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A Borel Extension Approach to Weakly Compact Operators on $C_o(T)$

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Abstract

Let X be a quasicomplete locally convex Hausdorff space. Let T be a locally compact Hausdorff space and let $C_o(T) = \{f : T \to \mathbb{C}, f$ is continuous and vanishes at infinity $\}$ be endowed with the supremum norm. Given a continuous linear map $u : C_o(T) \to X$, is given an alternative vector measure proof, based on the theorem of regular Borel extension of X-valued σ-additive Baire measures, to deduce most of the characterizations obtained in [14] for u to be weakly compact.

1. INTRODUCTION

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm. Then its dual $M(T)$ is the Banach space of all bounded complex Radon measures μ with norm given by $||\mu|| = var(\mu, T)$. Let X be a locally convex Hausdorff sapce (briefly, a lcHs) which is quasicomplete and let $u : C_o(T) \to X$ be a continuous linear map. When X is complete, Grothendieck gave in [6] some necessary and sufficient conditions for u to be weakly compact.

He studied in [6] some topological and range properties of the adjoint u^* and the biadjoint u^{**} of u, characterized weakly compact subsets of $M(T)$ and proved some deep results such as Theorems 1 and 3 and Proposition 11 of [6] to obtain the characterization theorem [6 , Theorem 6] for weakly compact operators $u : C(K) \to X$, where K is a compact Hausdorff space and X is a complete lcHs. Most of the results obtained in Sections 1.1, 1.2, 2.1 and 3.1 of [6] play a key role in the proof of the said theorem. Moreover, the major part of the results proved in [6] are given for compact Hausdorff spaces only and is remarked that they also hold for locally compact Hausdorff spaces.

Later, the results of Grothendieck [6] were proved in detail for the locally compact case in Sections 4.21, 4.22, and 9.1-9.4 of Edwards [5]. Among the results treated there, Theorem 9.4.10 of [5] extends Theorem 6 of Grothendieck [6] to locally compact Hausdorff spaces T and quasicomplete lcHs X. This result for complete lcHs X was mentioned without proof in Remark 2 on p.161 of [6]. However, the proof of (3) $\Rightarrow$ (2 bis) of the above theorem as given in Edwards [5] is incorrect (see Remarks 4 of [14]). Thus, as far as we know, Remark 2 on p.161 of [6] remained unestablished before we presented our work [14].

In [14] we used the Baire and σ-Borel characterizations of weakly compact subsets of $M(T)$ as given in [13] and obtained 35 characterizations for the continuous linear map $u : C_o(T) \to X$ to be weakly compact, where X is a quasicomplete lcHs. These include the characterizations mentioned in the above remark of Grothendieck [6] and in Theorem 9.4.10 of [5]. Moreover, the theorem on regular Borel extension of X-valued σ-additive Baire measures on T (breiefly, the Borel extension theorem) is obtained in [14] as a consequence of these characterizations. Further, Theorem 5.3 of Thomas [16] is also deduced as Theorem 13 in [14], where the proof is direct in the sense that the technique of reduction to metrizable compact case is dispensed with.

Using the Borel extension theorem, the first part of Theorem 1 of [14] and Lemma 1 and Theorem 2 of [6], recently with Dobrakov we obtained in [4] an elementary proof of the said theorem of Thomas [16] (also is given there in [4] a direct simple proof of the Borel extension theorem) and this proof is also devoid of the technique of reduction to metrizable compact case as in [14]. In the said proof no other result of [6] or [14] is used. In this context, arises the following question: Is it possible to give a vector measure proof based on the said results of [6] and [14] and the Borel extension theorem to obtain all the characterizations of weakly compact operators on $C_o(T)$ as given in [14]? The present paper answers the question in the affirmative upto 32 characterizations. The remaining 3 characterizations involving strong additivity are also obtained here, but we use Theorem 1 of [12] (where Lemma 1 of [6] was used for is proof) instead of Theorem 2 of Grothendieck [6].

In [10, 11] the Riesz representation theorem is used to obtain the regular Borel and σ-Borel extensions of a complex Baire measure on T. The paper [14] can be considered as its analogue for X-valued Baire measures, the Riesz representation theorem being replaced by the Bartle-Dunford-Schwartz representation of weakly compact operators. On the other hand, the regular σ-Borel extension of positive Baire measures on T is used in Halmos [7] to derive the Riesz representation theorem for positive linear forms on $C_o(T)$. Now the present proof can be considered as the vector analogue of the treatment of Halmos [7].

2. PRELIMINARIES

In this section we fix notation and terminology. For the convenience of the reader we also give some definitions and results from the literature.

In the sequel T will denote a locally compact Hausdorff space and $C_o(T)$ the Banach space of all complex valued continuous functions vanishing at infinity in T, endowed with the supremum norm $||f||_T = \sup_{t \in T} |f(t)|$.

Let $\mathcal{K}$ (resp. $\mathcal{K}_o$) be the family of all compacts (resp. compact $G_\delta s$) in T. The σ-ring $\mathcal{B}_o(T)$ (resp. $\mathcal{B}_c(T)$) of all Baire (resp. σ-Borel) sets in T is the σ-ring generated by $\mathcal{K}_o$ (resp. $\mathcal{K}$). The σ-algebra $\mathcal{B}(T)$ of all Borel sets in T is the σ-algebra generated by the class of all open sets in T. Note that a subset E of T is σ-Borel if and only if it is a σ-bounded Borel set in T.

$M(T)$ is the Banach space of all bounded complex Radon measures on T with their domain restricted to $\mathcal{B}(T)$. Thus each $\mu \in M(T)$ is a Borel regular (bounded) complex measure on T and has norm given by $||\mu|| = var(\mu, \mathcal{B}(T))(T)$. For $\mu \in M(T)$, $|\mu|(E) = var(\mu, \mathcal{B}(T))(E)$, $E \in \mathcal{B}(T)$.

We recall the following result from [13, Lemma 1].

PROPOSITION 1. *For $\mu \in M(T)$,*

$$|\mu|_{\mathcal{B}_o(T)}(\cdot) = var(\mu|_{\mathcal{B}_o(T)}, \mathcal{B}_o(T))(\cdot) and |\mu|_{\mathcal{B}_c}(T)(\cdot) = var(\mu|_{\mathcal{B}_c(T)}, \mathcal{B}_c(T))(\cdot).$$

A vector measure is an additive set function defined on a ring of sets with values in a lcHs. In the sequel X will denote a lcHs with topology τ. Let Γ be the set of all τ-continuous seminorms on X. The dual of X is denoted by X^*.

The srong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B : B$ bounded in $X\}$, where $p_B(x^*) = \sup_{x \in B} |x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the locally convex topology τ_e of uniform convergence in equicontinuous subsets of X^*. Note that $(X^*, \beta(X^*, X))$ and (X^*, τ_e) are lcHs.

It is well known that the canonical injection $J : X \to X^{**}$ given by $< Jx, x^* >= < x, x^* >$ for all $x \in X$ and $x^* \in X^*$, is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|_{JX} = \tau_e|_X = \tau$.

DEFINITION 1. A linear map $u : C_o(T) \to X$ is called a weakly compact operator on $C_o(T)$ if $\{uf : ||f||_T \leq 1\}$ is relatively weakly compact in X.

Let E anf F be lcHs and let $u : E \to F$ be a continuous linear map. Then the adjoint u^* and the biadjoint u^{**} of u are well defined linear maps and $u^* : (F^*, \sigma(F^*, F)) \to (E^*, \sigma(E^*, E))$ and $u^{**} : (E^{**}, \tau_e) \to (F^{**}, \tau_e)$ are continuous (see Corollary to Proposition 1, § 12, Chapter 3 of [8] and Proposition 8.7.2 of [5]).

