Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

Existence of Coexistence States: A Generic Property for Cyclic 3-Dimensional Competitive Systems.

Antonio Tineo

Notas de Matemática
Serie: Pre-Print
No. 200

Mérida - Venezuela
2000

Notas de Matemática, No. 200
Mérida, 2000.

Existence of Coexistence States: A Generic Property for Cyclic 3-Dimensional Competitive Systems.

Antonio Tineo

Abstract

In this paper we consider the class $C(T)$ of all dissipative 3-dimensional T-periodic Kolmogorov competitive and cyclic systems such that the trivial solution is a source, and we prove that "almost" every such system possesses a coexistence state. More precisely, we characterize an open and dense subset U of $C(T)$; with respect to the topology of the uniform convergence in compact sets; such that each member of U has a coexistence state.

1 Introduction

In this paper we prove that almost every dissipative, three dimensional competitive system possesses a coexistence state if it has a cyclic connection on the boundary of $\mathbb{R}^3$. The more celebrated example of a such system is due to May and Leonard [5]:

$$\begin{aligned} x_1' &= x_1[1 - x_1 - \alpha x_2 - \beta x_3] \\ x_2' &= x_2[1 - \beta x_1 - x_2 - \alpha x_3] \\ x_3' &= x_3[1 - \alpha x_1 - \beta x_2 - x_3]. \end{aligned} \tag{1.1}$$

This system has been extensively studied by several authors. For instance see [1] and the references therein.

Let S_i be the two-dimensional system obtained from (1.1) by letting $x_{i-1} = 0$. (Here and henceforth we shall use the *mod* 3 notation). We remark that if $0 < \beta < 1 < \alpha$ (resp. $0 < \alpha < 1 < \beta$) then, in system S_i; the species x_i (resp. x_{i+1}) is carried to extinction by x_{i+1} (resp. x_i), for all $i \in \mathbb{Z}$. In this case, we say that (1.1) is τ-cyclic (resp. σ-cyclic). Here, we are denoting by $\tau, \sigma : \mathbb{Z} \to \mathbb{Z}$ the permutations given by $\tau(i) = i+1$; $\sigma(i) = i-1$.

More generally, let us consider the system

$$x_i' = x_i F_i(t,x); x = (x_1, x_2, x_3);\ 1 \le i \le 3; \tag{1.2}$$

where $F_1, F_2, F_3 : \mathbb{R} \times \mathbb{R}_+^3 \to \mathbb{R}$ are continuous functions which are T-periodic in t and locally Lipschitz continuous in x. We shall assume that the following hypotheses hold:

H_1) System (1.2) is competitive. That is; $F_i(t,x)$ is decreasing with respect to x_j for all $i \neq j$.

H_2) System (1.2) is dissipative.

H_3) $\int_0^T F_i(t,0)dt > 0$ for all i. This condition implies that the trivial solution is a source.

We say that (1.2) is τ-cyclic (resp. σ-cyclic) if the species x_i (resp. x_{i+1}) is carried to extinction by x_{i+1} (resp. x_i) in the subsystem obtained from (1.2) by letting $x_{i-1} = 0$; $i \in \mathbb{Z}$. A more precise definition will be given in section 3.

Remark. If H_1) holds then H_2) is equivalent to say that the system

$$z' = zF_i(t, ze_i) \tag{1.3}$$

is dissipative for $1 \le i \le 3$. Here and henceforth, (e_1, e_2, e_3) denotes the canonical vector basis of $\mathbb{R}^3$. Thus, if $H_1) - H_3$) hold then (1.3) has a minimal positive T-periodic solution that we shall denote by v_i.

In section 3 we shall prove that if (1.2) is τ-cyclic then,

$$\int_0^T F_{i+1}(t, v_i(t)e_i)dt \ge 0 \ge \int_0^T F_i(t, v_{i+1}(t)e_{i+1})dt;\ i \in \mathbb{Z}$$

and the inequalities above are reversed if (1.2) is σ-cyclic. We shall prove the following results:

Theorem 1.1 *Assume $H_1) - H_3$) hold. If (1.2) is τ-cyclic and*

$$\int_0^T F_{i+1}(t, v_i(t)e_i)dt > 0;\ i \in \mathbb{Z}; \tag{1.4}$$

then the system has a coexistence state.

