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UNITARY INVARIANTS OF SPECTRAL MEASURES-I

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ABSTRACT

The present paper and the succeeding one offer a unified approach to the study of unitary invariants of (unbounded) normal and self-adjoint operators to obtain not only the inter-relations among various results known for operators on separable and arbitrary Hilbert spaces, but also to provide a comparative study of the varios notions of multiplicity employed in the literature. In this paper we introduce the concept of CGS-property of a spectral measure E(.) in a Hilbert space H and study the problem of determining complete systems of unitary invariants for E(.) when E(.) has the said property.

El presente trabajo y el que sigue ofrecen un enfoque unificado para el estudio de invariantes unitarias de los operadores normales (no acotados) y auto-adjuntos, con el fin de obtener no solamente las relaciones entre varios resultados conocidos para operadores en espacios de Hilbert separables y arbitrarios, sino también de dar un estudio comparativo de las varias nociones de multiplicidad desempeñadas en la literatura. En este trabajo introducimos el concepto de CGS-property de una medida espectral E(.) en un espacio de Hilbert y estudiamos el problema de determinar sistemas completos de invariantes unitarias para E(.) cuando E(.) tiene dicha propiedad.

UNITARY INVARIANTS OF SPECTRAL MEASURES-I

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T.V. PANCHAPAGESAN*

The results of Hellinger [6] and Hahn [4] on the problem of determining a complete system of unitary invariants of hermitian quadratic forms on ℓ^2 were generalized to the case of self-adjoint operators on an abstract separable Hilbert space in the treatise of Stone [12]. Ever since the publication of [12] many mathematicians worked on this problem and obtained complete systems of unitary invariants for self-adjoint and normal operators on arbitrary Hilbert spaces. Among them the works of Wecken [13], Nakano [18], Yosida [14], Plessner and Rohlin [10], Segal [11] and Halmos [5] are noteworthy. Besides, the results of Dunford and Schwartz [3] on this problem for self-adjoint and bounded normal operators on separable Hilbert spaces are closely related to those of [12].

The results for operators on separable Hilbert spaces as treated in [12] and [3] and those for operators on arbitrary Hilbert spaces as found in the said literature are apparently unrelated. Besides, various concepts of multiplicity and uni

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form multiplicity are employed in these works and an explicit study of the inter-relations among them is absent in the literature up to the knowledge of the author. Under these circumstances it is quite desirable to give a unified approach to the study of this problem so as to obtain not only the inter-relations among various results known for operators on separable and arbitrary Hilbert spaces but also to provide a comparative study of the various notions of multiplicity employed in the literature. The present paper and the subsequent one are devoted to such study.

In the present paper we introduce the concept of CGS-property of a spectral measure E(.) in a Hilbert space H and study the problem of unitary invariants for E(.) with the CGS-property in H. We obtain results extending those of [12] and [3] for such spectral measures and as a consequence, the classical results of [12] and [3] get extended to (unbounded) normal operators on separable Hilbert spaces. Also we introduce various concepts of multiplicity for E(.) such as OSD-multiplicity, OSR-multiplicity and total multiplicity and show that they all coincide. When E(.) is defined on the Borel sets $\mathcal{B}(X)$ of a Hausdorff space X, we introduce the mulplicity functions m_p and m_c generalizing those in Chapter VII of Stone [12] and the inter-relations between the total multiplicity and the functions m_p and m_c are studied.

In the subsequent paper [9] making use of the results of

Halmos [5] we study orthogonal representations of a Hilbert space H relative to an arbitrary spectral measure E(.) on H and obtain various complete systems of unitary invariants for E(.). Many known results for self-adjoint and normal operators on the problem of unitary invariants are extended to spectral measures. Spectral measures E(.) with the CGS-property in H are characterized in terms of the existence of a special hype of orthogonal representations (COBOTS-representations) of H relative to E(.) and are obtained inter-relations between CO BOTS-representations and the results of the present paper. Consequently, the inter-relations between the works of [12] and [3] and those of [5],[10],[11],[13] and [14] get established. In [9] we introduce other notions of multiplicity too and compare them with those given in this paper.

In the first section not only we fix the terminologies and notations, but also give some lemmas that are basic in the study of this paper. In Section 2 we introduce the notions of CGS-property in H for a spectral measure E(.), ordered spectral decompositions (OSDs) of H relative to E(.), equivalence of two OSDs, and the OSD-multiplicity of E(.) and show that two spectral measures $E_1(.)$ and $E_2(.)$ with the CGS property on H_1 and H_2 respectively are unitarily equivalent if and only if any two OSDs of H_1 and H_2 relative to $E_1(.)$ and $E_2(.)$ respectively are equivalent. Theorem 7.7 of Stone [12] is generalized to such spectral measures in Section 3.

When E(.) has the CGS-property in H and is defined on the Borel sets $\mathcal{B}(X)$ of a Hausdorff space X, we introduce the multiplicity functions m_p and m_c on X associated with E(.) and study their properties in Section 4. While Section 5 deals with the unitary invariants of spectral measures on product spaces, Section 6 obtains a generalization of Theorem 7.8 of Stone [12] to certain class of normal operators on separable Hilbert spaces.

In Section 7 we introduce the concept of total multiplicity of E(.) and show that the OSD-multiplicity and the total multiplicity of E(.) are the same. When E(.) is defined on $\mathcal{B}(X)$, X a Hausdorff space, the inter-relation between the total multiplicity of E(.) and the multiplicity functions m_p and m_c on X is studied in Section 8. In the last section we introduce the concepts of ordered spectral representations (OSRs) and special OSRs of H relative to E(.) and obtain the results in Chapters X.5 and XII.3 of [3] as very particular cases of those established here. Finally, we introduce the concept of OSR-multiplicity, generalizing that of [3] and show that the OSR-multiplicity and OSD-multiplicity of E(.) are the same.

1.- **PRELIMINARIES.** H, H_1, H_2 will denote (complex) Hilbert spaces of arbitrary dimensions (>0) unless otherwise mentioned. Their inner-products and norms are denoted by $(.,.)$ and $||.||$ respectively. An operator on H is a linear transformation whose domain and range are contained in H. If T is an operator on H, the domain $\mathcal{D}(T)$ of T is a linear manifold. A normal operator T on H is either bounded or unbounded according as its domain is the whole of H is a closed linear manifold in H. The subspace generated by a subset X of H is denoted by $[X]$. If U is an inner product preserving linear transformation with domain H_1 and range H_2, then we say that U is an isomorphismo from H_1 onto H_2.

The σ-algebras of the Borel subsets of $\mathbb{C}$, $\mathbb{R}$ and a topological space X are denoted respectively by $\mathcal{B}(\mathbb{C})$, $\mathcal{B}(\mathbb{R})$ and $\mathcal{B}(X)$.

S will be a fixed σ-algebra of subsets of a non-void set X. All spectral measures considered will have their domain S unless otherwise stated. The word 'measure' always signifies a finite positive measure. In the sequel E(.), $E_1(.)$, $E_2(.)$ will be spectral measures on S with values in projections of H, H_1, H_2 respectively. Σ is the collection of all measures on S. For μ_1, μ_2 in Σ we write $\mu_1 << \mu_2$ (or $\mu_2 << \mu_1$) if $\mu_1(\sigma) = o$ whenever $\mu_2(\sigma) = o$. If $\mu << \nu$ and $\nu << \mu$, then we write $\mu \equiv \nu$ and clearly, '$\equiv$' is an equivalence relation on Σ. If $\mu << \nu$, then the Radon-Nikodym derivative of μ with respect to ν is denoted by $\frac{d\mu}{d\alpha}$.

For a vector x in H, $Z(x)= [E(\sigma)x:\sigma \in S]$. Similarly, we use the symbols $Z_i(x_i)$, i=1,2. $\sum_{i \in J} \oplus M_i$ denotes the orthogonal direct sum of the subspaces M_i of some Hilbert space H. If K_i are Hilbert spaces, then $\Sigma \oplus K_i$ denotes their external direct sum. For $x \in H$, $\rho(x)$ denotes the measure $||E(.)x||^2$ on S. Similarly, we use the symbols $\rho_i(x_i)$ to denote $||E_i(.)x_i||^2$.

Other terminologies and notations will be introduced later in appropriate places.

We state the following lemma to be referred to later quite often.

LEMMA 1.1. Let x be a fixed vector in H and let $\mu = ||E(.)x||^2$. Then there exists an isomorphism U from $L_2(X,S,\mu)$ onto $Z(x)$ such that $U\chi_\sigma = E(\sigma)x$, $\sigma \in S$ and $U^{-1}E(.)Uf= \chi_{(.)}f$, $f \in L^2(X,S,\mu)$.

Vide p.95 of [5] for the proof.

LEMMA 1.2. Let x and y be vectors in H_1 and H_2 respectively. Then there exists an isomorphism V from $Z_1(x)$ onto $Z_2(y)$ such that $VE_1(.)V^{-1}= E_2(.)$ if and only if $\rho_1(x) \equiv \rho_2(y)$.

PROOF. Suppose V is an isomorphism from $Z_1(x)$ onto $Z_2(y)$ such that

$$VE_1(.)V^{-1}= E_2(.) \quad (1)$$

If $w = V^{-1}y$, then by (1) we have $\rho_1(w) = \rho_2(y)$. Let U_1 be the isomorphism from $L_2(X,\mathcal{S}, \rho_1(x))$ onto $Z_1(x)$ as described in Lemma 1.1. Then there exists $f \in L_2(X,\mathcal{S}, \rho_1(x))$ such that $U_1 f = w$ and hence, $\rho_2(y) = \rho_1(w) = ||E_1(.)U_1 f||^2 = ||U_1 \chi_{(.)} f||^2 = ||\chi_{(.)} f||_2^2$ from which it follows that $\rho_2(y) << \rho_1(x)$. By symmetry, $\rho_1(x) << \rho_2(y)$ and hence the necessity of the condition.

Conversely, let $\rho_1(x) \equiv \rho_2(y)$. Then by Theorem 65.3 of [5] there exists $w \in Z_1(x)$ such that $\rho_2(y) = \rho_1(w)$ and $Z_1(x) = Z_1(w)$. If U_1 and U_2 are the isomorphisms described in Lemma 1.1 with respect to w and y respectively, then clearly $V = U_2 \circ U_1^{-1}$ is an isomorphism from $Z_1(x)$ onto $Z_2(y)$ and satisfies (1).

LEMMA 1.3. Let T_i be a normal operator on H_i with the resolution of the identity $E_i(.)$, i=1,2. There exists an isomorphism U from H_1 onto H_2 such thar $U T_1 U^{-1} = T_2$ if and only if $UE_1(.)U^{-1} = E_2(.)$.

PROOF. If $F(.) = UE_1(.)U^{-1}$, then $F(.)$ is a spectral measure and $D = \{y: \int |\lambda|^2 \, d\, ||F(\lambda)y||^2 < \infty\} = U\,\mathcal{D}(T_1)$. Besides, for $y \in D$, $\int \lambda \, dF(\lambda) y = UT_1U^{-1}y$. By the uniquenen of the resolution of the identity of a normal operator we conclude that $F(.)$ is the resolution of the identity of UT_1U^{-1}. From this the lemma follows.

2.- **ORDERED SPECTRAL DECOMPOSITIONS (OSDs).** In order to give a unified approach to the study of unitary invariants of normal and self-adjoint operators on separable Hilbert spaces we introduce here the concept of ordered spectral decomposition of a Hilbert space relative to a spectral measure E(.). We show that H admits such a decomposition relative to E(.) if and only if E(.) has the CGS-property in H (vide Definition 2.3). Introducing the concept of equivalence between two such decompositions we prove that all such decompositions of H relative to the same spectral measure are equivalent and that two such decompositions of H_1 and H_2 relative to $E_1(.)$ and $E_2(.)$ respectively are equivalent if and only if $E_1(.)$ and $E_2(.)$ are unitarily equivalent.

DEFINITION 2.1. Let $\{x_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a countable set of non-zero vectors in H such that (i) $H = \sum_1^N \oplus Z(x_i)$ and (ii) $\rho(x_1) \gg \rho(x_2) \gg \ldots$. Then we say that $H = \sum_1^N \oplus Z(x_i)$ is an ordered spectral decomposition (an OSD, in abbreviation) of H relative to E(.). If E(.) is the resolution of the identity of a normal operator T on H (necessarily separable), then it is said to be an OSD relative to T.

The following proposition is immediate from the above definition.

PROPOSITION 2.2. If H has an OSD relative to E(.), then there exists a countable set X in H such that $[E(\sigma)X : \sigma \varepsilon S] = H$.

DEFINITION 2.3. E(.) is said to have the CGS-property (i.e.countable generating set-property) in H if there exists a countable set X in H such that $[E(\sigma)\ X: \sigma \varepsilon S] = H$.

In order to characterize the CGS-property of E(.) in H in terms of the existence of OSDs of H relative E(.) we give the following

LEMMA 2.4. Suppose E(.) has the CGS-property in H. Then given a vector y_o in H there exists a vector x in H such that $\rho(y) << \rho(x)$ for all $y \varepsilon H$ and $y_o \varepsilon Z(x)$.

PROOF. We can suppose $||y_o|| = 1$.

AFFIRMATION 1. If $n \varepsilon \mathbb{N}$ is the minimum of the dimensions of all generatining subspaces M of H (i.e.$[E(\sigma)M:\sigma\varepsilon S] = H$), then there exist non-zero elements $\{g_i\}_1^n$ in H such that $H = \sum_1^n \oplus\ Z(g_i)$.