The following result (Corollary 9.3.2 of [5], which is essentially due to Lemma 1 of [6]) plays a key role in Section 4.

PROPOSITION 2. *Let E and F be lcHs with F quasicomplete. If $u : E \to F$ is linear and continuous, then the following conditions are equivalent.*

(i) u maps bounded subsets of E into relatively weakly compact subsets of F.

(ii) $u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact for each equicontinuous subset A of F^*.

(iii) $u^{**}(E^{**}) \subset F$.

The following result is due to Theorem 2 of [6], which is the same as Theorem 4.22.1 of [5].

PROPOSITION 3. *Let A be a bounded set in $M(T)$. Then the following assertions are equivalent.*

(i) A is relatively weakly compact.

(ii) For each disjoint sequence $(U_n)_1^\infty$ of open sets in T,

$$\lim_n \sup_{\mu \in A} |\mu|(U_n) = 0.$$

(iii) Let $\epsilon > 0$.

(a) For each compact K in T, there exists an open set U in T such that $K \subset U$ and $\sup_{\mu\in A} |\mu|(U\backslash K) < \epsilon$; and

(b) there exists a compact C such that $\sup_{\mu\in A} |\mu|(T\backslash C) < \epsilon$.

For each τ-continuous seminorm p on X, let $p(x) = ||x||_p$, $x \in X$, and let $X_p = (X, ||\cdot||_p)$ be the associated seminormed space. The completion of the quotient normed space $X_p/p^{-1}(0)$ is denoted by $\tilde{X}_p$. Let $\Pi_p : X_p \to X_p/p^{-1}(0) \subset \tilde{X}_p$ be the canonical quotient map.

Let $\mathcal{S}$ be a σ-ring of subsets of a non empty set Ω. Given a vector measure $m : \mathcal{S} \to X$, for each τ-continuous seminorm p on X let $m_p : \mathcal{S} \to \tilde{X}_p$ be given by $m_p(E) = \Pi_p \circ m(E)$ for $E \in \mathcal{S}$. Then m_p is a Banach space valued vector measure on $\mathcal{S}$. We define p-semivariation $||m||_p$ of m by

$$||m||_p(E) = ||m_p||(E) \text{ for } E \in \mathcal{S}$$

and

$$||m||_p(\Omega) = ||m_p||(\Omega) = \sup_{E\in\mathcal{S}} ||m_p||(E)$$

where $||m_p||$ is the semivariation of the vector measure m_p and is given by $||m_p||(E) = \sup\{|x^* \circ m|(E) : x^* \text{ belongs to the dual of } \tilde{X}_p, \ ||x^*|| \leq 1\}$(see p.2 of [1]) .

An X-valued vector measure m on a σ-ring $\mathcal{S}$ of subsets of Ω is said to be bounded if $\{m(E) : E \in \mathcal{S}\}$ is bounded in X and equivalently, if $||m||_p(\Omega) < \infty$ for each τ-continuous seminorm p on X. When m is σ-additive, m_p is a Banach space valued σ-additive vector measure on the σ-ring $\mathcal{S}$ and hence by Corollary I.1.19 of [1], $||m||_p(\Omega) = ||m_p||(\Omega) \leq 4\sup_{E\in\mathcal{S}} ||m(E)||_p < \infty$.

We follow the theory of integration for bounded $\mathcal{S}$-measurable scalar functions with respect to a bounded X-valued vector measure on the σ-ring $\mathcal{S}$ as given in [12]. Note that a bounded scalar function on Ω is $\mathcal{S}$-measurable if and only if f is the uniform limit of a sequence of $\mathcal{S}$-simple functions.

PROPOSITION 4 (Lebesgue bounded convergence theorem). *Let X be a quasicomplete lcHs and let $m : \mathcal{S} \to X$ be σ-additive. If (f_n) is a bounded sequence of $\mathcal{S}$-measurable scalar functions with $\lim_n f_n(w) = f(w)$ for each $w \in \Omega$, then f is m-integrable and*

$$\int_E f dm = \lim_n \int_E f_n dm$$

for each $E \in \mathcal{S}$.

Since $\mathcal{S}_E = \mathcal{S} \cap E$ is a σ-algebra and $|| \int_E f_n dm - \int_E f dm||_p = || \int_E f_n dm_p - \int_E f dm_p||_p$, the above result is immediate from Theorem II.4.1 of [1].

The following result, which is due to the first part of Theorem 1 of [14], is needed in Sections 3 and 4.

PROPOSITION 5. *Let X be a lcHs. Let $u : C_o(T) \to X$ be a continuous linear map. Then there exists an X^{**}-valued vector measure m on $\mathcal{B}(T)$ satisfying the following properties:*

(i) $x^* \circ m \in M(T)$ for each $x^* \in X^*$ and consequently, $m : \mathcal{B}(T) \to X^{**}$ is σ-additive in $\sigma(X^{**}; X^*)$- topology.

(ii) The mapping $x^* \to x^* \circ m$ of X^* into $M(T)$ is weak-weak* continuous. Moreover, $u^*x^* = x^* \circ m, \ x^* \in X^*$.*

(iii) $x^*uf = \int_T fd(x^* \circ m)$ for each $f \in C_o(T)$ and $x^* \in X^*$.

(iv) $\{m(E) : E \in \mathcal{B}(T)\}$ is τ_e-bounded in X^{**}.

(v) $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$.

DEFINITION 2. Let $u : C_o(T) \to X$ be a continuous linear map. Then the vector measure m as given in Proposition 5 is called the representing measure of u.

DEFINITION 3. A σ-additive vector measure $m : \mathcal{B}_o(T) \to X$ (resp. $\mathcal{B}(T) \to X$, $\mathcal{B}_c(T) \to X$) is called an X-valued Baire (resp. Borel, σ-Borel) measure on T.

DEFINITION 4. Let $\mathcal{S}$ be a σ-ring of sets in T with $\mathcal{S} \supset \mathcal{K}$ or $\mathcal{K}_o$. Let $m : \mathcal{S} \to X$ be a vector measure. Then m is said to be $\mathcal{S}$-regular (resp. $\mathcal{S}$-outer regular, $\mathcal{S}$-inner regular) in $E \in \mathcal{S}$ if, given a τ-continuous seminorm p on X and $\epsilon > 0$, there exists a compact $K \in \mathcal{S}$ and an open set $U \in \mathcal{S}$ with $K \subset E \subset U$ (resp. an open set $U \in \mathcal{S}$ with $E \subset U$, a compact set $K \in \mathcal{S}$ with $K \subset E$) such that $||m||_p(U \backslash K) < \epsilon$ (resp. $||m||_p(U \backslash E) < \epsilon$, $||m||_p(E \backslash K) < \epsilon$). Even though T does not belong to $\mathcal{S}$ one can define $\mathcal{S}$-inner regularity of m in T as follows. Given $p \in \Gamma$ and $\epsilon > 0$, there exists a compact $K \in \mathcal{S}$ such that $||m||_p(B) < \epsilon$ for all $B \in \mathcal{S}$ with $B \subset T \backslash K$. The vector measure m is said to be $\mathcal{S}$-regular (resp. $\mathcal{S}$-outer regular, $\mathcal{S}$-inner regular) if it is so in each $E \in \mathcal{S}$. When $\mathcal{S} = \mathcal{B}(T)$ (resp. $\mathcal{B}_o(T)$, $\mathcal{B}_c(T)$), we use the terminology Borel (resp. Baire, σ-Borel) regularity or outer regularity or inner regularity.

Remark 1. In the above definition one can replace Γ by any other family of continuous seminorms on X which induces the topology τ.

The following proposition on regular Borel and σ-Borel extensions of an X-valued Baire measure is well known and it plays a key role in Section 4. It was first proved in [3,9] and extended to group valued measures in [15]. For a simple and direct proof of the proposition see [4]. Note that a highly technical operator theoretic proof is given in [14] as mentioned in the introduction.