Theorem 1.2 *Assume that $H_1) - H_3$) hold and that (1.2) is τ-cyclic. Then the system*

$$x_i' = x_i F_i^\epsilon(t,x); \tag{1.5}$$

satisfies the assumption in the above theorem, where

$$F_i^\epsilon(t,x) := F_i(t,x) + \epsilon[F_i(t,0) - F_i(t, x_{i-1}e_{i-1})];\ \epsilon \in (0,1).$$

Note that the minimal positive T-periodic solution of the logistic equation $z' = zF_i^c(t, ze_i)$ is also v_i, since $F_i^c(t, ze_i) \equiv F_i(t, ze_i)$.

Remark. Let $\mathcal{C}_\tau(T)$ be the class of all systems (1.2) which are τ-cyclic and satisfies $H_1) - H_3)$ and

$H_4)$ $F_i(t, ze_i)$ is decreasing in $z \geq 0$ for all $t \in \mathbb{R}$ and $F_i(s_i, z)$ is strictly decreasing in z, for some $s_i = s_i(F) \in \mathbb{R}$.

From the results in [9] it follows easily that the subclass $\mathcal{U}_\tau$ of $\mathcal{C}_\tau(T)$, determined by equation (1.4), is an open subset of $\mathcal{C}_\tau(T)$ in the topology of the uniform convergence in compact sets. Moreover, by Theorem 1.2, this set is also dense. We have parallel results for σ-cyclic systems.

This paper was motivated by an article of [1] in which it is proved that a large class of cyclic systems have coexistence states, and is divided in three sections. In section 1, we use some ideas in Hirsch [3] to show that the Poincare map of (1.2) has a compact invariant 2-cell. In section 3, we use a result by Campos, Ortega and Tineo [2] and a contradiction argument to show Theorem 1.1. Finally, in section 2 we find necessary conditions under which system (1.2) is τ-cyclic and we prove Theorem 1.2.

2 Existence of a Compact Invariant 2-cell

In this section we consider the n-dimensional system

$$x_i' = x_i F_i(t, x);\ x = (x_1, \cdots, x_n);\ 1 \leq i \leq n; \tag{2.1}$$

where $F_1, \cdots, F_n : \mathbb{R} \times \mathbb{R}_+^n \to \mathbb{R}$ are continuous functions which are T-periodic in t and locally Lipschitz continuous in x. We also assume that $H_1) - H_3)$ hold.

Let $\pi = (\pi_1, \cdots, \pi_n) : \mathbb{R}_+^n \to \mathbb{R}^n$ be the Poincare map of (2.1). By $H_1) - H_3)$, there exists $a_i > 0$ such that $\pi(a_i e_i) = a_i e_i$ and $\pi_i(ze_i) > z$ for all $z \in (0, a_i)$. In fact, $a_i = v_i(0)$, where v_i is the minimal positive T-periodic solution of (1.3). Analogously, there exists $b_i \geq a_i$ such that $\pi(b_i e_i) = b_i e_i$ and $\pi_i(ze_i) < z$ for all $z > b_i$.

Given $p \in \mathbb{R}_+^n$, we denote by $S(t, p)$ the solution of (2.1) determined by the initial condition $S(0, p) = p$. We also define D as the subset of $\mathbb{R}_+^n$ consisting of all points p such that $S(t, p)$ is defined on $\mathbb{R}$ and D_0 as the subset of D consisting of all points p such that

$$S(t, p) \to 0 \text{ as } t \to -\infty.$$

That is, D_0 is the domain of repulsion of $p = 0$. Note that by $H_3)$, D_0 is an open subset of $\mathbb{R}_+^n$. Using some ideas in [3], we shall prove the following result.

Theorem 2.1 *The boundary* $\Delta := \partial D_0$ *of* D_0 *relative to* $\mathbb{R}^n_+$ *is a compact* $(n-1)$*-cell invariant by* π*, such that* $a_i e_i \in \Delta;\ 1 \le i \le n$.

The proof requires two short results. We begin with the following well known fact, which we estate here for reference purposes.

Proposition 2.2 *If* u, v *are non negative solutions of (2.1) and* $u(0) \le v(0)$*, then* $u(t) \le v(t)$ *for all* $t \in (-\infty, 0) \cap domain(u) \cap domain(v)$.

Proposition 2.3 *If* $u = (u_1, \cdots, u_n)$ *is a non negative solution of (2.1) defined on* $\mathbb{R}$ *then* $u_i(0) \le b_i$*. Moreover, if* $u(0) \in D_0$*, then* $u_i(0) < a_i$*. In particular,* D *is compact and* $D_0 \subset [0, a_1) \times \cdots \times [0, a_n)$.

