, we have

$$\Delta \to r^4a^2(a + 2\tilde{\xi})^2. \tag{2.25}$$

Consider $\gamma := \beta^2$. According to Lemma 2.1, Equation (2.10) has a root with positive real part if, and only if, there exists β satisfying (2.21), with $\gamma \in (\pi^2, 4\pi^2) \cup (9\pi^2, 16\pi^2) \cup ([25\pi^2, 36\pi^2) \cup \ldots$, such that the 2-vector $v\left(\sqrt{\gamma}\right)$ is a positive multiple of $w\left(\sqrt{\gamma}\right)$.

The roots of $F(\beta)$ are given by

$$\gamma = \frac{1}{2}\left[-(P^2 - 2Q) \pm \sqrt{(P^2 - 2Q)^2 - 4(Q^2 - R^2)}\right], \tag{2.26}$$

and taking into account that

$$Q^2 - R^2 = r^4 a^2 \left[(p - \xi)^2 - \xi^2 e^{-2pr}\right] \to 0,$$

as $p \to 0$, we can ensure that the least root γ is positive and, if $p > 0$ is taken small enough, we have $\gamma \in (0, \pi^2)$. Taking $a > 0$ sufficiently large, we have

$$\sqrt{r^4 a^2 (a + 2\tilde{\xi})^2} > 4\pi^2 \tag{2.27}$$

and, therefore, decreasing p further if necessary, the limit (2.25) implies the distance $\sqrt{\Delta}$ between the two roots γ $(= \beta^2)$ is greater than $4\pi^2$.

Then, for all $\beta \in (\pi, 2\pi)$, the inequality (2.21) is satisfied, i.e., $|v(\beta)| < -R$. Recalling that there exists a unique $\beta \in (\pi, 2\pi)$ for which $v(\beta)$ is a positive multiple of $w(\beta)$, it is an immediate consequence of Lemma 2.1 that the characteristic Equation (2.10) has a root with positive real part. $\square$

Remark 2.4. It is a clear consequence of the proof of Theorem 2.3 that increasing further the parameters a, b, and therefore $\tilde{\xi}$ (since we can do it in such a way that x^* remains bounded away from 0 and 1), with arbitrary p in some interval $(0, p_0)$, the number of roots of Equation (2.10) to the right of the imaginary axis increases. That is, the (finite) dimension of the unstable eigenspace of the linearized equation (2.5) can be made arbitrarily large by a convenient choice of the parameters.

3. Local Hopf bifurcations

Let us suppose the parameters b and p together with the input f are fixed in such a way that the hypotheses of Theorem 2.3 can be fulfilled. If the parameter a is allowed to increase, starting at some value so small that the inequality (2.23) is not satisfied, then Theorem 2.3 guarantees the existence of a critical value a_0 such that when a crosses the point a_0 the equilibrium x^* looses the stability. It is natural to ask if in this circumstances a branch of periodic solutions of Equation (1.6) emanates from x^*. Now we are going to answer this question. We apply a version of the Hopf bifurcation Theorem for retarded functional differential equations due to Hale, see 4. We state below this theorem in order to be as self-contented as possible. In spite of some conflict with our notation we maintain the original formulation of the theorem, as no confusion can be expected from this.

Consider the one-parameter family of retarded functional differential equations of the form

$$\dot{x}(t) = F(\alpha, x_t) \tag{3.1}$$

where $F(\alpha, \phi)$ has continuous first and second derivatives in α, ϕ for $\alpha \in \mathbb{R}$ and $\phi \in C = C([-r, 0], \mathbb{R}^n)$, and $F(\alpha, 0) = 0$, for all α. Define $L : \mathbb{R} \times C \to \mathbb{R}^n$ by

$$L(\alpha)\psi = D_\phi F(\alpha, 0)\psi \tag{3.2}$$

where $D_\phi F(\alpha, 0)$ is the derivative of $F(\alpha, \phi)$ with respect to ϕ at $\phi = 0$. So, we have a family of linear retarded functional differential equations

$$\dot{x}(t) = L(\alpha)x_t. \tag{3.3}$$

Since $L(\alpha)$ is continuously differentiable in α, it is known that there is an $\alpha_0 > 0$ and a simple characteristic root $\lambda(\alpha)$ of the linear equation (3.3) that has a continuous derivative $\lambda'(\alpha)$ in α for $|\alpha| < \alpha_0$, see 4, Section 7.10, Lemma 10.1. The following hypotheses are needed:

(H1) The linear retarded functional differential equation $\dot{x}(t) = L(0)x_t$ has a simple pure imaginary characteristic root $\lambda_0 = i\gamma_0 \neq 0$ and all characteristic roots $\lambda_j \neq \lambda_0, \overline{\lambda_0}$, satisfy $\lambda_j \neq m\lambda_0$ for any integer m.

(H2) $\Re(\lambda'(0)) \neq 0$.