PROPOSITION 6 Let m be an X-valued Baire measure on T and let X be a quasicomplete lcHs. Then m is Baire regular in T. Moreover, there exists a unique X-valued Borel (resp. σ-Borel) regular σ-additive extension $\hat{m}$ (resp. $\hat{m_c}$) of m on $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$). Moreover, $\hat{m}|_{\mathcal{B}_c}(T) = \hat{m_c}$.

3. SOME LEMMAS

Throughout this section m will denote the representing measure (on $\mathcal{B}(T)$) of a continuous linear map $u : C_o(T) \to X$, where X is a quasicomplete lcHs. Let $m_o = m|_{\mathcal{B}_o(T)}$ and $m_c = m|_{\mathcal{B}_c(T)}$. Let $\mathcal{E} = \{A \subset X^* : A \text{ equicontinuous}\}$, and let $p_A(x) = \sup_{x^* \in A} |x^*(x)|$, for $A \in \mathcal{E}$ and $x \in X$. Then the family of seminorms $\{p_A : A \in \mathcal{E}\}$ induces the topology τ of X.

Let $Y = \widetilde{X_{p_A}}$. For $E \in \mathcal{B}(T)$,

$$\begin{aligned} ||m_{p_A}||(E) &= \sup\{|y^* \circ m|(E) : y^* \in Y^*, ||y^*|| \leq 1\} \\ &= \sup\{|x^* \circ m|(E) : x^* \in A, \, x^* \in X^*\} \end{aligned}$$

since $\{y^* \in Y^* : ||y^*|| \leq 1\}$ can be identified with $\{x^* \in X^* : x^* \in A\}$.

The semivariation $||m_o||_{p_A}(E) = \sup\{|x^* \circ m_o|(E) : x^* \in A\}$ for $A \in \mathcal{E}$ and $E \in \mathcal{B}_o(T)$, where $|x^* \circ m_o|(E) = var(x^* \circ m_o, \mathcal{B}_o(T))(E) = |x^* \circ m|_{\mathcal{B}_o(T)}(E)$, the last equality being due to Proposition 1. Thus we have

$$||m_o||_{p_A}(E) = \sup\{|x^* \circ m|(E) : x^* \in A\}$$

for $A \in \mathcal{E}$ and $E \in \mathcal{B}_o(T)$. Moreover,

$$||m_o||_{p_A}(E) \leq 4 \sup_{x^* \in A} \sup_{B \subset E, B \in \mathcal{B}_o(T)} |(x^* \circ m)(B)| = 4 \sup_{B \subset E, B \in \mathcal{B}_o(T)} ||m_o(B)||_{p_A} \quad (1)$$

for $E \in \mathcal{B}_o(T)$ and $A \in \mathcal{E}$.

LEMMA 1. *u^*A is bounded in $M(T)$ for each $A \in \mathcal{E}$.*

Proof. By Proposition 5(ii), $u^*A = \{x^* \circ m : x^* \in A\}$. As m has τ_e-bounded range by Proposition 5(v), we have

$$\sup_{x^* \in A} |x^* \circ m|(T) \leq 4 \sup_{E \in \mathcal{B}(T), x^* \in A} |(x^* \circ m)(E)| = 4 \sup_{E \in \mathcal{B}(T)} ||m(E)||_{p_A} < \infty.$$

Thus u^*A is bounded in $M(T)$.

Notation 1. $\mathcal{U}_o$ denotes the family of all open Baire sets in T.

LEMMA 2. *Suppose $m_o(\mathcal{U}_o) \subset X$. Then the following hold.*

(i) *m_o is σ-additive in $\mathcal{U}_o$ in τ. That is, given a disjoint sequence $(U_n)_1^\infty$ in $\mathcal{U}_o$, then $m_o(\cup_1^\infty U_n) = \Sigma_1^\infty m_o(U_n)$ (in topology τ).*

(ii) *If $(U_n)_1^\infty$ is a disjoint sequence in $\mathcal{U}_o$, then, for each $A \in \mathcal{E}$, $\lim_n ||m_o||_{p_A}(U_n) = 0$.*

Proof

(i) By Proposition 5(i), $x^* \circ m \in M(T)$ and hence

$$(x^* \circ m_o)(\cup_1^\infty U_n) = \Sigma_1^\infty (x^* \circ m_o)(U_n)$$

for each $x^* \in X^*$. By hypthesis, m_o has range in X and hence by the Orlicz-Pettis theorem we conclude that $m_o(\cup_1^\infty U_n) = \Sigma_1^\infty m_o(U_n)$ in topology τ. Thus (i) holds.

(ii) If possible, let $\inf_n ||m_o||_{p_A}(U_n) > 4\delta > 0$ for some equicontinuous subset A of X^*. Thus $\sup_{x^* \in A} |x^* \circ m_o|(U_n) > 4\delta$ for all n. Then there exists an $x_n^* \in A$ such that $|x_n^* \circ m_o|(U_n) > 4\delta$. Consequently, $\sup_{B \in \mathcal{B}_o(T), B \subset U_n} |(x_n^* \circ m_o)(B)| > \delta$ and hence there exists $B_n \subset U_n$ in $\mathcal{B}_o(T)$ such that $|(x_n^* \circ m_o)(B_n)| > \delta$. Since $x_n^* \circ m_o$ is a (σ-additive) scalar Baire measure, it is Baire regular and hence there exists an open Baire set G_n with $B_n \subset G_n \subset U_n$ such that $|(x_n^* \circ m_o)(G_n)| > \delta$. Consequently, $\inf_n |(x_n^* \circ m_o)(G_n)| > \delta$. This is absurd, since $|(x_n^* \circ m)(G_n)| \leq ||m_o(G_n)||_{p_A} \to 0$ by (i) as (G_n) is a disjoint sequence in $\mathcal{U}_o$.

The proofs of (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (v) of Theorem 1 of [13] are suitably modified to obtain Lemmas 3 and 4 below. For the sake of completeness we give their proofs.

LEMMA 3. *Let m_o satisfy the hypothesis of Lemma 2. Then the following hold.*

(i) *m_o is Baire inner regular (in τ_e) in each $U \in \mathcal{U}_o$.*

(ii) *For each $\epsilon > 0$ and for each equicontinuous subset A of X^*, there exists a $K \in \mathcal{K}_o$ such that $||m||_{p_A}(T \backslash K) = \sup_{x^* \in A}\{|x^* \circ m|(B) : B \subset T\backslash K, B \in \mathcal{B}(T)\} < \epsilon$.*

Proof. We shall prove both the assertions simultaneously. The proof of (iii) $\Rightarrow$ (iv) of Theorem 1 of [13] is suitably modified here. Let $U \in \mathcal{U}_o$ (resp. $U = T$). Let $\epsilon > 0$. If neither (i) nor (ii) is true, then without loss of generality we can suppose that there exists an equicontinuous subset A of X^* such that for no compact $K \in \mathcal{K}_o$, $||m_o||_{p_A}(U \backslash K) \leq \epsilon$ and $||m||_{p_A}(T\backslash K) \leq \epsilon$ hold. Then there exists $x_1^* \in A$ such that $|x_1^* \circ m_o|(U) > \epsilon$ (resp. $|x_1^* \circ m|(U) > \epsilon$), for otherwise $K = \emptyset$ will provide a contradiction. Then by the Baire regularity of $|x_1^* \circ m_o|$ (resp. by the Borel regularity of $|x_1^* \circ m|$) due to Proposition 5(i), there exists $K_1 \in \mathcal{K}_o$ with $K_1 \subset U$ (resp. $K \in \mathcal{K}$) such that $|x_1^* \circ m_o|(K_1) > \epsilon$ (resp. $|x_1^* \circ m|(K) > \epsilon$ and consequently, by Theorem 50.D of Halmos [7] there exists $K_1 \in \mathcal{K}_o$ with $K \subset K_1$ so that $|x_1^* \circ m_o|(K_1) \geq |x_1^* \circ m|(K) > \epsilon$). Since K_1 is a compact subset of U, by Theorem 50.D of Halmos [7] there exists $U_1 \in \mathcal{U}_o$ and $F_1 \in \mathcal{K}_o$ such that