Proof. Let us fix $1 \le i \le n$ and let w_i be the solution of (1.3) determined by the initial condition $w_i(0) = u_i(0)$. By H_1), u_i is a subsolution of (1.3) and hence, $w_i(t) \le u_i(t)$ if $t \in (-\infty, 0) \cap domain(w_i)$. Consequently, w_i is defined on $\mathbb{R}$ and so $u_i(0) \le b_i$. Moreover, if $u(0) \in D_0$, then $w_i(t) \to 0$ as $t \to -\infty$ and the proof follows easily.

Proof of Theorem 2.1. Since D is compact, we conclude that Δ is a compact subset of D. Note also that $0 \notin \Delta$. Moreover, $\Delta \cap \mathbb{R}_+ e_i = \{a_i e_i\}$ since $D_0 \cap \mathbb{R}_+ e_i = [0, a_i) e_i$. In particular, $a_i e_i \in \Delta$.

Given $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_+$, we define $|x| = x_1 + \cdots + x_n$ and we note that $K := \{x \in \mathbb{R}^n_+ : |x| = 1\}$ is a compact $(n-1)$-cell. We shall prove that the radial projection $R : \Delta \to K$; $R(x) = x/|x|$; is a bijection and consequently, a homeomorphism onto K since R is continuous and Δ is compact.

For each nonempty subset J of $\{1, \cdots, n\}$, let us define $\mathbb{R}^J_+ = \{x \in \mathbb{R}^n_+ : x_i = 0\ \forall\ i \notin J\}$ and note that $\mathbb{R}^J_+ = \mathbb{R}_+ e_j$ if $J = \{j\}$ is a singular set. In particular, the restriction $R_J : \mathbb{R}^J_+ \cap \Delta \to K$ is injective if J is singular.

We shall prove that R is injective. By induction, we can suppose that R_J is injective for all proper subsets of $\{1, \cdots, n\}$. On the other hand, $\mathbb{R}^J_+$ is invariant by π and so it suffices to show that the relations $x, y \in \Delta$; $x, y > 0$; $R(x) = R(y)$ implies $x = y$.

To do this suppose $x \ne y$ and define $w = R(x)$. Then $w > 0$ and $x - y = \lambda w$ for some $\lambda \in \mathbb{R}$. Without loss of generality, we can assume that $\lambda > 0$ and so $x > y$. Since $N := \{p \in \mathbb{R}^n_+ : 2p > x + y\}$ is a neighborhood of x, there exists $p \in D_0$ such that $p \in N$. On the other hand, $y < p$ and by Proposition 2.2, $S(t, y) \le S(t, p)$ for all $t \le 0$. From this, $S(t, y) \to 0$ as $t \to -\infty$ and hence, $y \in D_0$. This contradiction proves that R is injective.

As above, we can suppose that $R(\Delta) \supset K \cap \mathbb{R}^J_+$ for all proper subsets of $\{1, \cdots, n\}$. Thus it suffices to show that $R(\Delta)$ contains all positive vectors in K. To this end, let us fix $w > 0$ in K and note that $aw \in D_0$ for some $a > 0$, since D_0 is an open subset of $\mathbb{R}^n_+$ containing the origin. Now, let $\lambda := \sup\{a > 0 : aw \in D_0\}$, it is clear that $\lambda w \in \Delta$ and $R(\lambda w) = w$. ■

3 The Proof of Theorem 1.1

Let Δ be the 2-cell given by Theorem 2.1. Then $\pi : \Delta \to \Delta$ is an orientation preserving homeomorphism onto Δ, since $a_i e_i \in \Delta$; $1 \leq i \leq 3$. See proposition 3.2 of [7]. As above, π denotes the Poincaré map of (1.2).

Assume by contradiction that π has no positive fixed points. Since (1.2) is τ-cyclic, then $Fix(\pi) \cap \Delta = \{a_1e_1, a_2e_2, a_3e_3\}$. Now, let us quote the following result in [2] (theorem 2.1):

Theorem. Let $D \subset \mathbb{R}^2$ be a closed disk and let $h : D \to D$ be an orientation preserving homeomorphism such that $Fix(h) \subset \partial D$. Then, the ω-limit set of any orbit of h is a connected subset of $Fix(h)$.