In fact, let G be a generating subspaces of dimension n, with $\{h_i\}_1^n$ an orthonormal basis. Let $g_1 = h_1$. If $n = 1$, the affirmation is trivial. Let $n > 1$. By the definition of n, $Z(g_1) \neq H$ and $(I - p_1)\ h_2 \neq o$ where P_1 is the orthogonal projection on $Z(g_1)$. Taking $g_2 = (I - p_1)\ h_2$, we have evidently $Z(g_1) \perp Z(g_2)$ and $Z(g_1) \oplus Z(g_2) \neq H$ if $n > 2$. In the i^{th} stage $g_i = (I - \sum_1^{i-1} P_j)h_i \neq o$, if $n > i-1$ where P_j is the orthogonal projection on $Z(g_j)$ and $\sum_1^{i} \oplus Z(g_j) \neq H$ if $n > i$. Continuing this process, in the n^{th} stage we obtain non-zero vectors $\{g_i\}_1^n$ in H such that $Z(g_i) \perp Z(g_i')$ for $i \neq i'$ and $h_i \varepsilon \sum_1^n \oplus Z(g_i)$ for $i = 1,2,..,n$. Consequently, $H = \sum_1^n \oplus Z(g_i)$.

AFFIRMATION 2. Suppose for each subspace M of finite dimensión in H, $[E(\sigma)M : \sigma \varepsilon S] \neq H$. Then there exists a countable infinite orthogonal set $\{g_i\}_1^\infty$ of non-zero vectors in H such that $\sum_1^\infty \oplus Z(g_i) = H$.

In fact, let X be a countable set such that $[E(\sigma)X : \sigma \varepsilon S] = H$. If G is the subspace spanned by X, then clearly G is a generating subspace of H. Consequently, by hypothesis G is infinite dimensional. Let $\{h_i\}_1^\infty$ be an orthonormal basis of G. Let $g_1 = h_1$. Since $Z(g_1) \neq H$, for the orthogonal projection P_1 on $Z(g_1)$, $P_1 \neq I$. Let $i_1 > 1$ be the smallest integer such that $h_{i_1} \notin Z(g_1)$. By hyphotesis such i_1 exists. Let $g_2 = (I - P_1) h_{i_1}$. Then $g_2 \neq o$, $Z(g_1) \perp Z(g_2)$ and $Z(g_1) \oplus Z(g_2) \neq H$. In the k^{th} stage there exists the least integer $i_k > i_{k-1}$ such that $h_{i_k} \notin \sum_1^k \oplus Z(g_i)$. Let P_j be the orthogonal projection on $Z(g_j)$, $j = 1,..,k$. Then $g_{k+1} = (I - \sum_1^k P_j)h_{i_k} \neq o$ and $Z(g_i) \perp Z(g_i')$

$i \neq i'$, $i, i' = 1,2,\ldots, k+1$. Continuing this process indefinitely, we obtain a sequence $\{g_n\}_1^\infty$ of non-zero vectors in H such that $h_i \in \sum_1^\infty \oplus Z(g_n)$ for all $i \in \mathbb{N}$. Thus $H = \sum_1^\infty \oplus Z(g_n)$.

By affirmations 1 and 2 there exists a countable set of non-zero vectors $\{y_i\}_{i=o}^N$, $N \in \mathbb{N} \cup \{o\}$ or $N = \infty$, such that $H = \sum_{i=o}^N \oplus Z(y_i)$. If $N = o$, then clearly $x = y_o$ serves the purpose. Therefore let $N > o$. Replacing the Borel sets by members of $\mathcal{S}$, σ_o by X and $\sigma_n = \bigcup_{j=n}^N e_j$, $n \in \mathbb{N}$, $n \leq N$ and defining $x = \sum_{n=o}^N \frac{1}{2^n} E(\sigma_n) y_n$ in the proof of Lemma X.5.7 of [3] it can be shown that the vector x satisfies the properties mentioned in the lemma.

THEOREM 2.5. H has an OSD relative to E(.) if and only if E(.) has the CGS-property in H. If H is separable, then every spectral measure E(.) in H has the CGS-property.

PROOF. By Proposition 2.2 the condition is necessary. Conversely, let E(.) have the CGS-property in H. Then as shown in the proof of Lemma 2.4 there exists a countable set $\{y_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, of non-zero vectors in H such that $H = \sum_1^N \oplus Z(y_i)$. By Lemma 2.4 and an argument similar to that in the first part of the proof of Lemma X.5.8 of [3] it can be shown that H has an OSD $H = \sum_1^k \oplus$

$Z(x_i)$, $k \in \mathbb{N} \cup \{\infty\}$. Hence the condition is also sufficient.

COROLLARY 2.6. Let S be the σ-algebra generated by a countable family of sets. Then H has an OSD relative to E(.) defined on S if and only if H is separable. Consequently, if T is a normal operator on H, then H has an OSD relative to T if and only if H is separable.

DEFINITION 2.7. Let $H_i = \sum_{j=1}^{N_i} \oplus Z(x_j^{(i)})$, $N_i \in \mathbb{N} \cup \{\infty\}$ be OSDs relative to $E_i(.)$, i=1,2. We say that these OSDs are equivalent if $N_1 = N_2$ and $\rho_1(x_j^{(1)}) \equiv \rho_2(x_j^{(2)})$ for all j.

LEMMA 2.8. Suppose $H = \sum_1^N \oplus Z(x_k)$, $N \in \mathbb{N} \cup \{\infty\}$, is an OSD of H relative to E(.). Let $\{y_i\}_{i \in J}$ be a countable set of non-zero vectors in H such that $Z(y_i) \perp Z(y_{i'})$ for $i \neq i'$. Let y_{ik} be the orthogonal projection of y_i in $Z(x_k)$; $\mu_k = \rho(x_k)$, $\nu_i = \rho(y_i)$ and $\nu_{ik} = \rho(y_{ik})$. If U_k is the isomorphism described in Lemma 1.1 with respect to x_k, let $U_k^{-1} y_{ik} = f_{ik} \in L_2(X, S, \mu_k)$. Then:

(i) $\dfrac{d\nu_i}{d\mu_1} = \sum_{k=1}^N | f_{ik} |^2 \circ \dfrac{d\mu_k}{d\mu_1}$ μ_1 - a.e for $i \in J$.

(ii) $\sum_{k=1}^N f_{ik} \overline{f_{i'k}} \circ \dfrac{d\mu_k}{d\mu_1} = 0$ μ_1 - a.e. for $i \neq i', i, i' \in J$.

PROOF. By Lemma 1.1. $\rho(y_{ik}) = || E(.) U_k f_{ik} ||^2 = || U_k \chi_{(.)} f_{ik} ||^2 = || \chi_{(.)} f_{ik} ||_2^2$ and hence

$$\rho(y_{ik})(.) = \int_{(.)} | f_{ik} |^2 \circ \frac{d\mu_k}{d\mu_1} \, d\mu_1$$

as $\mu_k << \mu_1$ by hypothesis. Therefore, $\nu_{ik} << \mu_1$ and $\frac{d\nu_{ik}}{d\mu_1} = |f_{ik}|^2 \circ \frac{d\mu_k}{d\mu_1}$ μ_1 - a.e. Since $y_i = \sum_{k=1}^{N} y_{ik}$ and $Z(x_k) \perp Z(x_{k'})$ for $k \neq k'$, it follows that $\nu_i = \rho(y_i) = \sum_{k=1}^{N} \nu_{ik}$. As $\nu_{ik} << \mu_1$ for all k, we conclude that $\nu_i << \mu_1$. Then by the monotone convergence theorem $\int_\delta (\sum_{k=1}^{N} \frac{d\nu_{ik}}{d\mu_1}) \, d\mu_1 = \sum_{k=1}^{N} \int_\delta \frac{d\nu_{ik}}{d\mu_1} \, d\mu_1 = \sum_{k=1}^{N} \nu_{ik}(\delta) = \nu_i(\delta)$ for $\delta \in \mathcal{S}$ and hence,

$$\frac{d\nu_i}{d\mu_1} = \sum_{k=1}^{N} \frac{d\nu_{ik}}{d\mu_1} = \sum_{k=1}^{N} |f_{ik}|^2 \circ \frac{d\mu_k}{d\mu_1} \quad \mu_1 \text{ - a.e.}$$

Following an argument quite similar to that on p.260 of [12] we have

$$\int_\delta (\sum_{k=1}^{N} f_{ik} \bar{f}_{i'k} \circ \frac{d\mu_k}{d\mu_1}) \, d\mu_1 = \sum_{k=1}^{N} \int_\delta f_{ik} \bar{f}_{i'k} \circ \frac{d\mu_k}{d\mu_1} \, d\mu_1 \text{ for}$$

$\delta \in \mathcal{S}$.

As $Z(y_i) \perp Z(y_{i'})$ for $i \neq i'$, it follows that

$$\begin{aligned} 0 = (E(\delta)\, y_i, y_{i'}) &= \sum_{k=1}^{N} (E(\delta) y_{ik}, y_{i'k}) \\ &= \sum_{k=1}^{N} (E(\delta) U_k f_{ik}, U_k f_{i'k}) \\ &= \sum_{k=1}^{N} (\chi_{(\delta)} f_{ik}, f_{i'k}) \\ &= \sum_{k=1}^{N} \int_\delta f_{ik} \bar{f}_{i'k} \circ \frac{d\mu_k}{d\mu_1} \, d\mu_1 \\ &= \int_\delta (\sum_{k=1}^{N} f_{ik} \bar{f}_{i'k} \circ \frac{d\mu_k}{d\mu_1}) \, d\mu_1 . \end{aligned}$$

Since δ is arbitrary in $\mathcal{S}$, this proves (ii).

THEOREM 2.9. Let E(.) have the CGS-property in H. Then any two OSDs of H relative to E(.) are equivalent.

PROOF. Let $H=\sum_{1}^{N}\oplus Z(x_i)$, $H=\sum_{1}^{N}\oplus Z(y_i)$, N, N' ε ℕ ∪ {∞}, be OSDs of H relative to E(.). We augment the sets $\{x_i\}$ and $\{y_i\}$ by the introduction of countably infinite new elements, each equal to the null element, if N or N' is finite. Then it suffices to show that $\rho(x_i) \equiv \rho(y_i)$ for all i ε ℕ, since from the hypothesis that $\rho(x_1) \gg \rho(x_2) \gg \ldots$ and $\rho(y_1) \gg \rho(y_2) \gg \ldots$ it follows that N = N'.

Let $\mu_k = \rho(x_k)$ and $\nu_k = \rho(y_k)$, k ε ℕ. By Lemma 2.8 $\mu_1 \ll \nu_1$ and $\nu_1 \ll \mu_1$ so that $\mu_1 \equiv \nu_1$. Suppose we have known that $\mu_k \equiv \nu_k$ for k= 1,2,...,n. If possible, let $\mu_{n+1}(\sigma) = o$ and $\nu_{n+1}(\sigma) > o$. Because of (i) and (ii) of Lemma 2.8 the argument on pp.261-262 of Stone [12] can be suitably modified to conclude that there exists $s_o \in \sigma$ such that

$$(1)\quad o < \left(\frac{d\nu_i}{d\mu_1}\right)(s_o) = \sum_{k=1}^{n}\left(|f_{ik}|^2 \circ \frac{d\mu_k}{d\mu_1}\right)(s_o),\quad i=1,2,..,n+1$$

and

$$(2)\quad o = \sum_{k=1}^{n}\left(f_{ik}\,\overline{f}_{i'k} \circ \frac{d\mu_k}{d\mu_1}\right)(s_o),\quad i \neq i',\ i,i'=1,2,..,n+1.$$

Then the vectors $X_i = \left(f_{ik}(s_o)\left(\frac{d\mu_k}{d\mu_1}(s_o)\right)^{\frac{1}{2}}\right)_{k=1}^{n}$, i=1,2,..,n+1 are n+1 non-zero vectors in $\mathbb{C}^n$ by (1) and they are mutually orthogonal by (2). This contradiction proves that $\nu_{n+1} \ll$

μ_{n+1}. By symmetry, $\mu_{n+1} << \nu_{n+1}$. The proof is complete by induction.

The above theorem justifies the following terminology.

DEFINITION 2.10. If E(.) has the CGS-property in H, let $H = \sum_{1}^{N} \oplus Z(x_i)$ be an OSD of H relative to E(.). Then N is referred to as the OSD-multiplicity of E(.). When $N = \infty$, we say the E(.) has the OSD-multiplicity $\aleph_0$. If E(.) is the resolution of the identity of a normal operator T on H (necessarily separable) then the OSD-multiplicity of T is defined as that of E(.).

The following proposition is obvious.

PROPOSITION 2.11. If $E_1(.)$ and $E_2(.)$ are unitarily equivalent and if one of them has the CGS-property, then the other too has the CGS-property.

THEOREM 2.12. Let $E_1(.)$ and $E_2(.)$ have the CGS-property in H_1 and H_2 respectively. Then $E_1(.)$ and $E_2(.)$ are unitarily equivalent if and only if any two OSDs of H_1 and H_2 relative to $E_1(.)$ and $E_2(.)$ respectively are equivalent.

PROOF. Suppose U is an isomorphism from H_1 onto H_2 such that $UE_1(.)U^{-1} = E_2(.)$. Let $H_1 = \sum_{j=1}^{N_1} \oplus Z_1(x_j)$ (α) and $H_2 = \sum_{j=1}^{N_2} \oplus Z_2(y_j)$ (β) be OSDs relative to $E_1(.)$ and $E_2(.)$ respectively. If $w_j = Ux_j$ then clearly $H_2 = \sum_{j=1}^{N_1} \oplus Z_2(w_j)$ (γ)

and $\rho_2(w_j) = \rho_1(x_j)$ for all j. Consequently, (γ) is an OSD of H_2 relative to $E_2(.)$. Then by Theorem 2.9, $N_1 = N_2$ and $\rho_2(w_j) \equiv \rho_2(y_j)$ for all j. From this it follows that (α) and (β) are equivalent.