$$U \supset F_1 \supset U_1 \supset K_1.$$

Moreover, $|x_1^* \circ m_o|(U_1) \geq |x_1^* \circ m_o|(K_1) > \epsilon$. Since $F_1 \in \mathcal{K}_o$, by our assumption there exists $x_2^* \in A$ such that $|x_2^* \circ m_o|(U \backslash F_1) > \epsilon$ (resp. $|x_2^* \circ m|(U\backslash F_1) > \epsilon$). Then by the Baire regularity of $|x_2^* \circ m_o|$(resp. Borel regularity of $|x_2^* \circ m|$) we can choose $C_1 \in \mathcal{K}_o$ with $C_1 \subset U \backslash F_1$ such that $|x_2^* \circ m_o|(C_1) > \epsilon$ (resp. $C \in \mathcal{K}$ with $C \subset U \backslash F_1$ such that $|x_2^* \circ m|(C) > \epsilon$ and then, by Theorem 50.D of Halmos [7], we can choose $C_1 \in \mathcal{K}_o$ with $C \subset C_1 \subset U \backslash F_1$ so that $|x_2^* \circ m|(C_1) > \epsilon$). Let $K_2 = F_1 \cup C_1$. Then $K_2 \in \mathcal{K}_o$, $U \supset K_2 \supset F_1$ and $|x_2^* \circ m_o|(K_2 \backslash F_1) = |x_2^* \circ m_o|(C_1) > \epsilon$. Again by Theorem 50.D of Halmos [7] there exists $U_2 \in \mathcal{U}_o$ and $F_2 \in \mathcal{K}_o$ such that

$$U \supset F_2 \supset U_2 \supset K_2 \supset F_1 \supset U_1 \supset K_1.$$

Accordingly, $|x_2^* \circ m_o|(U_2 \backslash F_1) \geq |x_2 * \circ m_o|(K_2 \backslash F_1) > \epsilon$. Next by our assumption there exists $x_3^* \in A$ such that $|x_3^* \circ m_o|(U \backslash F_2) > \epsilon$ (resp. $|x_3^* \circ m|(U \backslash F_2) > \epsilon$). Then by the Baire regularity of $|x_3^* \circ m_o|$ (resp. by the Borel regularity of $|x_2^* \circ m|$ and then applying Theorem 50.D of Halmos [7]), we can choose $C_2 \in \mathcal{K}_o$ such that $C_2 \subset U \backslash F_2$ and $|x_3^* \circ m_o|(C_2) > \epsilon$. Let $K_3 = F_2 \cup C_2$. Then $K_3 \in \mathcal{K}_o$, $U \supset K_3 \supset F_2$ and $|x_3^* \circ m_o|(K_3 \backslash F_2) = |x_3^* \circ m_o|(C_2) > \epsilon$. Again by Theorem 50.D of Halmos [7] there exists $U_3 \in \mathcal{U}_o$ and $F_3 \in \mathcal{K}_o$ such that

$$U \supset F_3 \supset U_3 \supset K_3 \supset F_2 \supset U_2$$

and hence, $|x_3^* \circ m_o|(U_3 \backslash F_2) \geq |x_3^* \circ m_o|(K_3 \backslash F_2) > \epsilon$.

Thus proceeding step by step we produce an increasing sequence (U_n) in $\mathcal{U}_o$, another two increasing sequences (K_n) and (F_n) in $\mathcal{K}_o$ and a sequence (x_n^*) in A such that

$$U \supset \ldots \supset F_{n+1} \supset U_{n+1} \supset K_{n+1} \supset F_n \supset U_n \supset \ldots \supset K_2 \supset F_1 \supset U_1 \supset K_1$$

and

$$|x_{n+1}^* \circ m_o|(U_{n+1} \backslash F_n) > \epsilon$$

for all $n \geq 1$. Let $G_{n+1} = U_{n+1} \backslash F_n$, for $n \geq 1$. Then $\{G_{n+1}\}_1^\infty$ is a disjoint sequence in $\mathcal{U}_o$ and satisfies $||m_o||_{p_A}(G_{n+1}) \geq |x_{n+1}^* \circ m_o|(G_{n+1}) > \epsilon$. This contradicts Lemma 2 and hence (i) and (ii) hold.

LEMMA 4. *Suppose m_o is Baire inner regular in each $U \in \mathcal{U}_o$ with respect to the topology τ_e of X^{**} and for each $\epsilon > 0$ and for each equicontinuous subset A of X^* suppose there exists $K \in \mathcal{K}_o$ such that $||m_o||_{p_A}(T \backslash K) = \sup_{x^* \in A}\{|x^* \circ m|(B) : B \subset T \backslash K, B \in \mathcal{B}_o(T)\} < \epsilon$ (note that the range of m_o is contained in X^{**}). Then m_o is Baire inner regular in $\mathcal{B}_o(T)$ with respect to τ_e.*

Proof. As remarked earlier, we suitably modify the proof of (iv)$\Rightarrow$(v) of Theorem 1 of [13]. Let A be an equiconinuous subset of X^* and let $\epsilon > 0$. By hypothesis there exists $\Omega \in \mathcal{K}_o$ such that $||m_o||_{p_A}(T \backslash \Omega) < \frac{\epsilon}{2}$. Let $\mathcal{K}_o(\Omega)$ be the family of all compact G_δs in Ω. Clearly, $\mathcal{K}_o(\Omega) = \mathcal{K}_o \cap \Omega = \{K \subset \Omega : K \in \mathcal{K}_o\}$. Let $\mathcal{S}_A = \{E \in \mathcal{B}_o(\Omega) :$ for each $\epsilon' > 0$ there exists $K \in \mathcal{K}_o(\Omega)$ with $K \cap E$ compact and $||m_o||_{p_A}(\Omega \backslash K) \leq \epsilon'\}$. Clearly, $\mathcal{K}_o(\Omega) \subset \mathcal{S}_A$. Also we note by Theorem 5.E of Halmos [7] that $\mathcal{B}_o(\Omega) = \mathcal{B}_o(T) \cap \Omega$.

Affirmation 1. For each $U \in \mathcal{U}_o$, $U \cap \Omega \in \mathcal{S}_A$.

In fact, given $\epsilon' > 0$, by the Baire inner regularity of m_o in U (in τ_e), there exists $K \in \mathcal{K}_o$ with $K \subset U$ such that $||m_o||_{p_A}(U\backslash K) < \epsilon'$. Then $K_o = K \cap \Omega \in \mathcal{K}_o(\Omega)$, $K \cap \Omega \subset U \cap \Omega$, $U \cap \Omega \cap K_o = K_o$ is compact and $(U \cap \Omega)\backslash K_o \subset U\backslash K$. Therefore,

$$||m_o||_{p_A}((U \cap \Omega)\backslash K_o) \leq \epsilon'.$$

Let $K_1 = K_o \cup (\Omega\backslash U)$. Then $K_1 \in \mathcal{K}_o(\Omega)$, $K_1 \cap (U \cap \Omega) = K_o$ is compact and

$$||m_o||_{p_A}(\Omega\backslash K_1) = ||m_o||_{p_A}((U \cap \Omega)\backslash K_o) \leq \epsilon'.$$

Thus $U \cap \Omega \in \mathcal{S}_A$.