Let us fix a positive solution $u = (u_1, u_2, u_3)$ of (1.2), with $u(0) \in \Delta$. By the above theorem, there exists $i \in \mathbb{Z}$ such that

$$u(t) - v_{i-1}(t)e_{i-1} \to 0 \text{ as} t \to +\infty. \tag{3.1}$$

Integrating relation

$$\frac{u_i'}{u_i} = F_i(t, u(t))$$

over $[nT, nT + T]$; $n \in \mathbb{N}$; and letting $n \to \infty$, we obtain

$$\ln \frac{u_i(T + nT)}{u_i(nT)} = \int_{nT}^{nT+T} F_i(t, u(t))dt =$$

$$\int_0^T F_i(t, u(t + nT))dt \to \int_0^T F_i(t, v_{i-1}(t)e_{i-1})dt > 0.$$

From this, $\{u_i(nT)\}$ is an eventually strictly increasing sequence, which contradicts 3.1, and the proof of our first assertion is complete.

4 Planar Systems

In this section we prove some auxiliary results. We begin with a maybe well known result. All systems considered in this section are assumed to satisfy the hypotheses in sections 1 and 2. We say that x is carried to extinction by y in system (4.1) below

$$x' = xF(t, x, y);\ y' = yG(t, x, y); (x, y) \in \mathbb{R}^2_+; \tag{4.1}$$

if $\lim_{t\to+\infty} u(t) = 0$ for any positive solution (u, v) of the system.

Proposition 4.1 *Let us consider the system*

$$x' = xF_0(t,x,y);\ y' = yG_0(t,x,y); \tag{4.2}$$

and suppose that $F \geq F_0$ and $G \leq G_0$. If x is carried to extinction by y in (4.1) then the same holds for system (4.2).

Proof. Let (u_0, v_0) be a positive solution of (4.2) and fix a positive solution (u, v) of (4.1) such that $u(0) > u_0(0)$ and $v(0) < v_0(0)$. It suffices to show that

$$u(t) > u_0(t);\ t > 0. \tag{4.3}$$

Assume on the contrary that there exists $t_0 > 0$ such that

$$u(t) > u_0(t) \text{ for } t \in (0, t_0) \text{ and } u(t_0) = u_0(t_0). \tag{4.4}$$

Since $G_0(t,x,y)$ is decreasing with respect to x then,

$$v'(t) \leq v(t)G_0(t,u(t),v(t)) \leq v(t)G_0(t,u_0(t),v(t)); 0 \leq t \leq t_0;$$

and hence, the restriction of v to $[0, t_0]$ is a subsolution of the equation

$$z' = zG_0(t, u_0(t), z).$$

On the other hand, v_0 is a solution of this equation and so, $v \leq v_0$ in $[0, t_0]$. From this, in this interval we have,

$$u'(t) \geq u(t)F_0(t,u(t),v(t)) \geq u(t)F_0(t,u(t),v_0(t)).$$

If $u'(s) > u(s)F_0(s,u(s),v_0(s))$ for some $s \in (0, t_0)$ then, by Lemma A1 in the Appendix, $u > u_0$ on $[s, t_0]$, which contradicts (4.4) and proves that

$$u'(t) = u(t)F_0(t,u(t),v_0(t))\ \forall t \in [0, t_0].$$

Thus, u, u_0 are solutions of the equation $z' = zF_0(t, z, v_0(t))$ in the interval $[0, t_0]$ and by uniqueness $u \equiv u_0$ in this interval, since $u(t_0) = u_0(t_0)$. Hence, $u(0) = u_0(0)$ and this contradiction ends the proof.

Proposition 4.2 *Suppose that (4.1) satisfies the assumptions in section* 2 *and let ξ be the minimal positive T-periodic solution of*

$$x' = xF(t,x,0).$$

If x is carried to extinction by y in (4.1) then

$$\int_0^T G(t,\xi(t),0)dt \geq 0. \tag{4.5}$$

Proof. By the change of variables $(X,Y) = (x/\xi, y)$, we can suppose that $\xi \equiv 1$. Assume now that (4.5) is false; that is, suppose

$$\int_0^T G(t,1,0)dt < 0,$$

then we can write $G(t,1,0) = -\mu + A'(t)/A(t)$ for some positive T-periodic function A. ($-\mu$ is the average of $G(.,1,0)$). From the change of variables $(x,y) \mapsto (x, y/A)$, we can suppose that $G(t,1,0) < 0$ for all $t \in \mathbb{R}$. Thus, there exists $1 > \epsilon > 0$ such that

$$G(t,x,y) < 0 \text{ if } t \in \mathbb{R};\ |x-1| \leq \epsilon;\ 0 \leq y \leq \epsilon. \tag{4.6}$$

Let Δ be the compact 1-cell given by Theorem 2.1. As it was pointed in [6], it follows from Proposition 2.2 that Δ is a decreasing curve. That is, if $(x_1,y_1),(x_2,y_2) \in \Delta$ and $y_1 \leq y_2$ then, $x_1 \geq x_2$.