By taking α_0 sufficiently small, we may assume $\Im(\lambda(\alpha)) \neq 0$ for $|\alpha| < \alpha_0$ and obtain a function $\phi_\alpha \in C$ that is continuously differentiable in α and is a basis for the eigenspace of Equation (3.3) corresponding to $\lambda(\alpha)$. The functions

$$(\Re(\phi_\alpha), \Im(\phi_\alpha)) := \Phi_\alpha$$

form a basis for the eigenspace corresponding to the characteristic roots $\lambda(\alpha)$, $\overline{\lambda(\alpha)}$. If Q_α is the eigenspace corresponding to the remaining roots, the space C is decomposed as $P_\alpha \oplus Q_\alpha$. Given a function $\phi \in C$, we denote by ϕ^{P_α} and ϕ^{Q_α} the components of ϕ in P_α and in Q_α, respectively, relative to this decomposition of C.

We now state the Hopf Bifurcation Theorem as set in 4.

Theorem 3.1. *Suppose $F(\alpha, \phi)$ has continuous first and second derivatives with respect to α, ϕ, $F(\alpha, 0) = 0$ for all α, and hypotheses $(H1)$ and $(H2)$ are satisfied. Then there are constants $a_0 > 0$, $\alpha_0 > 0$, $\delta_0 > 0$, functions $\alpha(a) \in \mathbb{R}$, $\omega(a) \in \mathbb{R}$, and a $\omega(a)$-periodic function $x^*(a)$, with all functions being continuously differentiable in a for $|a| < a_0$, such that $x^*(a)$ is a solution of Equation (3.1) with*

$$x_0^*(a)^{P_\alpha} = \Phi_{\alpha(a)} y^*(a), \qquad x_0^*(a)^{Q_\alpha} = z_0^*(a), \tag{3.4}$$

where $y^*(a) = (a, 0)^T + o(|a|)$, as $a \to 0$, $z_0^*(a) = o(|a|)$, as $a \to 0$. Furthermore, for $|\alpha| < \alpha_0$ and $|\omega - (2\pi/\gamma_0)| < \delta_0$, every ω-periodic solution of Equation (3.1) with $|x_t| < \delta_0$ must be of this type except for a translation in phase.

Attempting to apply theorem above to our setting, we will need the following lemma:

Lemma 3.1. *If z is a double root of Equation* (2.10), *then* $\Re(z) \neq 0$.

Proof. In fact, if z is a double root of Equation (2.10), it must satisfy the system

$$\begin{cases} (z^2 + Pz + Q)\, e^z + R &= 0 \\ (2z + P)e^z - R &= 0. \end{cases} \tag{3.5}$$

So that if $z = i\beta$, $\beta \in \mathbb{R}$, then β satisfies

$$(P + 2)\beta i + P + Q - \beta^2 = 0 \tag{3.6}$$

and, therefore, $\beta = 0$. That is, $z = 0$ is a root of Equation (2.10). This is a contradiction because $-R < Q$. □

Theorem 3.2. *Suppose the parameters f, b and p are fixed in such a way the hypotheses of Theorem 2.3. Suppose, further, the inequality $r\xi e^{pr} \leq 1$ holds. Then, there exists a_0, $0 < a_0 < -2\tilde{\xi}$, such that Equation* (1.6) *undergoes a Hopf bifurcation at $a = a_0$.*

Proof. Suppose $a_1 > 0$ is sufficiently large, subjected to the bound $a_1 + \delta \leq -2a_1\tilde{\xi}$, for some $\delta > 0$ fixed. Choose a_1, in addition, in such a way that $v(\beta) < -R$, for all $\beta \in (\pi, 2\pi)$, if $a = a_1$. That is, there exists a root z of Equation (2.10) such that $\Re(z) > 0$, with $a = a_1$.

Now, keeping b fixed and noticing that $-\xi < b$, one sees that, for $a_2 > a_1$ sufficiently large, if $a = a_2$,

$$P^2 - 2Q = r^2(a^2 + p^2 + 2a\xi) > 0. \tag{3.7}$$

Thus, if $a = a_2$, the polynomial $F(\beta)$ defined in (2.21) is positive for every $\beta > 0$, that is, $v(\beta) > -R$, for $\beta > 0$. This means that x^* is asymptotically stable. So, there exists the least a_0, $a_1 < a_0 < a_2$, such that for $a = a_0$, Equation (2.10) has a pure imaginary root $z_0 = \beta_0 i$.