Affirmation 2. For each $K \in \mathcal{K}_o(\Omega)$, $\Omega\backslash K \in \mathcal{S}_A$.

In fact, by Proposition 14, §14, Chapter III of Dinculeanu [2], there exist $(V_n)_1^\infty \subset \mathcal{U}_o$ such that $K = \cap_1^\infty V_n$. Then $\Omega\backslash K = \cup_1^\infty(\Omega\backslash V_n)$. Then by Theorem 50.D of Halmos [7], for each n there exists an open Baire set $W_n \supset \Omega\backslash V_n$. Let $W = \cup_1^\infty W_n$. Then $W \in \mathcal{U}_o$, and $\Omega\backslash K = (\Omega\backslash K) \cap W = \Omega \cap (W\backslash K) \in \mathcal{S}_A$ by Affirmation 1.

To show that $\mathcal{S}_A$ is closed under countable intersections, let (E_n) be a sequence in $\mathcal{S}_A$ with $E = \cap_1^\infty E_n$. Let $\epsilon' > 0$. Then as $E_n \in \mathcal{S}_A$, there exists $K_n \in \mathcal{K}_o(\Omega)$ such that $K_n \cap E_n$ is compact and $||m_o||_{p_A}(E_n\backslash K_n) < \frac{\epsilon'}{2^n}$ for each $n \geq 1$. Then $K = \cap_1^\infty K_n \in \mathcal{K}_o(\Omega)$, $K \cap E = \cap_1^\infty(K_n \cap E_n)$ is compact and

$$||m_o||_{p_A}(E\backslash K) \leq \Sigma_1^\infty ||m_o||_{p_A}(E\backslash K_n) < \epsilon'$$

since $||m_o||_{p_A}$ is countably subadditive as $|x^* \circ m_o|$ is so for each $x^* \in A$. Thus $E \in \mathcal{S}_A$.

To verify that $\mathcal{S}_A$ is closed under complements, let $E \in \mathcal{S}_A$. Given $\epsilon' > 0$, there exists $K_1 \in \mathcal{K}_o(\Omega)$ such that $E \cap K_1$ is compact and $||m_o||_{p_A}(\Omega\backslash K_1) \leq \frac{\epsilon'}{2}$. Now by Theorem 51.D of Halmos [7], $E \cap K_1 \in \mathcal{K}_o(\Omega)$ and hence by Affirmation 2, $\Omega\backslash(E \cap K_1) \in \mathcal{S}_A$. Thus there exists $K_2 \in \mathcal{K}_o(\Omega)$ such that $(\Omega\backslash(E \cap K_1)) \cap K_2$ is compact and

$$||m_o||_{p_A}(\Omega\backslash K_2) \leq \frac{\epsilon'}{2}.$$

Then $(K_1 \cap K_2) \cap (\Omega\backslash E) = K_1 \cap K_2 \cap (\Omega\backslash(E \cap K_1))$ is compact and

$$||m_o||_{p_A}(\Omega\backslash(K_1 \cap K_2)) \leq \Sigma_1^2 ||m_o||_{p_A}(\Omega\backslash K_i) \leq \epsilon'.$$

Thus $\Omega\backslash E \in \mathcal{S}_A$. Therefore, $\mathcal{S}_A$ is a σ-algebra in Ω. Since $\mathcal{K}_o(\Omega) \subset \mathcal{S}_A$, we conclude that $\mathcal{S}_A = \mathcal{B}_o(\Omega)$. Thus, for $E \in \mathcal{B}_o(\Omega)$ and $\epsilon' > 0$, there exists $K \in \mathcal{K}_o(\Omega)$ such that $E \cap K$ is compact and $||m_o||_{p_A}(\Omega\backslash K) \leq \epsilon'$.

Now let $E \in \mathcal{B}_o(T)$. Then $E \cap \Omega \in \mathcal{B}_o(\Omega) = \mathcal{S}_A$ and hence there exists a compact $L \subset E \cap \Omega$ such that $||m_o||_{p_A}(\Omega\backslash L) \leq \frac{\epsilon}{2}$ and $||m_o||_{p_A}(E\backslash\Omega) \leq ||m_o||_{p_A}(T\backslash\Omega) < \frac{\epsilon}{2}$. Thus $||m_o||_{p_A}(E\backslash L) < \epsilon$. Consequently, as A is an arbitrry equicontinuous subset of X^* and E is an arbitrary Baire set in

T, we conclude that m_o is Baire inner regular in τ_e.

LEMMA 5. Suppose m (resp. m_c, m_o) is Borel (resp. σ-Borel, Baire) inner regular (in τ_e) in $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$). Then m (resp. m_c, m_0) is σ-additive in τ_e.

Proof. Let A be an equicontinuous subset of X^* and let $\epsilon > 0$. Let $\mathcal{S} = \mathcal{B}(T)$ and $\gamma = m$ (resp. $\mathcal{S} = \mathcal{B}_c(T)$ and $\gamma = m_c$; $\mathcal{S} = \mathcal{B}_o(T)$ and $\gamma = m_o$). Since $||\gamma(E)||_{p_A} \leq ||\gamma||_{p_A}(E)$ for $E \in \mathcal{S}$, it suffices to show that $\lim_n ||\gamma||_{p_A}(E_n) = 0$ whenever (E_n) is a decreasing sequence in $\mathcal{S}$ with $\cap_1^\infty E_n = \emptyset$. By hypothesis, for each n, there exists a compact $K_n \in \mathcal{S}$ with $K_n \subset E_n$ such that $||\gamma||_{p_A}(E_n \backslash K_n) < \frac{\epsilon}{2^n}$. Then adapting the proof at the end of p. 158 and in the top of p.159 of [1], we can show that there exists n_o such that $||\gamma||_{p_A}(E_n) < \epsilon$ for $n \geq n_o$. Hence the lemma holds.

4. CHARACTERIZATIONS OF WEAKLY COMPACT OPERATORS ON $C_o(T)$

Let X be a quasicomplete lcHs. Using Propositions 1-6 and Lemmas 1-5 of the earlier sections and Theorem 1 of [12] we shall obtain below all the 35 characterizations given in [14] for a continuous linear map $u : C_o(T) \to X$ to be weakly compact. As mentioned in the outset, the Borel extension theorem (Proposition 6) for σ-additive X-valued Baire measures on T plays a key role in the present proof in contrast to the proofs of the characterization theorems of [14].

THEOREM 1. *Let $u : C_o(T) \to X$ be a continuous linear map, where X is a quasicomplete lcHs. Let m be the representing measure of u and let $m_c = m|_{\mathcal{B}_c}(T)$ and $m_o = m|_{\mathcal{B}_o}(T)$. Then the following assertions are equivalent.*

(i) u is weakly compact.

(ii) The range of m is contained in X.

(iii) The range of m_c is contained in X.

(iv) The range of m_o is contained in X.

(v) $m(U) \in X$ for all open sets U in T.

(vi) $m(F) \in X$ for all closed sets F in T.

(vii) $m(U) \in X$ for all σ-Borel open sets U in T.

(viii) $m(U) \in X$ for all open Baire sets U in T.

(ix) $m(U) \in X$ for all open sets U in T which are σ-compact.

(x) $m(F) \in X$ for all closed sets F in T which are G_δ.

(xi) $m(U) \in X$ for all open sets U in T which are a countable union of closed sets in T.

(xii) *For each increasing sequence* $(f_n)_1^\infty \subset C_o(T)$, *with* $0 \leq f_n \leq 1$, (uf_n) *converges weakly in* X.

(xiii) m *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xiv) m_c *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xv) m_o *is* σ*-additive in the topology* τ_e *of* X^{**}.

(xvi) m *is strongly additive in the topology* τ_e *of* X^{**}.

(xvii) m_c *is strongly additive in the topology* τ_e *of* X^{**}.