Conversely, suppose the OSDs (α) and (β) given abve are equivalent. Then $N_1 = N_2$ and $\rho_1(x_j) \equiv \rho_2(y\ )$ for all j.Now by Lemma 1.2 there exists an isomorphism U_j from $Z_1(x_j)$ onto $Z_2(y_j)$ such that $U_j E_1(.) U_j^{-1} = E_2(.)$. If $U = \sum_1^N \oplus U_j$, then clearly U is an isomorphism from H_1 onto H_2 such that $UE_1(.)U^{-1} = E_2(.)$.

COROLLARY 2.13. Let H_1, H_2 be separable Hilbert spaces. If T_i, i= 1,2 are normal operators on H_i, then T_1 and T_2 are unitarily equivalent if and only if any two OSDs of H_1 and H_2 relative to T_1 and T_2 respectively are equivalent.

PROOF. This is immediate from the above theorem and Lemma 1.3.

3.- A GENERALIZATION OF HELLINGER'S THEOREM.

Making use of Theorem 2.12 we obtain an extension of Theorem 7.7 of Stone [12] to spectral measures with the CGS-property which are defined on the σ-algebra $\mathcal{B}(X)$ of a Hausdorff topological space X. Since the original version of the said theorem in [12] goes back to Hellinger, our extension is referred to as the generalized Hellinger's theorem.

DEFINITION 3.1. Let X be a Hausdorff topological space. Let E(.) be a spectral measure on $\mathcal{B}(X)$ with the CGS-property in H. The discrete part p_E of E(.) is the set $\{t \in X : E(\{t\}) \neq o\}$ and the continuos part c_E of E(.) is the set $X \setminus p_E$. $\mathfrak{M}(E)$ denotes the subspace $E(p_E)H$ and $\mathfrak{N}(E)$ denotes $E(c_E)H$. For spectral measures $E_i(.)$ on $\mathcal{B}(X)$ the corresponding subspaces will be denoted by $\mathfrak{M}(E_i)$ and $\mathfrak{N}(E_i)$.

Unless otherwise stated all the spectral measures will be assumed to have the CGS-property throughout the rest of this article. In this section E(.), $E_1(.)$, $E_2(.)$will have domain $\mathcal{B}(X)$, X a Hausdorff space.

PROPOSITION 3.2. If $p_E \neq \emptyset$, then there exists a countable orthonormal set $\{y_\alpha\}_{\alpha \in J}$ such that $\mathfrak{M}(E) = \sum_{\alpha \in J} \oplus Z(y_\alpha)$.

PROOF. If $t \in p_E$ and $x \in E(\{t\})H$, clearly $E(\sigma)x = o$ if $t \notin \sigma$ and $E(\sigma)x = x$ if $t \in \sigma$. Therefore, $[x] = Z(x)$. Since E(.) has the CGS-property in H, it follows that for each $t \in p_E$, $E(\{t\})H$ has at most dimension $\aleph_o$ and further, p_E is itself countable. From the countable additivity of E(.) it follows that $\sum_{t \in p_E} \oplus E(\{t\})H = E(p_E)H = \mathfrak{M}(E)$. From this the proposotion is immediate.

PROPOSITION 3.3. Let $p_E \neq \emptyset$, $E(c_E) \neq o$. Then there exist countable orthonormal sets $\{x_i\}_1^N$ and $\{y_i\}_1^{N'}$, $N, N' \in \mathbb{N} \cup \{\infty\}$, such that

(i) $\mathfrak{M}(E) = \sum_1^N \oplus Z(x_i)$

and

(ii) $\mathfrak{N}(E) = \sum_1^{N'} \oplus Z(y_i)$ is an OSD of $\mathfrak{N}(E)$ relative to $E(.)E(c_E)$.

PROOF.

(i) Is immediate from Proposition 3.2.

(ii) Follows from Theorem 2.5.

The following theorem is a generalization of Theorem 7.7 of [12].

THEOREM 3.4. (GENERALIZED HELLINGER'S THEOREM). For the spectral measures $E_1(.)$ and $E_2(.)$ on $\mathcal{B}(X)$ with the CGS-property in H_1 and H_2 respectively, let $\{x_i^{(j)}\}_{i=1}^{N_j}$ and $\{y_i^{(j)}\}_{i=1}^{N'_j}$ be the orthonormal sets respect to $E_j(.)$ as described in Proposition 3.3, j= 1,2, where some of these sets can be absent if $\mathfrak{M}(E_j)$ or $\mathfrak{N}(E_j)$ is the null vector, j= 1,2. Then $E_1(.)$ and $E_2(.)$ are unitarily equivalent if and only if all the following conditions hold

(i) $p_{E_1} \neq \emptyset$ if and only if $p_{E_2} \neq \emptyset$ and $E_1(c_{E_1}) \neq o$ if and only if $E_2(c_{E_2}) \neq o$.

(ii) There exists a biyective map Φ from $\{x_k^{(1)}\}_{k=1}^{N_1}$ onto

$\{x_k^{(2)}\}_1^{N_2}$ such that $x_k^{(1)}$ and $\Phi(x_k^{(1)})$ belong to $E_1(\{t\})H_1$ and $E_2(\{t\})H_2$ respectively for some $t \in p_{E_1}$, if $p_{E_1} \neq \emptyset$.

(iii) If $E_1(c_{E_1}) \neq o$, then $N_1' = N_2'$ and $\rho_1(y_k^{(1)}) \equiv \rho_2(y_k^{(2)})$ for all k.

PROOF. If there exists an isomorphism U from H_1 onto H_2 such that $UE_1(.)U^{-1} = E_2(.)$ it is easy to verify (i) and (ii). By Theorem 2.12 and by the fact that $E_i(.)E_i(c_{E_i})$ are unitarily equivalent spectral measures on $E_i(c_{E_i})H_i$, i= 1,2, the condition (iii) holds.

Conversely, let (i), (ii) and (iii) hold. If $\mathfrak{M}(E_1) \neq o$, then by (i) $\mathfrak{M}(E_2) \neq o$ and by (ii) and Proposition 3.3 there exists an isomorphism U_1 from $\mathfrak{M}(E_1)$ onto $\mathfrak{M}(E_2)$ such that $U_1 x_k^{(1)} = \Phi\, x_k^{(2)}$. From the discrete nature of p_{E_i} it follows that $U_1 E_1(\sigma)x = E_2(\sigma)U_1 x$, $x \in \mathfrak{M}(E_1)$. If $E_1(c_{E_1}) \neq o$, then by (i) $\mathfrak{N}(E_2) \neq o$. Now by Theorem 2.12 and (iii) there exists an isomorphism, U_2 from $\mathfrak{N}(E_1)$ onto $\mathfrak{N}(E_2)$ such that.

$$U_2 E_1(.)\, E_1(c_{E_1}) = E_2(.)E_2(c_{E_2})U_2$$

If $U = U_1 \oplus U_2$, then U is an isomorphism from H_1 onto H_2 such that $UE_1(.)U^{-1} = E_2(.)$.

COROLLARY 3.5. Let T_i be normal operators on H_i, H_i separable, with the corresponding resolutions of the identify $E_i(.)$, i= 1,2. Then T_1 and T_2 are unitarily equivalent if and only if the conditions (i), (ii) and (iii) of Theorem 3.4 hold for $E_1(.)$ and $E_2(.)$.

NOTE 3.6. Corollary 3.5. is the generalization of Theorem 7.7. of [12] to normal operators.

4.- **MULTIPLICITY FUNCTIONS.** Because of the availablity of Theorem 3.4 we are able to generalize Definition 5.2 of Stone [12] to spectral measures E(.) on $\mathcal{B}(X)$. Also we introduce multiplicity functions m_p and m_c on X with respect to p_E and c_E respectively and study some of their elementary properties. X is again a Hausdorff space.

DEFINITION 4.1. The multiplicity function m_p on X relative to p_E is defined by

$$m_p(t) = \begin{cases} o \text{ if } t \notin p_E \\ \dim E(\{t\})H, \text{ if } t \in p_E \end{cases}.$$

When $E(\{t\})H$ is infinite dimensional we say $m_p(t) = \aleph_o$. When E(.) is the resolution of the identify of a normal operator T on a separable Hilbert space H, the function m_p on $\mathcal{C}$ is called the multiplicity function relative to the point spectrum of T.

NOTE 4.2. Since $E(\{t\})H$ is of countable dimension for

$t \in p_E$, $m_p(t)$ cannot assume a value greater than $\aleph_o$.

Theorem 3.4 can be reformulated as follows for its subsequent applications.

THEOREM 4.3. Let $m_p^{(j)}$ be the multiplicity function relative to p_{E_j}, j= 1,2. Then $E_1(.)$ and $E_2(.)$ on $\mathcal{B}(X)$ are unitarily equivalent if and only if

(i) $m_p^{(1)} = m_p^{(2)}$

(ii) Any two OSDs of $\mathfrak{M}(E_1)$ and $\mathfrak{M}(E_2)$ relative to $E_1(.)E(c_{E_1})$ and $E_2(.)E_2(c_{E_2})$ respectively are equivalent.

DEFINITION 4.4. The element $t= t_o$ in X is called (i) a point of constancy of E(.) if there exists an open set U containing t_o such that E(U) = o; (ii) a point of continuity of E(.) if $E(\{t_o\})$= o and for every open set U containing t_o, $E(U) \neq o$ and (iii) a point of discontinuity of E(.) if $E(\{t_o\}) \neq o$. The set of all points of continuity of E(.) is denoted by $\mathcal{C}_E$; that of all points of discontinuity of E(.) by $\mathcal{P}_E$ and that of all points of constancy by ρ_E.

NOTE 4.5. If E(.) is the resolution of the the identity of a self-adjoint operator T on a separable Hilbert space H, then $\lambda = \lambda_o$ in $\mathbb{R}$ is a point of constancy (respectivively, a point of continuity, a point of discontinuity) of E(.) if and anly if it is so with respect to $E_\lambda = E((-\infty, \lambda])$ in the sense of Definition 5.2 of [12].

PROPOSITION 4.6. Let X be a locally compact Hausdorff space. If E(.) is a regular spectral measure on $\mathcal{B}(X)$ with its spectrum $\Lambda(E)$ (see Definitions 15 and 17 of [2]), then

(i) $p_E = \mathcal{P}_E$;

(ii) $\rho_E = X \setminus \Lambda(E)$;

(iii) $\mathcal{C}_E = \Lambda(E) \setminus p_E$.

Consequently, when E(.) is the resolution of the identity of a normal operator T on separable Hilbert space H, then $p_E = \sigma_p(T)$, $\rho_E = \rho(T)$ and $\mathcal{C}_E = \sigma_c(T)$.

PROOF. Since $E(\{t\}) \neq o$ if and only if $t \in p_E$, (i) holds. By Theorem 2.3 of [2], $E(X \setminus \Lambda(E)) = o$. As $\Lambda(E)$ is closed it follows that $X \setminus \Lambda(E) \subset \rho_E$. On the other hand, if $t \in \rho_E$ then there exists an open neighbourhood U of t such that $E(U) = o$. Thus by Definition 17 of [2], $U \subset X \setminus \Lambda(E)$ and hence (ii) hods. By the definition of $\mathcal{C}_E$ and Theorem 2.3 of [2], $\mathcal{C}_E \subset \Lambda(E)$. But, for $t \in \mathcal{C}_E$, $E(\{t\}) = o$ and hence $\mathcal{C}_E \subset \Lambda(E) \setminus p_E$. Conversely, if $t \in \Lambda(E) \setminus p_E$, then $E(\{t\}) = o$ and for every open neighbourhood U of t, $E(U) \neq o$ by the definition of $\Lambda(E)$. Hence $t \in \mathcal{C}_E$.

Clearly, $p_E = \sigma_p(T)$ and E(.) is regular. As $\sigma_r(T) = \emptyset$ and $\Lambda(E) = \sigma(T)$, the second part follows from the first.

Motivated by Definition 7.1. of [12] we give the following

DEFINITION 4.7. If $\mathfrak{M}(E) = \sum_{1}^{N} \oplus Z(y_i)$ is an OSD of $\mathfrak{M}(E)$ relative to $E(.)E(c_E)$, then the multiplicity function m_c relative to c_E is defined on X as follows:

$m_c(t) = o$ if $\mathfrak{M}(E) = o$ or if $\mathfrak{M}(E) \neq o$, and there exists an open neighbourhood U of t such that $E(U)y_1 = o$; $m_c(t) = n \in \mathbb{N}$ if y_k exists for $k = 1,2,\ldots,n$ and for every open neighbourhood U of t, $E(U)y_k \neq o$ for $k = 1,2,\ldots,n$ while $N = n$ or y_{n+1} exists and $E(U)y_{n+1} = o$ for some open neighbourhood U of t; $m_c(t) = \aleph_o$ if $N = \infty$ and for every open neighbourhood U of t, $E(U)y_k \neq o$ for all $k \in \mathbb{N}$.

Since $\mathfrak{M}(E)$ is invariant with respect to $E(.)$, by Theorem 2.9 any two OSDs of $\mathfrak{M}(E)$ relative to $E(.)E(c_E)$ are equivalent and hence m_c is well-defined. When $E(.)$ is the resolution of the identity of a self-adjoint operator T on a separable Hilbert space H, clearly m_c coincides with the multiplicity function given in Definition 7.1 of [12].