Given $p \in \mathbb{R}^2_+$, let $S(t,p) = (u(t,p), v(t,p))$ be the solution of (4.1) determined by the initial condition $S(0,p) = p$. Since $S(t,1,0) = (1,0) \in (1-\epsilon, 1+\epsilon) \times [0,\epsilon)$ for all $t \in \mathbb{R}$, there exists $p \in \Delta$ such that $S(t,p) \in (1-\epsilon, 1+\epsilon) \times [0,\epsilon)$ for all $t \in [0,T]$. From this and (4.6), $v'(t,p) < 0$ in $[0,T]$ and hence, $v(T,p) < v(0,p)$. Therefore, $u(T,p) \geq u(0,p)$ since Δ is decreasing.

Using H_1) we conclude that $u(t+T,p) \geq u(t,p)$ and $v(t+T,p) \leq v(t,p)$ for all $t \geq 0$. In particular, $\{u(nT,p)\}$ is an increasing sequence of positive numbers, which is a contradiction since $u(t,p) \to 0$ as $t \to +\infty$. This contradiction ends the proof.

Remark. Assume the hypotheses of Proposition 4.2 hold and let η be the minimal positive T-periodic solution of the logistic equation $z' = zF(t,0,z)$. Using the arguments in that proposition, we can prove that

$$\int_0^T F(t,0,\eta(t))dt \leq 0.$$

From this, we obtain the following result.

Corollary 4.3 *If (1.2) is τ-cyclic then*

$$\int_0^T F_{i+1}(t, v_i(t)e_i)dt \geq 0 \geq \int_0^T F_i(t, v_{i+1}(t)e_{i+1})dt;\ i \in \mathbb{Z}.$$

The inequalities are reversed if (1.2) is σ-cyclic.

Proof of Theorem 1.2. If $x_{i-1} = 0$ we have, $F_i^\epsilon(t,x) = F_i(t,x)$ and $F_{i+1}^\epsilon(t,x) = F_{i+1}(t,x) + \epsilon[F_{i+1}(t,0) - F_{i+1}(t,x_ie_i)] \geq F_{i+1}(t,x)$; and by Proposition 4.1, (1.5) is τ-cyclic. On the other hand,

$$F_{i+1}^\epsilon(t, v_i(t)e_i) = (1-\epsilon)F_{i+1}(t, v_i(t)e_i) + \epsilon F_{i+1}(t,0),$$

and the proof follows from H_3), Corollary 4.3 and Theorem 1.1.

Examples.

1. We first show an example that satisfies the assumptions of Theorem 1.1, but does not satisfies the assumptions in [1]:

$$x' = x[(1-x)(x-2)^2 - y - 16z]; \;\; y' = y[1-x-2y-z]; \;\; z' = z[3-x-7y-4z].$$

2. We shall sketch the construction of a τ-cyclic autonomous system satisfying $H_1) - H_3)$, which has no positive equilibria. To this end, let us first show that $F(x,y) := (1 - x^2 - y^2 - 2y(1-y), 2x(1-y))$ is a C^∞-vector field in the unitary disk $D = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$, such that the limit sets of any trajectory are equal to $\{(0,1)\}$. To this end, note first that $(0,1)$ is the unique equilibrium of F and let S_h be the sphere of center $(0,(1+h)/2)$ and radius $(1-h)/2$, for $h \in [-1,1]$. Then, D is the union of S_h and $S_h \cap S_k = \{(0,1)\}$ if $h \neq k$. On the other hand, the restriction of F to S_h is a vector field on S_h, and our assertion about the trajectories of F follows easily. From this, $(1-x^2)F(x,y)$ is a smooth vector field on D, which only has three equilibria and these equilibria belongs the boundary of D. Moreover, F has a cycle in ∂D. Using the above ideas, we can construct a continuously differentiable vector field G in the standard two-simplex $\Delta := \{(x,y,z) \in \mathbb{R}^3_+ : x + y + z = 1\}$ such that $\{e_1, e_2, e_3\}$ is the set of all equilibria of G and G has a cycle in the relative boundary of Δ. Our example follows now from the Smale's construction [8].