(xviii) m_o *is stongly additive in the topology* τ_e *of* X^{**}.

(xix) m *is Borel regular in* τ_e *of* X^{**}.

(xx) m *is Borel inner regular in* τ_e *of* X^{**}.

(xxi) m *is Borel inner regular (in* τ_e*) in each open set* U *in* T.

(xxii) m *is Borel outer regular (in* τ_e*) in each compact set* K *in* T *and Borel inner regular (in* τ_e*) in the set* T.

(xxiii) m_c *is* σ*-Borel regular in* τ_e *of* X^{**}.

(xxiv) m_c *is* σ*-Borel inner regular in* τ_e *of* X^{**}.

(xxv) m_c *is* σ*-Borel inner regular (in* τ_e*) in each* σ*-Borel open set* U *in* T *and in the set* T.

(xxvi) m_c *is* σ*-Borel outer regular (in* τ_e*) in each compact set* K *in* T *and* σ*-Borel inner regular (in* τ_e*) in the set* T.

(xxvii) m_o *is Baire regular in* τ_e *of* X^{**}.

(xxviii) m_o *is Baire inner regular in* τ_e *of* X^{**}.

(xxix) m_o *is Baire inner regular (in* τ_e*) in each open Baire set* U *in* T *and in the set* T.

(xxx) m_o *is Baire outer regular (in* τ_e*) in each compact* G_δ *in* T *and Baire inner regulat (in* τ_e*) in the set* T.

(xxxi) *All bounded Borel measurable scalar functions* f *on* T *are* m*-integrable and* $\int_T f dm \in X$.

(xxxii) *All bounded* $\mathcal{B}_c(T)$*-measurable scalar functions* f *on* T *are* m_c*-integrable and* $\int_T f dm_c \in X$.

(xxxiii) *All bounded Baire measurable scalar functions* f *on* T *are* m_o*-integrable and* $\int_T f dm_o \in X$.

(xxxiv) *All bounded scalar functions* f *belonging to the first Baire class in* T *are* m_o*-integrable and* $\int_T f dm_o \in X$.

(xxxv) $u^{**}f \in X$ *for all bounded scalar functions* f *belonging to the first Baire class in* T.

Proof.

(i)$\Rightarrow$(ii) By (i) and Proposition 2, $u^{**}C_o^{**}(T) \subset X$ and by Proposition 5(v), $m(E) = u^{**}(\chi_E)$ for $E \in \mathcal{B}(T)$. As $\mathcal{B}(T) \subset C_o^{**}(T)$, (ii) holds.

Obviously, (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (viii).

(viii) $\Rightarrow$ (iv) . In fact, by hypothesis (viii) and by Lemmas 3 and 4, m_o is Baire inner regular in τ_e of X^{**}. Given $K \in \mathcal{K}_o$, by Theorem 50.D of Halmos [7] there exists $U \in \mathcal{U}_o$ such that $K \subset U$ and hence $m_o(K) = m_o(U) - m_o(U \backslash K) \in X$. Thus $m_o(\mathcal{K}_o) \subset X$. Let $E \in \mathcal{B}_o(T)$. Let $D(E) = \{K \in \mathcal{K}_o : K \subset E\}$ and let $K_1 \geq K_2$ for $K_1, K_2 \in D(E)$ if $K_1 \supset K_2$. Then by the Baire inner regularity of m_o in E, $\lim_{D(E)} m_o(K) = m_o(E)$ so that the net $\{m_o(K) : K \in D(E)\}$ is τ_e-Cauchy with limit $m_o(E)$. Since by Proposition 5(iv), m has τ_e-bounded range in X^{**}, $m_o(\mathcal{K}_o)$ has τ-bounded range in X. Thus there exists a τ-bounded closed set H in X such that $m_o(\mathcal{K}_o(T)) \subset H$. Since X is quasicomplete, we conclude that $m_o(E) \in X$. Thus m_o has range in X.

(iv) $\Rightarrow$ (i) In fact, by hypothesis, Proposition 5(i) and the Orlicz-Pettis theorem m_o is σ-additive in τ. Then by Proposition 6 there exists a unique X-valued Borel regular σ-additive extension $\hat{m}$ of m_o on $\mathcal{B}(T)$. As each $f \in C_o(T)$ is a bounded Baire measurable function by Theorem 51.B of Halmos [7], by Proposition 5(iii) we have

$$x^* u f = \int_T f d(x^* \circ m) = \int_T f d(x^* \circ m_o) = \int_T f d(x^* \circ \hat{m}), \ \ f \in C_o(T).$$

Since $x^* \circ m \in M(T)$ by Proposition 5(i) and since $x^* \circ \hat{m} \in M(T)$ as $\hat{m}$ is Borel regular and σ-additive, it follows by the uniqueness part of the Riesz representation theorem that $x^* \circ m = x^* \circ \hat{m}$ for each $x^* \in X^*$. Since m has range in X^{**} and $\hat{m}$ has range in X we conclude that $m = \hat{m}$ and hence m not only has range in X but also is σ-additive in $\mathcal{B}(T)$ in τ. Then, given a disjoint sequence (U_n) of open sets in T, $m(\cup_1^\infty U_n) = \Sigma_1^\infty m(U_n)$ and in particular, $\lim_n m(U_n) = 0$. Thus, for each equicontinuous subset A of X^*, we have $\lim_n ||m(U_n)||_{p_A} = \lim_n \sup_{x^* \in A} |(x^* \circ m)(U_n)| = 0$. Moreover, by Lemma 1, u^*A is bounded in $M(T)$. Therefore, by Proposition 5(ii) and Proposition 3, u^*A is relatively weakly compact in $M(T)$. Consequently, by Proposition 2, u is weakly compact. Thus (i) holds.

Clearly, (v) and (vi) are equivalent. Moreover, (ii) $\Rightarrow$ (v) $\Rightarrow$ (vii) $\Rightarrow$ (viii). Thus the assertions (i)-(viii) are equivalent.

Clearly, (vi) $\Rightarrow$ (x).

(x) $\Rightarrow$ (xi) Let U be an open set in T such that it is a countable union of closed sets. Then $T \backslash U$ is a closed set which is a G_δ and hence by hypothesis (x) we have $m(U) = m(T) - m(T \backslash U) \in X$. Hence (xi) holds.

Obviously (xi) $\Rightarrow$ (ix) and (ix) $\Rightarrow$ (viii) by § 14, Chapter III of Dinculeanu [2]. Thus (i)-(xi) are equivalent.

(ii) $\Rightarrow$ (xii) Let (f_n) be as in (xii). Then $\lim_n f_n(t) = f(t)$ exists in [0,1] for each $t \in T$ and f is Borel measurable. Then the hypothesis (ii) combined with Proposition 5(i) and the Orlicz- Pettis theorem implies that m is σ-additive in $\mathcal{B}(T)$. Consequently, by Proposition 4

$$\lim_n \int_T f_n dm = \int_T f dm \in X.$$

Then by Proposition 5(iii) we have

$$\lim_n x^* u f_n = \lim_n \int_T f_n d(x^* \circ m) = x^*(\lim_n \int_T f_n dm) = x^*(\int_T f dm)$$

for all $x^* \in X^*$. Thus (xii) holds.

(xii) $\Rightarrow$ (viii) Let $U \in \mathcal{U}_o$. Then by § 14, Chapter III of Dinculeanu [2], there exists a sequence $(K_n) \subset \mathcal{K}_o$ such that $K_n \nearrow U$. By Urysohn's lemma we can choose an increasing sequence g_n of non negative continuous functions with compact support such that $g_n \nearrow \chi_U$. Then by hypothesis there exists a vector $x_o \in X$ such that $\lim_n x^* u g_n = x^* x_o$ for all $x^* \in X^*$. Therefore, by Propositions 4 and 5 we have $x^* x_o = \lim_n \int_T g_n d(x^* \circ m) = x^* m(U)$ for all $x^* \in X^*$. Since $m(U) \in X^{**}$, it follows that $m(U) = x_o \in X$. Hence (viii) holds.