DEFINITION 4.8. If $E(.)$ is the resolution of the identity of a normal operator T on a separable Hilbert space H, then m_c is called the multiplicity function of T relative to its continuous spectrum.

By Proposition 4.6, $E(c_E) = E(\sigma_c(T))$ and hence we are justified in using the above terminology. Clearly, the multiplicity functions m_p and m_c are unitarily invariant.

PROPOSITION 4.9. Let X be a locally compact Hausdorff space. For a regular spectral measure E(.) on $\mathcal{B}(X)$ the following assertions hold:

(i) $p_E = \{ t \in X: m_p(t) > o \}$.

(ii) $\{t \in X: m_p(t) = o \text{ and } m_c(t) > o\} \subset \Lambda(E) \setminus p_E$.

(iii) $X \setminus \Lambda(E) \subset \{ t: m_p(t) = o = m_c(t) \} \subset (X \setminus \Lambda(E)) \cup (\bar{p}_E \setminus p_E)$ (In the above, $\Lambda(E)$ is the spectrum of E(.)).

Consequently, if E(.) is the resolution of the identity of a normal operator T on a separable Hilbert space H, then (i), (ii) and (iii) hold if we replace p_E, $\Lambda(E)$ and $X \setminus \Lambda(E)$ by $\sigma_p(T)$, $\sigma(T)$ and $\rho(T)$ respectively.

PROOF. Let $\mathfrak{M}(E) = \sum_{1}^{N} \oplus Z(y_i)$ be an OSD of $\mathfrak{M}(E)$ relative to $E(.)E(c_E)$. (i) is obvious. If $m_p(t) = o$ and $m_c(t) > o$, then there exists an open neighbourhood U of t such that $E(U)y_1 \neq o$ and hence by Theorem 2.3. of [2], $t \in \Lambda(E)$ and $t \notin p_E$. Thus (ii) holds. Since $\Lambda(E)$ is closed, $X \setminus \Lambda(E)$ is open. By Theorem 23 of [2], $E(X \setminus \Lambda(E)) = o$ and hence, $X \setminus \Lambda(E) \subset \{ t: m_p(t) = o = m_c(t) \}$. If $m_p(t) = o = m_c(t)$, then clearly $t \in X \setminus \Lambda(E)$ whenever $\mathfrak{M}(E) = o$. Suppose $\mathfrak{M}(E) \neq o$. Then there exists an open neighbourhood U of t such that $E(U)y_1 = o$. If there exists an open neighbourhood V of t such that $E(V) = o$, then $t \in X \setminus \Lambda(E)$ by Proposition 4.6. On the other hand, if, for every open neighbourhood V of t, $E(V \cap U) \neq o$, then as $E(V \cap U)y_1 = o$ it follows that $E(V \cap U)\mathfrak{M}(E) = o$ and therefore, $E(V \cap U)\mathfrak{M}(E) \neq o$. This proves that $V \cap U \cap p_E \neq \emptyset$ so that $t \in \bar{p}_E \setminus p_E$.

The last part is obvious from the first and Proposition 4.6.

5.- **UNITARY INVARIANTS OF SPECTRAL MEASURES ON PRODUCT SPACES.** Suppose X_1 and X_2 are Hausdorff topological spaces and E(.) is a spectral measures on $\mathcal{B}(X_1) \times \mathcal{B}(X_2)$ with the CGS-property in H. When E(.) satisfies some additional properties, we obtain a complete set of unitary invariants of E(.) in terms of the induced spectral measures $E_{X_1}(.)$ and $E_{X_2}(.)$ on $\mathcal{B}(X_1)$ and $\mathcal{B}(X_2)$ respectively. As a consequence of this study we obtain a complete set of unitary invariantes for certain class of normal operators on separable Hilbert spaces.

Let $\mathcal{S}_i$ be a σ-algebra of subsets of X_i, i=1,2. E(.) will be assumed to be defined on the σ-algebra $\mathcal{S}_1 \times \mathcal{S}_2$. No topological properties on X_i are assumed unless otherwise stated.

NOTATIONS 5.1. $E_{X_1}(.): \mathcal{S}_1 \to \mathcal{B}(H)$ is defined by $E_{X_1}(\sigma) = E(\sigma \times X_2)$. $E_{X_2}(.): \mathcal{S}_2 \to \mathcal{B}(H)$ is defined by $E_{X_2}(\sigma) = E(X_1 \times \sigma)$.

Clearly, $E_{X_i}(.)$ is a spectral measure on $\mathcal{S}_i$. When E(.)

is the resolution of the identity of a normal operator T on H then as $\mathcal{B}(\mathbb{R}^2) = \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R})$, we define the spectral measure $\hat{E}(.)$ on $\mathcal{B}(\mathbb{R}^2)$ by $\hat{E}(\sigma) = E(\{u + iv : (u,v) \in \sigma\})$ and the spectral measures $E_R(.)$ and $E_I(.)$ as below:

$$E_R(\sigma) = \hat{E}(\sigma \times \mathbb{R})$$

$$E_I(\sigma) = \hat{E}(\mathbb{R} \times \sigma), \quad \sigma \in \mathcal{B}(\mathbb{R}).$$

Then ReT and ImT have their resultions of the identity $E_R(.)$ and $E_I(.)$ respectively.

For a spectral measure $E(.)$ on $S_1 \times S_2$ and for a vector $w \in H$ we define

$$R^w_{X_i} = \{ e \in S_i : E_{X_i}(e)w = o \}, \ i = 1,2.$$

In the case of two spectral measures $E^{(1)}_X(.)$ and $E^{(2)}_X$ o $S_1 \times S_2$ we define similarly $R^{w_i}_{X_i,j} = \{ e \in S_i : E^{(j)}_{X_i}(e)w_i = o \}$, i= 1,2, j=1,2. By the countable additivity of $E_{X_i}(.)$ evidently $R^w_{X_i}$ are σ-rings.

DEFINITION 5.2. A vector w in H is said to be well-behaving with respect to $E(.)$ on $S_1 \times S_2$ if, given $\sigma \in S_1 \times S_2$ with $E(\sigma)w = o$, there exists $\delta \in \Sigma_w = S(C_w)$ such that $\sigma \subset \delta$, where $C_w = \{ A \subset X_1 \times X_2 : A \in R^w_{X_1} \times S_2$ or $A \in S_1 \times R^w_{X_2} \}$ and $S(C_w)$ is the σ-ring generated by C_w. When $E(.)$ is the resolution of the identity of a normal operator T on a separable Hilbert space H, tehn w is said to be well-behaving relative to T if it is so relative to $\hat{E}(.)$.

PROPOSITION 5.3. Let $E^{(1)}(.)$ and $E^{(2)}(.)$ be spectral measures on $S_1 \times S_2$. If U is an isomorphism from H_1 onto H_2 such that $UE^{(1)}(.)U^{-1} = E^{(2)}(.)$, then a vector w in H_1 is well-behaving relative to $E^{(1)}(.)$ if and only if Uw is so relative to $E^{(2)}(.)$.

PROOF. It is easy to verify that $C_w^{(1)} = C_{Uw}^{(2)}$, from which the result follows.

PROPOSITION 5.4. If E(.) is a spectral measure on $S_1 \times S_2$ and if for $\sigma_o \in S_1 \times S_2$ there exist a vector w in H and a set $\delta_o \in \Sigma_w$ such that $\sigma_o \subset \delta_o$ then $E(\sigma_o)w = o$.

PROOF. $R_{X_i}^w$ are σ-rings contained in S_i, i=1,2. Besides, for $e \in R_{X_1}^w$ and $\sigma \in S_2$

$$E(e \times \sigma)w = E_{X_2}(\sigma)E_{X_1}(e)w = o.$$

Then by the additivity of E(.) we have $E(\sigma)w=o$ for σ in the ring R_1 generated by the semi-ring $\{e \times \delta : e \in R_{X_1}^w, \delta \in S\}$. Let $M=\{\sigma \in R_{X_1}^w \times S_2 : E(\sigma)w = o\}$. Then $R_1 \subset M \subset R_{X_1}^w \times S_2$. If $\{\sigma_n\}$ is a monotone sequence in M with $\sigma = = \lim \sigma_n$, then $\| E(\sigma)w \|^2 = \lim_n \|E(\sigma_n)w\|^2 = o$ so that

Let $E^{(1)}(.)$ and $E^{(2)}(.)$ be well-behaving in H_1 and H_2 respectively. Let $\sum_{k=1}^{N_j} \oplus Z_j(y_k^{(j)})$ be an OSD of $\mathfrak{M}(E^{(j)})$, relative to $E^{(j)}(.)$ $E^{(j)}(c_{E^j})$, j= 1,2. Then $E^{(1)}(.)$ and $E^{(2)}(.)$ are unitarily equivalent if and only if

(i) $m_p^{(1)} = m_p^{(2)}$; $m_c^{(1)} = m_c^{(2)}$ where $m_p^{(j)}$, $m_c^{(j)}$ are the multiplicity functions of $E^{(j)}(.)$, j= 1,2.

(ii) $\rho_{E_{X_1}^{(1)}}(y_k^{(1)}) \equiv \rho_{E_{X_1}^{(2)}}(y_k^{(2)})$; $\rho_{E_{X_2}^{(1)}}(y_k^{(1)}) \equiv \rho_{E_{X_2}^{(2)}}(y_k^{(2)})$

for all those k for which they are significant.

Consequently, if $E^{(j)}(.)$ is the resolution of the identity of the well-behaving normal operator T_j in H_j (separable) j=1,2 then T_1 and T_2 are unitarily equivalent if and only if (i) and (ii) hold, where $E_{X_1}^{(j)}(.)$ and $E_{X_2}^{(j)}(.)$ are replaced by $E_R^{(j)}(.)$ and $E_I^{(j)}(.)$ respectively and $E^{(j)}(.)$ by $\hat{E}^{(j)}(.)$ for j= 1,2.

PROOF. From the proof of Theorem 3.4 it is clear that there exists an isomorphism U_1 from $\mathfrak{M}(E^{(1)})$ onto $\mathfrak{M}(E^{(2)})$ such that $U_1 E^{(1)}(.) U_1^{-1} = E^2(.)$ if and only if $m_p^{(1)} = m_p^{(2)}$. In the light of 2.12 there exists an isomorphism U_2 from $\mathfrak{N}(E^{(1)})$ onto $\mathfrak{N}(E^{(2)})$ such that $U_2 E^{(1)}(.) U_2^{-1} = E^{(2)}(.)$ if and only if $N_1 = N_2$ and $\rho_{E^{(1)}}(y_k^{(1)}) \equiv \rho_{E^{(2)}}(y_k^{(2)})$ for all k. Thus it suffices to show that $N_1 = N_2$ and $\rho_{E^{(1)}}(y_k^{(1)}) \equiv \rho_{E^{(2)}}(y_k^{(2)})$ for all k if and only if $m_c^{(1)} = m_c^{(2)}$ and (ii) holds. Clearly, $m_c^{(1)} = m_c^{(2)}$ implies that $N_1 = N_2$. If (ii) holds, let $E^{(1)}(\sigma)y_k^{(1)} = o$. As $E^{(1)}(.)$ is well-behaving, by Proposition 5.6 $y_k^{(1)}$ are well-behaving relative to $E^{(1)}(.)$ and hence there exists $\delta \in \Sigma^{(1)}_{y_k^{(1)}}$ such that $\sigma \subset \delta$.

On the other hand, as (ii) holds, $E^{(1)}_{X_1}(e)\ y_k^{(1)} = o \Longleftrightarrow E^{(2)}_{X_1}(e)\ y_k^{(2)} = o$ and hence $R^{y_k^{(1)}}_{X_1,1} = R^{y_k^{(2)}}_{X_1,2}$. Similarly, $R^{y_k^{(1)}}_{X_2,1} \doteq R^{y_k^{(2)}}_{X_2,2}$. Consequently, $C^{(1)}_{y_k^{(1)}} = C^{(2)}_{y_k^{(2)}}$ and hence $\Sigma^{(1)}_{y_k^{(1)}} = \Sigma^{(2)}_{y_k^{(2)}}$.

Thus $\delta \in \Sigma^{(2)}_{y_k^{(2)}}$ and therefore, by Proposition 5.4. $E^{(2)}(\sigma)y_k^{(2)} = o$ so that $\rho_{E^{(2)}}(y_k^{(2)}) << \rho_{E^{(1)}}(y_k^{(1)})$. By symmetry, $\rho_{E^{(1)}}(y_k^{(1)}) << \rho_{E^{(2)}}(y_k^{(2)})$. Hence $\rho_{E^{(1)}}(y_k^{(1)}) \equiv \rho_{E^{(2)}}(y_k^{(2)})$ for all k. The converse is easy to prove and the details are omitted.

NOTE 5.8. If T_i are self-adjoint in the above Theorem , clearly they are well-behaving and condition (ii) is the same as $\rho_{E^{(1)}}(y_k^{(1)}) \equiv \rho_{E^{(2)}}(y_k^{(2)})$ for those k for which they are significant.

6.- UNITARY INVARIANTS OF SPECTRAL MEASURES WITH PROPERLY INTERTWINED DISCRETE PARTS.

The object of this section is to give a generalization of Theorem 7.8 of [12] to suitably restricted spectral measures on $\mathcal{B}(\mathbb{R}^2)$ and consequently, to certain class of normal operators on separable Hilbert spaces.