The characteristic equation (2.10) can be written in its vector counterpart, $\Lambda(\alpha, \beta) = -Rw(\beta)$, that is equivalent to

$$e^\alpha \left[(2\alpha\beta, \alpha^2 + P\alpha) + (P\beta, Q - \beta^2)\right] = -R(-\sin\beta, \cos\beta) \tag{3.8}$$

and, in terms of the original parameters,

$$e^\alpha \left[\alpha\,(2\beta, \alpha + r(a + p)) + r\left(r(a + p)\beta, r^2 a(p - \xi) - \beta^2\right)\right] = - R(-\sin\beta, \cos\beta). \tag{3.9}$$

Equation (3.9) is equivalent to the system

$$\begin{cases} F^1(\alpha, \beta, a) := & e^\alpha[2\alpha\beta + r(a + p)\beta] - r^2 a\xi e^{-pr} \sin\beta = 0 \\ F^2(\alpha, \beta, a) := & e^\alpha[\alpha^2 + \alpha r(a + p) + r^2 a(p - \xi) - \beta^2] + \\ & + r^2 a\xi e^{-pr} \cos\beta = 0. \end{cases} \tag{3.10}$$

We know that, for some $beta_0 \in (\pi, 2\pi)$, System (3.10) has a solution of the form $(\alpha, \beta, a) = (0, \beta_0, a_0)$, i.e.,

$$
(3.11) \qquad \begin{cases} r(a+p)\beta_0 = r^2 a\xi e^{-pr} \sin\beta_0 \\ r^2 a(p-\xi) - \beta_0^2 = -r^2 a\xi e^{-pr}\cos\beta_0. \end{cases}
$$

It is a matter of routine to express the partial derivatives of F^1 and F^2 with respect to α and β as

$$
(3.12) \qquad \begin{aligned} F^1_\alpha &= e^\alpha \beta[2\alpha + r(a+p) + 2], \\ F^1_\beta &= e^\alpha[2\alpha + r(a+p)] - r^2 a\xi e^{-pr}\cos\beta, \\ F^2_\alpha &= e^\alpha[\alpha^2 + (\alpha+1)r(a+p) + r^2 a(p-\xi) - \beta^2 + 2\alpha], \\ F^2_\beta &= -2e^\alpha\beta - r^2 a\xi e^{-(pr+\alpha)}\sin\beta. \end{aligned}
$$

The reasoning below depends on the fact that the number ξ does not depend on a. If $F(\alpha, \beta, a) := \big(F^1(\alpha,\beta,a), F^2(\alpha,\beta,a)\big)$, it follows from Equation (3.12), together with the formulae (3.11), that the partial derivative of $F(\alpha,\beta,a)$ with respect to (α,β), in $(0,\beta_0,a_0)$, is a linear isomorphism represented by the nonsingular matrix

$$
(3.13) \quad \mathcal{M} = \begin{bmatrix} [2+r(a_0+p)]\beta_0 & r(a_0+p) + r^2 a_0\xi e^{-pr} - \beta_0^2 \\ r(a_0+p) + r^2 a_0\xi e^{-pr} - \beta_0^2 & -[2+r(a_0+p)]\beta_0 \end{bmatrix}.
$$

Note that $\det\mathcal{M} < 0$. As a consequence of the Implicit Function Theorem, there are smooth functions $a \mapsto \alpha^*(a)$ and $a \mapsto \beta^*(a)$, defined for a in a neighborhood, $J(a_0)$, of a_0, such that $\alpha^*(a_0) = 0$, $\beta^*(a_0) = \beta_0$ and

$$
(3.14) \qquad F\left(\alpha^*(a), \beta^*(a), a\right) = 0, \qquad \forall a \in J(a_0).
$$

Moreover, if

$$
(3.15) \qquad \mathcal{M}_\alpha := \begin{bmatrix} r\beta_0 & r(a_0+p) + r^2 a_0\xi e^{-pr} - \beta_0^2 \\ r^2 p & -[2+r(a_0+p)]\beta_0 \end{bmatrix},
$$

the Implicit Function Theorem also gives

$$
(3.16) \qquad \frac{d\alpha^*}{da}(a_0) = -\frac{\det\mathcal{M}_\alpha}{\det\mathcal{M}}.
$$

According to formulae (3.11) and to the hypothesis $r\xi e^{pr} \le 1$, one sees that

$$
r(a_0+p) + r^2 a_0\xi e^{-pr} - \beta_0^2 = r(a_0+p) - r^2 a_0 \xi e^{-pr}\cos\beta_0 > 0.
$$

Therefore, $\det\mathcal{M}_\alpha < 0$ and this implies $\left[d\alpha^*/da\right]_{a=a_0} < 0$. That is, hypothesis (H2) is satisfied for Equation (1.6).

According to Lemma 3.1, the root $i\beta_0$ is simple and the non-resonance condition ($\lambda_j \neq \lambda_0$ for any root $\lambda_j \neq m\lambda_0$, $\overline{\lambda}_0$ and any integer m) is trivially satisfied because, according to the proof of Lemma 2.1, there is no more than two imaginary roots of Equation (2.10). So, hypothesis (H1) is satisfied for Equation (1.6). Theorem 3.2 follows now as a consequence of Theorem 3.1 □

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