A Appendix

In this section we prove a partially well known result about extension of inequalities in O.D.E.'s. Let us consider the scalar system

$$x' = f(t,x) \tag{1.1}$$

where $f : U \to \mathbb{R}$ is a continuous function, defined on an open subset of $\mathbb{R}^2$, which is locally Lipschitz continuous in x.

Lemma A.1 *Let $u, v : [a, b] \to \mathbb{R}$ be continuously differentiable functions such that u (resp. v) is a subsolution (resp. supersolution) of (1.1). If $u(a) \leq v(b)$, then $u \leq v$. Moreover, if*

$$v'(s) - u'(s) > f(s, v(s)) - f(s, u(s)) \text{ for some } s \in (a, b), \tag{1.2}$$

then $u < v$ in $[s, b]$.

Proof. The first assertion is well known [4]; Th. 1.1.1; but we include the proof of it by completeness. Since f is locally Lipschitz continuous and $[a, b]$ is compact, there exists $M > 0$ such that

$$|f(t, v(t)) - f(t, u(t))| \leq M|v(t) - u(t)|; \; t \in [a, b]. \tag{1.3}$$

Now, let us write $w = v - u$ and note that by (1.3),

$$w' + M|w| \geq 0 \text{ on } [a, b]. \tag{1.4}$$

Assume now that $w(t_0) < 0$ for some $t_0 \in (a, b]$. Since $w(a) \geq 0$, there exists $t_1 \in [a, t_0)$ such that $w < 0$ in (t_1, t_0) and $w(t_1) = 0$. From this and (1.4), we have $w' - Mw \geq 0$ in $[t_1, t_0]$ and hence $e^{-Mt}w(t)$ is increasing in this interval. Consequently,

$$0 = \exp(-Mt_1)w(t_1) \leq \exp(-Mt_0)w(t_0) < 0$$

and this contradiction proves that $w \geq 0$. Assume now that (1.2) holds. Then $w'(s) + M|w(s)| > 0$. If $w(s) = 0$ then, $w'(s) > 0$ and so $w < 0$ in $(s - \epsilon, s)$ for some $\epsilon > 0$. This contradiction proves that $w(s) > 0$. Suppose that there exists $t_0 \in (s, b]$ such that $w > 0$ on $[s, t_0)$ and $w(t_0) = 0$. By (1.4), $w' + Mw \geq 0$ on $[s, t_0]$ and by the above argument, $0 = \exp(Mt_0)w(t_0) \geq \exp(Ms)w(s) > 0$. This contradiction ends the proof.

References

[1] Battauz A. and Zanolin F., *Coexistence States for Periodic Competitive Kolmogorov Systems* . J. Math. Anal. and Appl. 219, (1998), 179-199.

[2] Campos J. Ortega R. and Tineo A., *Homeomorphisms od the Disk with Trivial Dynamics and Extinction of Competitive Systems.* J.D.E., Vol. 138, N^{0} 1 (1997), 157-170.

[3] Hirsch M., *Systems of Differential Equations which are competitive or Cooperative. III: Competing species.* Nonlinearity, (1988), 1, 51-71.

[4] Ladde G. S., Laksmikantham V. Vatsala A. S., *Monotone Iterative Techniques for Nonlinear Differential Equations.* Pitman Advanced Publishing Program. Boston-London-Melbourne (1985).

[5] May R. and Leonard W., *Nonlinear Aspects of Competition Between Three Species.* SIAM J. Appl. Math. Vol. 29, No. 2, (1975), 243-253.

[6] P. de Mottoni and A. Schiaffino, *Competition Systems which Periodic coefficients: A Geometric Approach.* J. Math. Biol., 11 (1981), 319-335.

[7] Ortega R. and Tineo A., *An Exclusion Principle for Periodic Competitive Systems in Three Dimensions.* Nonlinear Analysis T.M.A., 193 (1997), 975-978.

[8] S. Smale, *On the differential equations of species in competition.* J. Math. Biol., 3 (1976),5-7.

[9] Tineo A., *Iterative Schemes for some Population Models.* Nonlinear World, 3, (1996), 695-708.