DEFINITION 6.1. Let E(.) be a spectral measure on $\mathcal{B}(\mathbb{R}^2)$ with the CGS-property in H. Let $E_{(1)}(\sigma) = E(\sigma \times \mathbb{R})$ and $E_{(2)}(\delta) = E(\mathbb{R} \times \delta)$, $\sigma, \delta \in \mathcal{B}(\mathbb{R})$. We say that E(.) has properly intertwined discrete part if E(.) is well-behaving in H and

$$\mathfrak{M}(E) = [\mathfrak{M}(E_{(1)}), \mathfrak{M}(E_{(2)})].$$

If T is normal on H, H separable, with the resolution of the identity E(.) and is well-behaving in H then we say that T has properly intertwines point spectrum if

$$\mathfrak{M}(E) = [\mathfrak{M}(E_R), \mathfrak{M}(E_I)].$$

EXAMPLES 6.2.

(i) Every self-adjoint operator on a separable Hilbert space has properly intertwined point spectrum.

(ii) If X is locally compact, Hausrdorff and second countable, then for a finite measure μ on $\mathcal{B}(X)$ such that $\mu(\{t\}) = o$ for all $t \in X$, the multiplication operator M_g on $L_2(X, \mathcal{B}(X), \mu)$ for $g \in L_\infty(X, \mathcal{B}(X), \mu)$ has properly intertuined point spectrum in case M_g is well-behaving.

PROPOSITION 6.3. A well-behaving spectral measure $E(.)$ on $\mathcal{B}(\mathbb{R}^2)$ with $p_E \neq \emptyset$ has properly intertwined point spectrum if and only if

(i) $E_{(1)}(\{t\}) = E\{(t,u): (t,u) \in p_E\}$ for $t \in p_{E_{(1)}}$

and

(ii) $E_{(2)}(\{u\}) = E\{(t,u): (t,u) \in p_E\}$ for $u \in p_{E_{(2)}}$

hold.

PROOF. The conditions are necessary. In fact, because of symmetry, it suffices to verify(i). As $p_E \neq \emptyset$ and $E(.)$ has the CGS-property in H, let $p_E = \{(t_j,u_j): j \in J\}$, J countable. For $t \in p_{E_{(1)}}$, by hypoyhesis, $E_{(1)}(\{t\})H \subset E(p_E)H$. Therefore, $E_{(1)}(\{t\}) = \sum_{j \in J} E_{(1)}(\{t\})E\{(t_j, u_j)\} = \sum_{j \in J} E(t \times \mathbb{R})E\{(t_j,u_j)\} = E\{(t_j,u_j): t_j = t\}$.

Conversely, suppose (i) and (ii) hold. If $(u,v) \in p_E$, then $E\{(u,v)\}H \subset E(u \times \mathbb{R})H = E_{(1)}(\{u\})H$. Thus $u \in p_{E_{(1)}}$. Similarly, $v \in p_{E_{(2)}}$ and $E\{(u,v)\}H \subset E_{(2)}(\{v\})H$. Consequently, $\mathfrak{M}(E) \subset [\mathfrak{M}(E_{(1)}), \mathfrak{M}(E_{(2)})]$ since p_E is countable.

On the other hand, by (i)

$$\mathfrak{M}(E_{(1)}) = \sum_{u \in p_{E_{(1)}}} \oplus\, E_{(1)}(\{u\})H \subset E(p_E)H.$$

Similarly, by (ii)

$$\mathfrak{M}(E_{(2)}) \subset E(p_E)H.$$

Hence the conditions are also sufficient.

COROLLARY 6.4. If T is a well-behaving normal operator on a separable Hilbert space with $\sigma_p(T) \neq \emptyset$ and with the resolution of the identity E(.), then T has properly intertwined point spectrum if and only if

(i) $E_R(\{t\}) = E(\lambda \in \sigma_p(T): \text{Re}\lambda = t)$ for $t \in \sigma_p(\text{ReT})$; and

(ii) $E_I(\{u\}) = E(\lambda \in \sigma_p(T): \text{Im}\lambda = u)$ for $u \in \sigma_p(\text{ImT})$

PROPOSITION 6.5. For a spectral measure E(.) on $\mathcal{B}(\mathbb{R})$ with the CGS-property in H, the function $||E_\lambda \; x \; ||^2$ is continuous and non-decreacing in $\mathbb{R}$ for $x \in \mathfrak{N}(E)$.

PROOF. We shall prove only the continuity of $||E_\lambda \; x \; ||^2$. Since $||E(.)x \; ||^2$ is countably additive on $\mathcal{B}(\mathbb{R})$, clearly $E_\lambda x$ is continuos on the right. For $x \in \mathfrak{N}(E)$ if $||E_\lambda x||^2$ is nor continuos at λ_o then $E(\{\lambda_o\})x = E_{\lambda_o}x - E_{\lambda_o^-}x \neq o$ so that $\lambda_o \in p_E$ and $x \in \mathfrak{M}(E)$. This contradiction proves the continuity of $||E_\lambda x \; ||^2$.

THEOREM 6.6. Suppose $E^{(1)}(.)$ and $E^{(2)}(.)$ are sepectral measures on $\mathcal{B}(\mathbb{R}^2)$ with properly intertwined discrete parts. Let $m_p^{(j)}$, $m_c^{(j)}$ be the associated multiplicity functions of $E^{(j)}(.)$, j= 1,2. Let $\sum_{k=1}^{N_j} \oplus Z(y_k^{(j)})$ be an OSD of $\mathfrak{N}(E^{(j)})$ relative to $E^{(j)}(.)E(c_{E^{(j)}})$, j= 1,2.

Then:

(i) $F^{(j)}_{k,(1)}(\lambda) = ||E^{(j)}_{(1)}((-\infty, \lambda])y^{(j)}_k||^2$ and $F^{(j)}_{k,(2)}(\lambda) = ||E^{(j)}_{(2)}((-\infty, \lambda])y^{(j)}_k||^2$ are continuos non-decreasing real functions on $\mathbb{R}$ for j= 1,2 and for those k for which they are significant.

(ii) Let $f^{(j)}_{k,(1)}$ and $f^{(j)}_{k,(2)}$ be the real non-decreasing functions on [o,1] given by

$$f^{(j)}_{k,(1)}(x) = F^{(j)}_{k,(1)}(\lambda) \quad \text{for } x = F^{(j')}_{k,(1)}(\lambda) \;;$$

$$f^{(j)}_{k,(2)}(x) = F^{(j)}_{k,(2)}(\lambda) \quad \text{for } x = F^{(j')}_{k,(2)}(\lambda)$$

for $j, j' \in \{1,2\}$, $j \neq j'$ whenever k is the value for which they are significant. Then $E^{(1)}(.)$ and $E^{(2)}(.)$ are unitarily equivalent if and only if

(a) $m^{(1)}_p = m^{(2)}_p$; $m^{(1)}_c = m^{(2)}_c$;

(b) $f^{(j)}_{k,(1)}$, $f^{(j)}_{k,(2)}$ are continuos such that

$$\int_o^1 f^{(j)}_{k,(1)}\, dx = 1 = \int_o^1 f^{(j)}_{k,(2)}\, dx, \; j = 1,2.$$

PROOF. By the hypothesis that $E^{(j)}(.)$ have properly intertwined discrete parts and by Proposition 6.5, evidently (i) holds. The necessity of (a) and (b) can be proved as on p.273 of [12] once we note that $\rho_{E^{(1)}}(y^{(1)}_k) \equiv \rho_{E^{(2)}}(y^{(2)}_k)$ implies that $\rho_{E^{(1)}_{(j)}}(y^{(1)}_k) \equiv \rho_{E^{(2)}_j}(y^{(2)}_k)$ j= 1,2. The argument on p.274 of [12] can be used to prove that (a) and (b) imply

$$\rho_{E_j^{(1)}}(y_k^{(1)}) \equiv \rho_{E_j^{(2)}}(y_k^{(2)}), \quad j = 1,2$$

for all k. This together with Theorem 5.7 shows that the conditions are also sufficient.

COROLLARY 6.7. Let T_i be normal operators on separable Hilbert spaces H_i with properly intertwined point spectra, and the corresponding resolutions of the identity $E^{(i)}(.)$, $i = 1,2$. Then T_1 and T_2 are unitarily equivalent if and only if conditions (a) and (b) of Theorem 6.6(ii) hold when we replace $E^{(j)}_{(1)}(.)$ by $E^{(j)}_R$ and $E^{(j)}_{(2)}(.)$ by $E^{(j)}_I(.)$.

7.- **TOTAL MULTIPLICITY OF SPECTRAL MEASURES.** The concepts of total multiplicity given for self-adjiont operators in [1] is extended to spectral measures and is shown that for a spectral measure E(.) with the CGS-property in H its OSD-multiplicity given in Section 2 coincides with its total multiplicity.

DEFINITION 7.1. A subspace G of H is called a generating subspace of E(.) if $[E(\sigma)g : \sigma \in S, g \in G] = H$. If E(.) has a finite dimensional generating subspace in H, then the minimum dimension of all the generating subspace of E(.) is called the total multiplicity of E(.). If E(.) has no generating subspace of finite dimension and if there exists a generating subspace of dimension $\aleph_0$, then the total multiplicity of E(.) is said to be $\aleph_0$. In all other cases the total multiplicity of E(.) is said to be uncountably infinite. If E(.) is the resolution of the identity of a normal operator T, then the total multiplicity of T is defined as that of E(.).

The following proposition is immediate from Affirmations 1 and 2 in the proof of Lemma 2.4.

PROPOSITION 7.2. The total multiplicity of E(.) is less than or equal to $\aleph_0$ if and only if E(.) has the CGS-property in H. Then a normal operator on H has total multiplicity $N \leq \aleph_0$ if and only if H is separable.

THEOREM 7.3. If E(.) has the CGS-property in H, then its total multiplicity and OSD-multiplicity are the same. Consequently, for a normal operator T on a separable Hilbert space its OSD-multiplicity and total multiplicity coincide.

PROOF. Suppose the total multiplicity of E(.) is $n \in \mathbb{N}$. Then by Affirmation 1 in the proof of Lemma 2.4 there exists an orthonormal set $\{g_i\}_1^n$ such that $H = \sum_1^n \oplus Z(g_i)$.

Then following an argument similar to that in the first part of the proof of Lemma X.5.8 of [3] we obtain a finite set $\{y_i\}_1^k$ of orthonormal vectors such that $k \leq n$, $H = \sum_1^k \oplus Z(y_i)$ and $\rho(y_1) \gg \rho(y_2) \gg \ldots \gg \rho(y_k)$. On the other hand, from Definiton 7.1 it follows that the OSD-multiplicity of $E(.) = k \geq n$. Thus in this case both the multiplicities coincide. If the total multiplicity of $E(.)$ is $\aleph_0$, then as $E(.)$ has the CGS-property in H, $E(.)$ has the OSD-multiplicity, which can not be finite by Definition 7.1. Thus its OSD-multiplicity is also $\aleph_0$.

Theorem 2.5 establishes the existence of an OSD of H relative to $E(.)$ if $E(.)$ has the CGS-property in H. If the total multiplicity of $E(.)$ is finite, starting with a generating subspace of minimum dimension it is possible to construct an OSD of H as is discussed below.

Let the total multiplicity of $E(.)$ be $n, n \in \mathbb{N}$. Let $\{g_i'\}_1^n$ be an orthonormal basis of a generating subspace G. Then by Affirmation 1 in the proof of Lemma 2.4 there exist orthonormal vectors $\{g_i\}_1^n$ such that $H = \sum_1^n \oplus\ Z(g_i)$. Let $y_1 = \sum_1^n g_i$. It is easy to verify that $\rho(y) << \rho(y_1)$ for all $y \in H$. If $n=1$, $H = Z(y_1)$ is an OSD of H. If $n > 1$, let us suppose we have constructed a set of non-zero orthogonal vectors $\{y_i\}_1^m$ $1 < m < n$ such that

(i) $\rho(y_1) >> \rho(y_2) >> \ldots >> \rho(y_m)$;

(ii) $Z(y_i) \perp Z(y_{i'})$ for $i \neq i'$; and

(iii) $\rho(y) << \rho(y_m)$ for $y \in H \ominus (\sum_1^m \oplus\ Z(y_i))$.

If P_j is the perpendicular projection on $Z(y_j)$, by Definition 7.1, $\sum_{j=1}^m P_j \neq I$ and hence $(I - \sum_1^m P_j)g_k \neq 0$ for some k.

Let $h'^{(m)}_k = (I - \sum_1^m P_j) g_k$, $k = 1,2,\ldots,n$. Then by Affirmation 1 of Lemma 2.4 we can construct the orthogonal vectors $\{h_k^{(m)}\}_{k=1}^n$ such that $H \ominus K = \sum_1^n \oplus Z(h_k^{(m)})$, where $K = \sum_1^m \oplus Z(y_i)$. If $y_{m+1} = \sum_{k=1}^n h_k^{(m)}$, then $y_{m+1} \neq o$, $Z(y_{m+1}) \perp Z(y_i)$ for $i = 1,2,\ldots,m$ and $\rho(y) << \rho(y_{m+1})$ for all $y \in H \ominus K$. Thus continuing this process, in the n^{th} step we obtain non-zero orthogonal vectors $\{y_i\}_1^n$ such that (i), (ii) and (iii) remain valid for m=n. If $H \neq \sum_1^n \oplus Z(y_i)$, then by Theorem 2.5 E(.) will have its OSD-multiplicity greater than its total multiplicity. This contradiction shows that $H = \sum_1^n \oplus Z(y_i)$ is an OSD relative to E(.).