(ii) $\Rightarrow$ (xiii) By (ii), Proposition 5(i) and the Orlicz-Pettis theorem m is σ-additive and has range in X. Since $\tau_e|_X = \tau$, (xiii) holds.

Clearly (xiii) $\Rightarrow$ (xiv) $\Rightarrow$ (xv).

(xv) $\Rightarrow$ (i) Let Y be the completion of (X^{**}, τ_e). Then by hypothesis $m_o : \mathcal{B}_o(T) \to Y$ is σ-additive in τ_e and hence by Proposition 6 there exists a unique Y-valued Borel regular σ-additive (in τ_e) extension $\tilde{m}$ of m_o on $\mathcal{B}(T)$. Each $f \in C_o(T)$ is a bounded Baire measurable function by Theorem 51.B of Halmos [7] and consequently, by Proposition 5(iii) we have

$$x^* u f = \int_T f d(x^* \circ m) = \int_T f d(x^* \circ m_o) = \int_T f d(x^* \circ \tilde{m})$$

for each $f \in C_o(T)$. By Proposition 5(i), $x^* m \in M(T)$. Since each $x^* \in X^*$ is τ_e-continuous in X^{**}, it follows that $x^* \circ \tilde{m}$ is a σ-additive regular Borel complex measure on T and hence $x^* \circ \tilde{m} \in M(T)$. Thus the continuous linear functional $x^* u$ on $C_o(T)$ is represented by both $x^* \circ m, x^* \circ \tilde{m} \in M(T)$ and hence $x^* \circ m = x^* \circ \tilde{m}$. Since x^* is arbitrary in X^*, m takes values in X^{**} and $\tilde{m}$ takes values in Y, it follows that $m = \tilde{m}$ so that $\tilde{m}$ has values in X^{**}. Moreover, m $(= \tilde{m})$ is σ-additive in τ_e. Consequently, given a disjoint sequence (U_n) of open sets in T, we have $||m(U_n)||_{p_A} \to 0$ as $n \to \infty$ for each equicontinuous subset A of X^*. Moreover, for such A, by Lemma 1, $u^* A$ is bounded in $M(T)$. Then by an argument similar to that in the end of the proof of (iv)$\Rightarrow$(i) we conclude that u is weakly compact. Hence (i) holds.

Clearly, (xiii) $\Rightarrow$ (xvi) $\Rightarrow$ (xvii) $\Rightarrow$ (xviii).

(xviii) $\Rightarrow$ (i) Let $\Sigma(\mathcal{B}_o(T))$ be the Banach space of all bounded complex functions which are uniform limits of sequences of $\mathcal{B}_o(T)$-simple functions, with pointwise addition and scalar multiplication and with norm the supremum norm $||\cdot||_T$. Let

$$Vf = \int_T f dm_o, \; f \in \Sigma(\mathcal{B}_o(T)).$$

By Proposition 5(iv) m_o is a τ_e-bounded vector measure and hence, by Lemma 6 of [12], V is a well defined X^{**}-valued continuous linear map. Then as the representing measure m_o of V (see Definition 2 of [12]) is strongly additive by hypothesis (xviii), by Theorem 1 of [12] V is a weakly compact operator. By Theorem 51.B of Halmos [7] each $f \in C_o(T)$ is Baire measurable and bounded and hence is the uniform limit of a sequence of Baire simple functions. Hence $C_o(T) \subset \Sigma(\mathcal{B}_o(T))$. In particular, $V|_{C_o(T)}$ is weakly compact. Besides, by Lemma 6(iii) of [12] and by Proposition 5(iii), we have

$$x^*Vf = \int_T fd(x^* \circ m_o) = \int_T fd(x^* \circ m) = x^*uf, \;\; f \in C_o(T)$$

for each $x^* \in X^*$. Since $Vf \in X^{**}$ and $uf \in X$, we conclude that $Vf = uf$ for each $f \in C_o(T)$. Consequently, $u = V|_{C_o(T)}$ and hence $\{uf : ||f||_T \leq 1\}$ is relatively $\sigma(X^{**}, X^{***})$- compact. Since $u(C_o(T)) \subset X$, it follows that $\{uf : ||f||_T \leq 1\}$ is relatively weakly compact in X. Thus u is weakly compact. Hence (i) holds.

(ii) $\Rightarrow$ (xix) By (ii), Proposition 4(i) and the Orlicz-Pettis theorem, m is σ-additive in $\mathcal{B}(T)$ in the topology τ of X. Then m_o is σ-additive in $\mathcal{B}_o(T)$ and has range in X. Therefore, by Proposition 6 there exists a unique Borel regular X-valued σ-additive (in τ) extension $\hat{m}$ of m_o on $\mathcal{B}(T)$. Then by Proposition 5(iii) and by the fact that each $f \in C_o(T)$ is bounded and Baire measurable (by Theorem 51.B of [7]), we have

$$x^*uf = \int_T fd(x^* \circ m) = \int_T fd(x^* \circ m_o) = \int_T fd(x^* \circ \hat{m})$$

for each $x^* \in X^*$ and $f \in C_o(T)$. Since $x^* \circ m \in M(T)$ by Proposition 5(i) and since $x^* \circ \hat{m} \in M(T)$ as $\hat{m}$ is Borel regular and σ-additive with values in X, we conclude that $x^* \circ m = x^* \circ \hat{m}$ for each $x^* \in X^*$. Since m has range in X^{**} and $\hat{m}$ in X, it follows that $m = \hat{m}$. Thus m is Borel regular in τ and hence m is Borel regular in τ_e as $\tau_e|_X = \tau$. Thus (xix) holds.

Clearly, (xix) $\Rightarrow$ (xx) $\Rightarrow$(xxi).

(xxi)(resp. (xxv), (xxix))$\Rightarrow$(xxviii) Let $U \in \mathcal{U}_o$ or let $U = T$. Let A be an equicontinuous set in X^* and $\epsilon > 0$. Then by hypothesis and by Theorem 50.D of Halmos [7] there exists a compact G_δ K such that $K \subset U$ and $||m||_{p_A}(U \backslash K) < \epsilon$ (resp. $||m_c||_{p_A}(U\backslash K) < \epsilon$, $||m_o||_{p_A}(U\backslash K) < \epsilon$). Thus, in particular, $||m_o||_{p_A}(E) < \epsilon$ for all $E \in \mathcal{B}_o(T)$ with $E \subset U\backslash K$. Since this holds for all $U \in \mathcal{U}_o$ and for $U = T$, conditions (i) and (ii) of Lemma 4 are satisfied by m_o. Therefore, m_o is Baire inner regular in $\mathcal{B}_o(T)$. Hence (xxviii) holds.

(xxviii)$\Rightarrow$(xv) by Lemma 5.

(xix)$\Rightarrow$(xxii) Obvious.

(xxii)$\Rightarrow$(i) Let $K \in \mathcal{K}$ and let A be an equicontinuous subset of X^*. Given $\epsilon > 0$, by hypothesis there exists an open set U in T such that $||m||_{p_A}(U\backslash K) < \epsilon$. Since $u^*x^* = x^* \circ m$ by Proposition 5(ii) and since u^*A is bounded in $M(T)$ by Lemma 1, condition (iii)(a) of Proposition 3 is satisfied by u^*A. Since m is inner regular in T, there exists a compact set C such that $||m||_{p_A}(T\backslash C) < \epsilon$ and hence condition (iii)(b) of the said proposition also holds for u^*A. Hence by Proposition 3, u^*A is relatively weakly compact in $M(T)$ and consequently, by Proposition 2, u is weakly compact. Hence (i) holds.