DEFINITION 7.4. If E(.) is a spectral measure on $\mathcal{S}$, the total multiplicity relative to E(.) of a projection P commuting with E(.) is defined as that of E(.)P in PH. If $\mathcal{S} = \mathcal{B}(X)$, X a Hausdorff space then the limiting total multiplicity of E(.) at a point $t \in X$ is defined as $\lim_{\substack{t \in U \\ U \text{ open}}} m(U)$, where m(U) is the total multiplicity of E(U) relative to E(.). Similarly are defined the total multiplicity of a projection P and the limiting total multilplicity at $t \in \mathbb{C}$ relative to a normal operator T whenever P commutes with T.

8.- **TOTAL MULTIPLICITY AND MULTIPLICITY FUNCTIONS.** In this section E(.) denotes a spectral measure on $\mathcal{B}(X)$, X a

Hausdorff space, with the CGS-property in H. We study the inter-relation between the total multiplicity of E(.) and the associated multiplicity functions m_p and m_c of E(.).

LEMMA 8.1. If the total multiplicity of E(.) is $n_o \varepsilon \mathbb{N} \cup \aleph_o$, then $\sup_{t \varepsilon X}(m_p(t),\ m_c(t)) \geq n_o$.

PROOF. As E(.) has the CGS-property in H, the discrete part p_E is countable. Without loss of generality, let $p_E \neq \emptyset$. Let $p_E = \{t_i\}_1^N$, $N \varepsilon \mathbb{N} \cup \{\infty\}$. For each t_i there exists an orthonormal basis $\{x_j^{(i)}\}_{j=1}^{m_p(t_i)}$. We define $x_j^{(i)} = o$ if $m_p(t_i) < j \leq k$, $j \varepsilon \mathbb{N}$, where $k = \sup_{t \varepsilon X}(m_p(t), m_c(t))$. Since $m_c(t) \leq k$ for all $t \varepsilon X$, if $\mathfrak{M}(E) \neq o$ and $\mathfrak{M}(E) = \sum_{i=1}^{p} \oplus Z(y_i)$ is an OSD of $\mathfrak{M}(E)$ then $o < p = \sup_{t \varepsilon X} m_c(t) \leq k$. If $\mathfrak{M}(E) = o$, $m_c \equiv o$, in which case we take $y_1 = o$. For $p < j \leq k$, $j \varepsilon \mathbb{N}$ we define $y_j = o$. Let

$$x_j = \sum_{n=1}^{N} \frac{1}{n} x_j^{(n)} + y_j$$

for $j \leq k$, $j \varepsilon \mathbb{N}$. Then $x_j \neq o$ for all j and clearly $H = \sum_{j=1}^{k} \oplus Z(x_j)$. Hence $k \geq n_o$.

LEMMA 8.2. If E(.) has the total multiplicity $n_o \varepsilon \mathbb{N}$, then $m_p(t) \leq n_o$ for all $t \varepsilon X$.

PROOF. Suppose for some $t \varepsilon X$, $m_p(t) > n_o$. By hypothesis and Affirmation 1 in the proof of Lemma 2.4 there exists an orthonormal set $\{g_i\}_1^{n_o}$ in H such that $H=\sum_1^{n_o} \oplus Z(g_i)$. Since $m_p(t) > n_o$ there exists $x_o \neq o$ in $E(\{t\})H$ such that $(x_o, E(\{t\})g_i) = o$ for $i=1,2,\ldots,n_o$. Then for $\sigma \in \mathcal{B}(X)$ we have

$$(E(\sigma)g_i, x_o) = (g_i, E(\sigma)x_o) = (g_i, E(\sigma \cap \{t\})x_o)$$

$$= (E(\sigma \cap \{t\})g_i, x_o) = o$$

for all i. Hence $x_o \perp H$. This contradiction proves that $m_p(t) \leq n_o$.

THEOREM 8.3. If E(.) is a spectral measure on $\mathcal{B}(X)$ and has the CGS-property in H, then the total multiplicity of E(.) is equal to $\sup_{t \in X}(m_p(t), m_c(t))$. Consequently, if T is normal in a separable Hilbert space H then its total multiplicity coincides with $\sup_{\lambda \in \mathbb{C}}(m_p(\lambda), m_c(\lambda)) = \sup_{\lambda \in \sigma(T)}(m_p(\lambda), m_c(\lambda))$.

PROOF. Suppose the total multiplicity of E(.) is $n \in \mathbb{N}$. By Lemma 8.1 $k=\sup_{t \in X}(m_p(t), m_c(t)) \geq n$. By Lemma 8.2 $\sup_{t \in X} m_p(t) \leq n$. Thus it sufficies to show that $\sup_{t \in X} m_c(t) \leq n$. If $M=\sup_{t \in X} m_c(t)$, then clearly $E(.)E(c_E)$ has OSD-multiplicity M and hence by Theorem 7.3 and Definition 7.4 $E(c_E)$ has total multiplicity M. Obviously, the total multiplicity of $E(c_E)$ is less than or equal to that of E(.) and hence

$M \leq n$.

Conversely, if $\sup(m_p(t), m_c(t)) = k \in \mathbb{N}$, then by Lemma 8.1 the total multiplicity n of E(.) is finite and from the above it is immediate that k=n. Consequently, the total multiplicity of E(.) is $\aleph_0$ if and only if $k=\aleph_0$.

The last part is a trivial consequence of the first if one observes that $m_p(\lambda) = o = m_c(\lambda)$ for $\lambda \in \rho(T)$ by Proposition 4.9.

The following theorem gives a construction of an OSD of H relative to E(.), from the subspace $\mathfrak{M}(E)$ and the given OSD of $\mathfrak{M}(E)$.

THEOREM 8.4. If E(.) has the CGS-property in H and is defined on $\mathcal{B}(X)$, then let $p_E = \{t_i\}_1^N$, $N \in \mathbb{N} \cup \{\infty\}$ and let $\mathfrak{M}(E) = \sum_1^p \oplus Z(y_k)$ be an OSD of $\mathfrak{M}(E)$. Then for the vectors $\{x_j\}_{j=1}^k$ defined in the proof of Lemma 8.1

$$H = \sum_1^k \oplus Z(x_j)$$

is an OSD of H relative to E(.), where $k = \sup_{t \in X} (m_p(t), m_c(t)) = \sup_{t \in X} (m_p(t), p)$.

PROOF. It sufficies to verify

$$\rho(x_1) >> \rho(x_2) >> \ldots$$

Let $E(\sigma)x_j = o$, $1 \leq j \leq k$. Then

$$\sum_{n=1}^{N} \frac{1}{n} E(\sigma)x_j^{(n)} + E(\sigma)y_j = o.$$

Since $E(\sigma)x_j^{(n)} = E(\sigma \cap \{t_n\})x_j^{(n)} = x_j^{(n)}$ or o it follows that $\sum_{n=1}^{N} \frac{1}{n} E(\sigma)x_j^{(n)} \varepsilon\ \mathfrak{M}(E)$, whereas $E(\sigma)y_j\ \varepsilon\ \mathfrak{N}(E)$. Therefore, $E(\sigma)y_j = o = \sum_{n=1}^{N} \frac{1}{n} E(\sigma)x_j^{(n)}$. Then, $E(\sigma)y_{j+1} = o$. Since $\left|\left|\sum_{n=1}^{N} \frac{1}{n} E(\sigma)x_{j+1}^{(n)}\right|\right|^2 = \sum_{n=1}^{N} \frac{1}{n^2} \left|\left|E(\sigma)x_{j+1}^{(n)}\right|\right|^2 \leq \sum_{n=1}^{N} \frac{1}{n^2} \left|\left|E(\sigma)x_j^{(n)}\right|\right|^2 = \left|\left|\sum_{n=1}^{N} \frac{1}{n} E(\sigma)x_j^{(n)}\right|\right|^2 = o$, we have

$||E(\sigma)x_{j+1}||^2 = o$. Thus $\rho(x_{j+1}) << \rho(x_j)$.

The following theorem deals with the limiting total multiplicity at a point t in a locally compact Hausdorff space.

THEOREM 8.5. Let E(.) be a regular spectral measure on $\mathcal{B}(X)$, X locally compact and Hausdorff. If E(.) has the CGS-property in H then the following assertions hold.

(i) Let t_o be an isolated point of the spectrum $\Lambda(E)$ of E(.). Then the limiting total multiplicity $\hat{m}(t_o) = \max(m_p(t_o), m_c(t_o))$. If $t_o\ \varepsilon\ c_E$ then $\hat{m}(t_o) = o = m_c(t_o) = m_p(t_o)$.

(ii) For $t_o\ \varepsilon\ X \setminus \Lambda(E)$, $\hat{m}(t_o) = \max(m_p(t_o), m_c(t_o)) = o$.

(iii) If $t_o\ \varepsilon\ \Lambda(E)$ such that $m_p(t_o) = \sup_{t \varepsilon X} (m_p(t), m_c(t))$, then $\hat{m}(t_o) = m_p(t_o)$.

(iv) If $t_o \varepsilon \Lambda(E)$ such that $m_c(t_o) = \sup_{t \varepsilon X} (m_p(t), m_c(t))$, then $\hat{m}(t_o) = m_c(t_o)$.

PROOF. Since $E(.)$ is regular, $\Lambda(E)$ is compact and by Theorem 23 of [2] $E(X \setminus \Lambda(E)) = o$.

(i) Let t_o be an isolated point of $\Lambda(E)$.

Case 1. $t_o \varepsilon p_E$.

Then there exists an open neighbourhood U of t_o in X such that $U \cap \Lambda(E) = \{t_o\}$. As $E(U \setminus \Lambda(E)) = o$, we have $E(U) = E(\{t_o\})$. If $\{x_j\}_{j=1}^{m_p(t_o)}$ is an orthonormal basis of $E(\{t_o\})H$, then $E(U)H = \sum_{j=1}^{m_p(t_o)} \oplus Z(x_j)$ is an OSD of $E(U)H$ and hence by Theorem 6.3 the total multiplicity of $E(U)$ is $m_p(t_o)$. Consequently, $m(t_o) = \lim_{\substack{t_o \varepsilon V \\ V \text{ open}}} m(V) = m_p(t_o)$. As $E(U)\mathcal{M}(E) = E(\{t_o\})\mathcal{M}(E) = o$, $m_c(t_o) = o$.

Case 2. $t_o \varepsilon c_E$.

Let us take U as in case 1. Then $E(U) = E(\{t_o\}) = o$. Therefore, $m_c(t_o) = o$. Also, $m_p(t_o) = o$. Further, for every open neighbourhood V of t_o such that $V \subset U$, $E(V)H = o$ and hence $\hat{m}(t_o) = o$.

(ii) This is immediate from Proposition 4.9 and the fact that $X \setminus \Lambda(E)$ is open and $E(X \setminus \Lambda(E)) = o$.

(iii) Let $t_o \varepsilon \Lambda(E)$ such that $m_p(t_o) = \sup_{t \varepsilon X}(m_p(t), m_c(t))$.

Case 1. $t_o \varepsilon p_E$.

If U is an open neighbourhood of t_o, then $E(U)H = E(U \cap p_E)H \oplus E(U \cap c_E)H$. Then by Theorem 8.4 the OSD-multiplicity of $E(.)E(U)$ is equal to $m_p(t_o)$ since $t_o \varepsilon U \cap p_E$ and $m_p(t_o) = \sup_{t \varepsilon X}(m_p(t), m_c(t))$. Therefore, $E(U)$ has total multiplicity $m_p(t_o)$ by Theorem 7.3. Since U is arbitrary, it follows that $\hat{m}(t_o) = m_p(t_o)$.

Case 2. $t_o \varepsilon \Lambda(E) \setminus p_E$.

Then by hypothesis $m_p = m_c \equiv o$ and hence $H = \{o\}$. Since $H \neq o$, this case is impossible.

(iv) Let $t_o \varepsilon \Lambda(E)$ such that $m_c(t_o) = \sup_{t \varepsilon X}(m_p(t), m_c(t)) = k$ (say). Then let $\mathfrak{M}(E) = \sum_1^k \oplus Z(y_i)$ be an OSD of $\mathfrak{M}(E)$ relative to $E(.)E(c_E)$. By hypothesis for every open neighbourhood U of t_o, $E(U)y_i \neq o$ for all i so that $E(U)\mathfrak{M}(E) = \sum_1^k \oplus Z(E(U)y_i)$ is an OSD of $E(U)\mathfrak{M}(E)$. Since $m_p(t) \leq k$ for all $t \varepsilon X$, by Theorem 8.4 it follows that $E(U)H = E(U \cap p_E)H \oplus E(U)\mathfrak{M}(E)$ has an OSD of the form $\sum_1^k \oplus Z(x_i)$ so that the OSD-multiplicity of $E(.)E(U)$ is k. Now by Theorem 7.3 we conclude that $\hat{m}(t_o) = k$.

NOTE 8.6. In [9] we introduce the concept of total H-multiplicity of E(.) and show that the total multilplicity and the total H-multiplicity of E(.) coincide when E(.) has the CGS-property.

9.- **ORDERED SPECTRAL REPRESENTATIONS (OSRs).** An allied concept of OSDs, known as ordered spectral representations (OSRs in abbreviation) of a Hilbert space H relative to a spectral measure E(.) is introduced and results analogous to Theorem 2.5, 2.9, 2.12 are obtained for them. Also we develop some auxiliary results to show that our concept of OSRs subsumes that of Dunford and Schwartz [3] and consequently, Theorems X.5.10, X.5.12 and X.II.3.16 of [3] are particular cases of our results in this section.

NOTATION 9.1. Let $\{\mu_j\}_{j \varepsilon J}$ be a non-void family of non-zero finite measures on $\mathcal{S}$. If $\tilde{H} = \sum_{j \varepsilon J} \oplus L_2(X, \mathcal{S}, \mu_j)$, then we denote by $\tilde{E}(.)$ the set function on $\mathcal{S}$ defined by

$$\tilde{E}(.)(f_i)_{j \varepsilon J} = (\chi_{(.)} f_j)_{j \varepsilon J}, \quad (f_j)_{i \varepsilon J} \varepsilon \tilde{H}.$$

DEFINITION 9.2. Let $\{\mu_n\}_1^N$, $N \varepsilon \mathbb{N} \cup \{\infty\}$, be non-zero measures of Σ such that $\mu_1 >> \mu_2 >> \ldots\ldots$ An isomorphism U from H onto $K = \sum_1^N \oplus L_2(X, \mathcal{S}, \mu_n)$ is said to be an ordered spectral representation (an OSR, in abbreviation) of H relative to E(.) if

$$UE(.)U^{-1} = \tilde{E}(.).$$

Besides, if $E(.)$ is the resolution of the identity of a normal operator T on H then we say that U is an OSR of H relative to T. The sequence $(\mu_n)_1^N$ is called the measure sequence of the OSR U.

PROPOSITION 9.3. If H has an OSR U relative to $E(.)$ then $E(.)$ has the CGS-property in H.

PROOF. If $(\mu_n)_1^N$ is the measure sequence of U, let $U_n^{-1} = U^{-1}|L_2(X,S,\mu_n)$. By hypothesis, $x_n = U_n^{-1} 1 \neq \sigma$ and $U_n E(\sigma) U_n^{-1} 1 = \chi_\sigma, \sigma \in S$. In other words, $U_n E(\sigma) x_n = \chi_\sigma$ from which it follows that $UZ(x_n) = L_2(X,S,\mu_n)$. Consequently, $H = \sum_1^N \oplus Z(x_n)$ and hence the proposition.

THEOREM 9.4. H has an OSR relative to $E(.)$ if and only if $E(.)$ has the CGS-property in H.

PROOF. The condition is necessary by Proposition 9.3. If $E(.)$ has the CGS-property in H, then by Theorem 2.5 there exists an OSD $H = \sum_1^N \oplus Z(x_i)$ relative to $E(.)$. If $\rho(x_i) = \mu_i$, U_i the isomorphism from $Z(x_i)$ onto $L_2(X,S,\mu_i)$ given in Lemma 1.1. and $U = \sum_1^N \oplus U_i$, then evidently $U: H \to \sum_1^N \oplus L_2(X,S,\mu_i)$ is an OSR of H relative to $E(.)$. Thus the condition is also sufficient.

The following proposition is almost immediate.

PROPOSITION 9.5. If S is the σ-algebra generated by a countable family of sets then for the spectral measure $E(.)$ on S, H has an OSR relative to $E(.)$ if and only if H is separable. Consequently, H has an OSR relative to a normal operator defined on it if and only if H is separable.

The next theorem deals with the inter-relation between OSDs and OSRs.

THEOREM 9.6. Given an OSD $H=\sum_{1}^{N} \oplus Z(x_i)$ relative to $E(.)$ then there exists an OSR $U: H \to \sum_{1}^{N} \oplus L_2(X,S,\mu_i)$ relative to $E(.)$ where $\mu_i = \rho(x_i)$. Conversely, given an OSR $U: H \to \sum_{1}^{N} \oplus L_2(X,S,\mu_i)$ relative to $E(.)$ there exists an OSD $H=\sum_{1}^{N} \oplus Z(x_i)$, where $x_i = U_i^{-1}1$, $U_i^{-1} = U^{-1}| L_2(X,S,\mu_i)$. The OSD thus obtained will be called the OSD induced by the given OSR U.

PROOF. The first part is immediate from the proof of sufficiency of Theorem 9.4. Conversely, if U is an OSR of H with the measure sequence $(\mu_n)_1^N$, by the proof of Proposition 9.3, $H=\sum_{1}^{N} \oplus Z(x_n)$. Besides, $\rho(x_n)=\|E(.)x_n\|^2 = \|E(.)U_n^{-1}1\|^2 = \|U_n E(.)U_n^{-1}1\|^2 = \|\chi_{(.)}\|_2^2$ and hence

$$\rho(x_n)(\sigma) = \int_\sigma d\mu_n = \mu_n(\sigma), \quad \sigma \in S. \quad (1)$$

Then by hypothesis and (1) evidently $\rho(x_1) >> \rho(x_2) >> \ldots$ and hence $H=\sum_{1}^{N} \oplus Z(x_n)$ is an OSD relative to $E(.)$.

DEFINITION 9.7. Let U_i be OSRs of H_i relative to $E_i(.)$ with the corresponding measure sequences $\{\mu_n^{(i)}\}_{n=1}^{N_i}$, i=1,2, We say that U_1 and U_2 are equivalent if $N_1 = N_2$ and $\mu_n^{(1)} \equiv \mu_n^{(2)}$ for all n.

THEOREM 9.8. Any two OSRs of H relative to a spectral measure E(.) with the CGS-property in H are equivalent. Consequently, any two OSRs of a separable Hilbert space H relative to a normal operator T defined on it are equivalent.

PROOF. If U_i are OSRs of H relative E(.) with the corresponding measure sequences $\{\mu_n^{(i)}\}_{n=1}^{N_i}$, let $H = \sum_{n=1}^{N_i} \oplus Z(x_n^{(i)})$ be the OSD induced by U_i, i= 1,2. Then by Theorem 2.9, $N_1 = N_2$ and $\rho(x_n^{(1)}) \equiv \rho(x_n^{(2)})$ for all n. But, as in the proof of Theorem 9.6, we have $\rho(x_n^{(i)}) = \mu_n^{(i)}$ for all n,i=1,2. Hence the result.

THEOREM 9.9. Let $E_1(.)$ and $E_2(.)$ have the CGS-property in H_1 and H_2 respectively. Then $E_1(.)$ and $E_2(.)$ are unitarily equivalent if and only if any two OSRs of H_1 and H_2 relative to $E_1(.)$ and $E_2(.)$ respectively are equivalent.

PROOF. Let U be an isomorphism from H_1 onto H_2 such that $UE_1(.)U^{-1} = E_2(.)$. Let $\{\mu_n^{(i)}\}_{n=1}^{N_i}$ be the measure sequence of the OSR U_i of H_i relative to $E_i(.)$, i= 1,2. Let $H_i = \sum_{n=1}^{N_i} \oplus Z(x_n^{(i)})$ be the OSD induced by U_i, i=1,2. Then by

Theorem 2.12, $N_1=N_2$ and $\rho_1(x_n^{(1)}) \equiv \rho_2(x_n^{(2)})$ for all n. On the other hand, by (1) in the proof of Theorem 9.6 we have $\rho_i(x_n^{(i)}) = \mu_n^{(i)}$, for all n, i= 1,2. Hence U_1 and U_2 are equivalent.

Conversely, let U_1 and U_2 be given as in the above. Suppose U_1 and U_2 are equivalent. Then modifying slightly the above argument it can be shown that the OSDs induced by U_1 and U_2 are equivalent. Consequently, by Theorem 2.12 $E_1(.)$ and $E_2(.)$ are unitarily equivalent.

COROLLARY 9.10. If T_1 and T_2 are normal operators on separable Hilbert spaces H_1 and H_2 respectively, then T_1 and T_2 are unitarily equivalent if and only if any two OSRs of H_1 and H_2 relative to T_1 and T_2 respectively are equivalent.

In order to compare our concept of OSRs of a separable Hilbert space H relative to a normal operator T defined on it with that of [3] in Chapters X and XII we develop some results below.

LEMMA 9.11. Let $H= \sum_1^N \oplus Z(z_n)$ be an OSD of H relative to E(.). Then there exist non-zero vectors $\{x_n\}_1^N$ in H and a decreasing sequence $\{e_n\}_1^N$ in S, $e_1 = X$ such that

(i) $Z(z_n)= Z(x_n)$

(ii) $\rho(z_n) \equiv \rho(x_n)$

and

(iii) $\rho(x_n)(e) = \rho(x_1)(e \cap e_n)$, $e \in S$.

PROOF. Let $\mu_n = \rho(z_n)$. Modifying the notations and arguments on pp.915-916 of [3] suitably, we can define the vectors $x_n = \lim_k x_{n_k}$, $n > 1$ and $x_1 = z_1$ as in [3] and show that (iii) holds for them, where $\{e_n\}_1^N$ is a suitably chosen decreasing sequence in S .

Since $x_n \in Z(z_n)$, evidently $Z(x_n) \subset Z(z_n)$ and $\rho(x_n) \ll \rho(z_n)$. Now, let $\rho(x_n)(\sigma) = o$. Then by (iii) $\rho(x_1)(\sigma \cap e_n) = o$. Since $\rho(z_n) \ll \rho(z_1)$, we have $\rho(z_n)(\sigma \cap e_n) = o$. On the other hand, from the definition of e_n it follows that $\rho(z_n)(X \setminus e_n) = o$ and hence $\rho(z_n)(\sigma) = o$. Thus $\rho(x_n) \equiv \rho(z_n)$. (i) es immediate from Theorem 65.3 of [5].

PROPOSITION 9.12. Let U be an OSR of H relative to E(.) with the measure sequence $\{\mu_n\}_1^N$. Then there exist a decreasing sequence $\{e_n\}_1^N$ in S with $e_1 = X$ and a non-zero finite measure ν on S such that

(i) $\nu(e_n) > o$ for all n;

(ii) $\nu_n \equiv \mu_n$ for all n, where $\nu_n(e) = \nu(e \cap e_n)$, $e \in S$;

and

(iii) H is isomorphic with $\sum_1^N \oplus L_2(X, S, \nu_n)$ under an isomorphism V such that $VE(.)V^{-1} = E(.)$ and V is an

OSR of H relative to E(.).

Conversely, let ν be a non-zero finite measure on S and let $\{e_n\}_1^N$ be a decreasing sequence in S with $e_1 = X$ and $\nu(e_n) > o$ for all n. If there exists an isomorphism U from H onto $K = \sum_1^N \oplus L_2(e_n, \nu)$ such that $UE(.)U^{-1} = \tilde{E}(.)$, where $L_2(e_n,\nu) = L_2(e_n, S \cap e_n, \nu)$, then U is an OSR of H relative to E(.) with the measure sequence $\{\nu_n\}_1^N$, where $\nu_n(e) = \nu(e \cap e_n)$, $e \in S$.

PROOF. Let U be an OSR of H relative to E(.) with the measure sequence $(\mu_n)_1^N$. If $H = \sum_1^N \oplus Z(x_n)$ is the OSD of H induced by U, then by Lemma 9.11 there exits a decreasing sequence $\{e_n\}_1^N$ in S and vectors $\{y_n\}_1^N$ such that $Z(x_n) = Z(y_n)$, $\rho(x_n) \equiv \rho(y_n)$ and $\rho(y_n)(e) = \rho(y_1)(e \cap e_n)$, $e \in S$. Consequently, by Lemma 1.2 there exists an isomorphism V from H onto $L = \sum_{n=1}^N \oplus L_2(e_n, \nu)$ such that $VE(.)V^{-1} = \tilde{E}(.)$, where $\nu = \rho(y_1)$. Since (e_n) es decreasing, it follows that $\nu >> \nu_2 >> \nu_3 >> \ldots$, where $\nu_n = \rho(y_n)$ and hence V is an OSR of H onto L.

Conversely, if U is an isomorphism from H onto $K = \sum_1^N \oplus L_2(e_n, \nu)$ such that $UE(.)U^{-1} = \tilde{E}(.)$, then on defining $\nu_n(e) = \nu(e \cap e_n)$, $e \in S$ it follows that $L_2(e_n, \nu) = L_2(X, S, \nu_n)$ and that U is an OSR of H onto K with the measure sequence $(\nu_n)_1^N$.

DEFINITION 9.13. If ν, $\{e_n\}_1^N$, and U are as in the second part of Proposition 9.12, then we say that U is a special OSR of H relative to E(.) (respectively, relative to T if E(.) is the resolution of the identity of a normal operator T on H, H separable).

THEOREM 9.14. Let E(.) have the CGS-property in H. Then:

(i) Every special OSR of H relative to E(.) is, in particular, an OSR.

(ii) H has special OSRs relative to E(.).

(iii) If U and V are two special OSRs of H relative to E(.) then U and V are equivalent as OSRs.

PROOF. The results are immediate from Theorems 2.5 and 2.9 and Proposition 9.12.

PROPOSITION 9.15. Let U_i be special OSRs of H relative to $E_i(.)$ with the corresponding measures μ_i and decreasing sequences $\{e_n^{(i)}\}_{n=1}^{N_i}$, i= 1,2. Then the following statements are equivalent.

(i) U_1 and U_2 are equivalent as OSRs.

(ii) $N_1 = N_2$ and $\mu_1(e_n^{(1)} \Delta e_n^{(2)}) = o = \mu_2(e_n^{(1)} \Delta e_n^{(2)})$ for all n and $\mu_1 \equiv \mu_2$.

(iii) $N_1 = N_2$ and $(\mu_1)_n \equiv (\mu_2)_n$ for all n, where $(\mu_i)_n^{(e)} =$

$= \mu_i(e \cap e_n^{(i)}),\ e \in \mathcal{S},\ i = 1,2.$

PROOF. The easy proof is left to the reader.

We give the following lemma for an arbitrary family of measures on $\mathcal{S}$, even though a countable family would suffice for the purpose of the present section. This we do for our later need in the study of orthogonal spectral representations in [9].

LEMMA 9.16. Let $\{\mu_j\}_{j \in J}$ be a non-void family of non-zero members of Σ. Let $\tilde{H} = \sum_{j \in J} \oplus L_2(X, \mathcal{S}, \mu_j)$. Then:

(i) $\tilde{E}(.)$ is a spectral measure on $\mathcal{S}$.

(ii) If g is an $\mathcal{S}$-measurable function, let $e_n = \{t \in X: |g(t)| \leq n\},\ n \in \mathbb{N}$.

Then the operator $T(g)$ defined by

$$T(g)\mathbf{f} = \lim_n \int_{e_n} g\, d\tilde{E}\mathbf{f}$$

is normal, has its domain $\mathcal{D}(T(g)) = \{\mathbf{f} = (f_j)_{j \in J} \in \tilde{H}: \sum_{j \in J} \int_X |g|^2 |f_j|^2 d\mu_j < \infty\}$ and its resolution of the identity $\tilde{E}_g(.)$ is given by

$$\tilde{E}_g(\sigma) = \tilde{E}(g^{-1}(\sigma)),\ \sigma \in \mathcal{B}(\mathbb{C}).$$

(iii) $T(g)\mathbf{f} = (gf_j)_{j \in J},\ \mathbf{f} = (f_j)_{f \in J} \in \mathcal{D}(T(g)).$

PROOF.

(i) By the Lebesgue dominated convergence theorem one can easily prove that $\tilde{E}(.)$ is countably additive in the strong operator topology.

(ii) By Theorem XVIII.2.17 of [3] and observing that the projections are hermitian, we conclude that T(g) is normal and has its resolution of the identity $\tilde{E}_g(.)$. Then the domain $\mathcal{D}(T(g))$ is given by

$$\mathcal{D}(T(g)) = \{f \in \tilde{H}: \int_{\mathbb{C}} |\lambda|^2 d\|\tilde{E}_g(\lambda)f\|^2 < \infty\}$$
$$= \{f \in \tilde{H}: \int_X |g(t)|^2 d\|\tilde{E}(t)f\|^2 < \infty\}.$$

Let $D = \{f=(f_j)_{j \in J}: \sum_{j \in J} \int_X |g(t)|^2 |f_j(t)|^2 d\mu_j(t) < \infty\}$. Let $f = (f_j)_{j \in J}$ be a fixed vector in $\tilde{H}$. Clearly,

$$\|\tilde{E}(.)f\|^2 = \sum_{j \in J} \|\chi_{(.)} f_j\|_2^2 = \sum_{j \in J} \int_{(.)} |f_j|^2 d\mu_j.$$

Let $\nu_j \in \Sigma$ be given by

$$\nu_j(.) = \int_{(.)} |f_j|^2 d\mu_j .$$

Then $\frac{d\nu_j}{d\mu_j} = |f_j|^2$. Since $\sum_{j \in J} \nu_j(X) = \|\tilde{E}(X)f\|^2 = \|f\|^2 < \infty$, $J_f = \{j \in J: \nu_j(X) \neq 0\}$ is countable. Consequently,

$$\int_X |g(t)|^2 d\|\tilde{E}(t)f\|^2 = \sum_{j \in J_f} \int_X |g(t)|^2 d\nu_j(t)$$
$$= \sum_{j \in J} \int_X |g(t)|^2 d\nu_j(t)$$
$$= \sum_{j \in J} \int_X |g(t)|^2 |f_j(t)|^2 d\mu_j(t).$$

Thus $f \in \mathcal{D}(T(g))$ if and only if $f \in D$. This proves (ii).

(iii) Let f be a fixed vector in $\mathcal{D}(T(g))$. Given $\varepsilon > 0$, there exists an S-simple function $s=\sum_{1}^{k}\lambda_j\chi_{\sigma_j}$, $\sigma_j\cap\sigma_{j'}=\emptyset$ for $j\neq j'$, $\sigma_j\subset e_n$ and $\sigma_j\in S$ for all j such that

$$||f||\sup_{t\in e_n}|s(t)-g(t)|<\frac{\varepsilon}{2}. \quad (1)$$

Obviously,

$$\int_{e_n}s(t)d\tilde{E}(t)f=(sf_j)_{j\in J}. \quad (2)$$

Then by (1) and (2) we have

$$||\int_{e_n}gd\tilde{E}f-(g\chi_{e_n}f_j)_{j\in J}||<||\int_{e_n}gd\tilde{E}f-\int_{e_n}sd\tilde{E}f||+$$

$$||\int_{e_n}sd\tilde{E}f-(g\chi_{e_n}f_j)_{j\in J}||=(\int_{e_n}|g(t)-s(t)|^2d||\tilde{E}(t)f||^2)^{\frac{1}{2}}+$$

$$+(\sum_{j\in J}\int_X|s-\chi_{e_n}g|^2|f_j|^2d\mu_j)^{\frac{1}{2}}<\varepsilon.$$

Thus

$$\int_{e_n}g\,d\tilde{E}f=(g\chi_{e_n}f_j)_{j\in J}. \quad (3)$$

By the definition of $T(g)f$ there exists $n_o\in\mathbb{N}$ such that

$$||\int_X g\,d\tilde{E}f-\int_{e_n}g\,d\tilde{E}f||<\frac{\varepsilon}{3} \quad (4)$$

for $n\geq n_o$. Then by (3) and (4) we have

$$||\int_X g\,d\tilde{E}f-(gf_j)_{j\in J}||<\frac{\varepsilon}{3}+||((1-\chi_{e_n})gf_j)_{j\in J}|| \quad (5)$$

for $n \geq n_o$. Since $f \in \mathcal{D}(T(g)) = D$, by the convergence of $\sum_{j \in J} \int_X |g|^2 |f_j|^2 \, d\mu_j$, there exists a finite subset L of J such that

$$\sum_{j \in J \setminus L} \int_X (1 - \chi_{e_n})^2 |g|^2 |f_j|^2 \, d\mu_j$$

$$\leq \sum_{j \in J \setminus L} \int_X |g|^2 |f_j|^2 \, d\mu_j < \frac{\varepsilon^2}{9} \qquad (6)$$

Now, by the Lebesgue dominated convergence theorem there exists $n_1 > n_o$ such that

$$\sum_{j \in L} \int_X (1 - \chi_{e_n})^2 |g|^2 |f_j|^2 \, d\mu_j < \frac{\varepsilon^2}{9} \quad (7)$$

for $n \geq n_1$. Then by (5), (6) and (7)

$$\left\| \int_X g \, d\tilde{E} f - (g f_j)_{j \in J} \right\| < \varepsilon .$$

Thus (iii) holds.

THEOREM 9.17. Suppose T is a normal operator on a separable Hilbert space H with the resolution of the identity E(.). Let $\mu \in \Sigma$ and $(e_n)_1^N$, $N \in \mathbb{N} \cup \{\infty\}$, be a decreasing sequence of sets in $\mathcal{B}(\mathbb{C})$ with $e_1 = \mathbb{C}$ and $\mu(e_n) > o$ for all n. Then an isomorphism U from H onto $K = \sum_1^N \oplus L_2(e_n, \mu)$ is a special OSR of H relative to T if and only if the following two conditions are satisfied:

(i) $\mu(\sigma) = o$ for $\sigma \in \mathcal{B}(\mathbb{C})$ with $\sigma \cap \sigma(T) = \emptyset$.

(ii) If $V = U T U^{-1}$, let $g(V)f = \lim_n \int_{\sigma_n} g \, dF f$, where F(.)

is the resolution of the identity of V, $\sigma_n = \{\lambda \in \sigma(T): |g(\lambda)| \leq n\{$, and g is a Borel measurable function on $\sigma(T)$. Then for all such g the domain $\mathcal{D}(g(V))$ of $g(V)$ is given by

$$\mathcal{D}(g(V)) = \{(f_n)_1^N \in K: \sum_1^N \int_{e_n} |g|^2 |f_n|^2 d\mu < \infty\}$$

and

$$g(V)(f_n)_1^N = (gf_n)_1^N, \quad (f_n)_1^N \in \mathcal{D}(g(V)).$$

PROOF. By Lemma 1.3 V is a normal operator on K with the resolution of the identity $F(.) = UE(.)U^{-1}$. Besides, by Lemma 9.16(i) $\tilde{E}(.)$ is a spectral measure on $\mathcal{B}(\mathbb{C})$.

Suppose U is a special OSR of H relative to T. Then $UE(.)U^{-1} = \tilde{E}(.)$ and hence $\tilde{E}(.)$ is the resolution of the identity $F(.)$ of V. Since $E(\rho(T)) = o$ we can extend g to the whole of $\mathbb{C}$ such that g remains Borel measurable. Then clearly, $g(V) = T(g)$ and $\mathcal{D}(g(V)) = \mathcal{D}(T(g))$, where $T(g)$ is as given in Lemma 9.16 with S being replaced by $\mathcal{B}(\mathbb{C})$. Hence by Lemma 9.16, (ii) holds. For $\sigma \in \mathcal{B}(\mathbb{C})$, with $\sigma \cap \sigma(T) = \emptyset$ we have $E(\sigma) = o$ and hence $(\chi_\sigma f_n)_1^N = o$ for all $(f_n)_1^N \in K$. In particular, for $f_1 = 1$, $f_n = o$ $n \geq 2$, we have

$$o = \|(\chi_\sigma f_n)_1^N\|^2 = \int_\sigma |f_1|^2 d\mu = \mu(\sigma).$$

Thus (i) holds.

Conversely, if (i) and (ii) hold, let us show that $UE(.)U^{-1} = \tilde{E}(.)$. If $\sigma \in \mathcal{B}(\sigma(T))$, then χ_σ is Borel measurable and if g= χ_σ, then

$$g(V)=\int_\sigma dF(.)=\int_\sigma d\,U\,E(.)U^{-1}=UE(\sigma)U^{-1}.$$

Then by (ii) $\mathcal{D}(g(V))=K$ and for $(f_n)_1^N \varepsilon K$ we have

$$UE(\sigma)U^{-1}(f_n)_1^N = g(V)(f_n)_1^N = (gf_n)_1^N = (\chi_\sigma f_n)_1^N \quad . \qquad (1)$$

For $\sigma \varepsilon \mathcal{B}(\mathbb{C})$, by (i) $\mu(\sigma \setminus \sigma(T))=o$ and hence by (1)

$$\tilde{E}(\sigma)(f_n)_1^N = (\chi_\sigma f_n)_1^N = (\chi_{\sigma\cap\sigma(T)} f_n)_1^N = UE(\sigma\cap\sigma(T)U^{-1}(f_n)_1^N =$$

$UE(\sigma)U^{-1}(f_n)_1^N$ since $E(\rho(T))=o$.

This complete the proof.

NOTE 9.18. In the light of Theorem 9.17, Definitions X.5.9. and XII.3.4 of Dunford and Schwartz [3] are implied by our Definition 9.13 for a bounded normal operator or an (unbounded) self-adjoint operator on a separable Hilbert space. Then by Theorem 9.14 and Proposition 9.15 , it is obvious that Theorems X.5.10, X.5.12 and XII.3.16 of [3] are particular cases of Theorem 9.8 and Corollary 9.10.

Due to Theorem 9.8 we are justified in introducing the following concept.

DEFINITION 9.19. If E(.) has the CGS-property in H and $(\mu_n)_1^N$ is the measure sequence of an OSR of H relative to E(.) then N is called the OSR-multiplicity of E(.).When $N=\infty$, we say that the OSR-multiplicity of E(.) is $\aleph_o$.The

ORS-multiplicity relative to E(.) of a projection P commuting with E(.) is defined as that of E(.)P on PH. If T is normal on H, H separable, then the OSR-multiplicity of T is defined as that of its resolution of the identity. Similarly is defined the OSR-multiplicity of a projection P relative to T if P commutes with T.

THEOREM 9.20. If E(.) has the CGS-property in H then its OSR-multiplicity, OSD-multiplicity, and the total multiplicity are the same.

PROOF. Follows from Theorems 9.6, 9.8 and 7.3.

NOTE 9.21. The concept of multiplicity given in Chapters X an XII of [3] coincides with that of Definition 9.19 if T is a bounded normal operator or a self-adjoint operator on a separable Hilbert space.

THEOREM 9.22. Let T be a normal operator on a separable Hilbert space H. If the OSD-multiplicity of T is $N \in \mathbb{N} \cup \aleph_0$, then there exists an OSR U of H relative to T with the measure sequence $\{\mu_n\}_1^N$. Besides, if $K = \sum_1^N \oplus L_2(\mathbb{C}, \mathcal{B}(\mathbb{C}), \mu_n)$ then

$$U\,T\,U^{-1} = M_\lambda$$

where $\mathcal{D}(M_\lambda) = \{f = (f_n)_1^N \in K: \sum_1^N \int_{\mathbb{C}} |\lambda|^2 \; |f_n(\lambda)|^2 \, d\mu_n(\lambda) < \infty\}$

and

$$M_\lambda (f_n)_1^N = (\lambda f_n)_1^N \;,\; (f_n)_1^N \in \mathcal{D}(M_\lambda).$$

The operator M is called the canonical ordered representation of T on K.

PROOF. The result is immediate from Theorem 9.20 and 9.17 if we take $g(\lambda)=\lambda$, $\lambda \in \sigma(T)$.

NOTE 9.23. Further discussion of spectral measures with the OSR-multiplicity $\aleph_0$ will be given in [9].

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