(ii)$\Rightarrow$(xxiii) Proceeding as in the proof of (ii) $\Rightarrow$ (xix), we have $m = \hat{m}$ on $\mathcal{B}(T)$. Since $\hat{m}|_{\mathcal{B}_c(T)}$ is σ-Borel regular by Proposition 6, we conclude that m_c is σ-Borel regular in τ and hence in τ_e. Thus (xxiii) holds.

(xxiii)$\Rightarrow$(xxiv) Obvious.

(xxiv)$\Rightarrow$(xiv) by Lemma 5.

(xxiii)implies the first part of (xxv) and (xix) implies the second part of (xxv). As (xxv)$\Rightarrow$(xxviii), it follows that (i)$\Leftrightarrow$(xxv).

(xix)$\Rightarrow$(xxvi) Given $K \in \mathcal{K}$, $A \in \mathcal{E}$ and $\epsilon > 0$, then by hypothesis there exists an open set U with $U \supset K$ such that $||m||_{p_A}(U\backslash K) < \epsilon$. By Theorem 50.D of Halmos [7] we can choose a $V \in \mathcal{U}_0$ such that $K \subset V \subset U$ so that $||m_c||_{p_A}(V\backslash K) < \epsilon$. Thus m_c is σ-Borel outer regular in K. Clearly, m_c is σ-Borel inner regular in T as m is Borel regular in T. Hence (xxvi) holds.

(xxvi)$\Rightarrow$(i) Let $K \in \mathcal{K}$. Proceeding as in the proof of (xxii)$\Rightarrow$(i), we have $||m_c||_{p_A}(U\backslash K) < \epsilon$, where U is a σ-Borel open set containing K. Thus $\sup_{x^*\in A} |x^* \circ m_c|(U\backslash K) < \epsilon$. Now by Proposition 1 it follows that $\sup_{x^*\in A} |x^* \circ m|(U\backslash K) < \epsilon$, where the $|x^* \circ m$ is the variation of $x^* \circ m$ with respect to $\mathcal{B}(T)$. Since u^*A is bounded in $M(T)$ by Lemma 1, condition (iii)(a) of Proposition 3 is satisfied by u^*A. Again by hypothesis, there exists a compact C such that $||m_c||_{p_A}(T\backslash C) < \epsilon$. Thus for each compact $K \subset T\backslash C$, by Proposition 1 we have $\sup_{x^*\in A} |x^* \circ m|(K) < \epsilon$. As $|x^* \circ m|$ is Borel regular by Proposition 5(i) for each $x^* \in A$. it follows that $\sup_{x^*\in A} |x^* \circ m|(T\backslash C) \leq \epsilon$. Thus condition (iii)(b) of Proposition 3 is also satisfied by u^*A. Therefore, u^*A is relatively weakly compact in $M(T)$. Now by Proposition 2 we conclude that u is weakly compact. Hence (i) holds.

(xv)$\Rightarrow$(xxvii) Since m_o is σ-additive in τ_e, by the first part of Proposition 6, m_o is Baire regular in τ_e. Thus (xxvii) holds.

Clearly, (xxvii)$\Rightarrow$(xxviii).

(xxviii)$\Rightarrow$ (xv) by Lemma 5.

(xxix)⇒(xxviii) by Lemma 4.

(xix)⇒(xxix) Let $U \in \mathcal{U}_o$, $A \in \mathcal{E}$ and $\epsilon > 0$. By hypothesis, there exists a compact $K \subset U$ such that $||m||_{p_A}(U \backslash K) < \epsilon$. By Theorem 50.D of Halmos [7] there exists a compact $C \in \mathcal{K}_o$ such that $K \subset C \subset U$. Then $||m_o||_{p_A}(U \backslash C) < \epsilon$. Hence m_o is Baire inner regular in U. As m is Borel inner regular in T, there exists $K \in \mathcal{K}$ such that $||m||_{p_A}(T \backslash K) < \epsilon$. By Theorem 50.D of Halmos[7] there exists $C \in \mathcal{K}_o$ such that $K \subset C$ and hence $||m_o||_{p_A}(B) < \epsilon$ for all $B \in \mathcal{B}_o(T)$ with $B \subset T \backslash C$. Thus m_o is Baire inner regular in T. Hence (xxix) holds.

(xix)⇒(xxx) Let $K \in \mathcal{K}_o$, $A \in \mathcal{E}$ and $\epsilon > 0$. By hypothesis, and by Theorem 50.D of Halmos [7] there exists $U \in \mathcal{U}_o$ with $K \subset U$ such that $||m||_{p_A}(U \backslash K) < \epsilon$. En particular, $||m_o||_p(U \backslash K) < \epsilon$. Similarly, we can show that m_o is Baire inner regular in T. Hence (xxx) holds.

(xxx)⇒(xxix) Given $A \in \mathcal{E}$ and $\epsilon > 0$, by the hypothesis of Baire inner regularity in T and by Theorem 50.D of Halmos [7] there exists a compact $\Omega \in \mathcal{K}_o$ such that $||m_o||_{p_A}(T \backslash \Omega) < \frac{\epsilon}{2}$. Let $U \in \mathcal{U}_o$ such that U is relatively compact.

Affirmation 1. m_o is Baire inner regular in U.

In fact, by Theorem 50.D of Halmos [7] we can choose a compact $C \in \mathcal{K}_o$ such that $\bar{U} \subset C$. Then $U = C \backslash (C \backslash U)$ and $C \backslash U \in \mathcal{K}_o$. Therefore, by hypothesis there exists $W \in \mathcal{U}_o$ with $W \supset C \backslash U$ such that $||m_o||_{p_A}(W \backslash (C \backslash U)) < \epsilon$. Now $U = C \backslash (C \backslash U) \supset C \backslash W$ and $C \backslash W \in \mathcal{K}_o$. Moreover, $U \backslash (C \backslash W) = U \cap ((T \backslash C) \cup W) = U \cap W$. On the other hand, $W \backslash (C \backslash U) \supset W \cap U$. Therefore, $||m_o||_{p_A}(U \backslash (C \backslash W)) < \epsilon$. Thus the affirmation holds.

Now let $U \in \mathcal{U}_o$. Choose by Theorem 50.D of Halmos [7] a relatively compact open Baire set V such that $\Omega \subset V$. Then $U \cap V$ is relatively compact and belongs to $\mathcal{U}_o$. Therefore, by Affirmation 1, m_o is Baire inner regular in $U \cap V$ and hence there exists a compact $K \in \mathcal{K}_o$ with $K \subset U \cap V$ such that $||m_o||_{p_A}((U \cap V) \backslash K) < \frac{\epsilon}{2}$. Then $K \subset U$ and $||m_o||_{p_A}(U \backslash K) \leq ||m_o||_{p_A}((U \cap V) \backslash K) + ||m_o||_{p_A}(U \backslash \Omega) < \epsilon$. Therefore, m_o is Baire inner regular in each open Baire set and hence (xxix) holds.

(ii)⇒(xxxi), (xxxii) and (xxxiii). By (ii), Proposition 5(i) and the Orlicz-Pettis theorem m is X-valued and σ-additive in τ. Since every bounded Borel (resp. σ-Borel, Baire) measurable scalar function is the uniform limit of a sequence of Borel (σ-Borel, Baire) simple functions and m is a τ-bounded X-valued vector measure, f is m-integrable (see Definition 1 of [12]) and $\int_T f dm \in X$ (resp. $\int_T f dm_c \in X$, $\int_T f dm_o \in X$).

Clearly, (xxxiii)⇒(xxxiv).

(xxxi) (resp. (xxxii), (xxxiii)) ⇒(ii) (resp. (iii), (iv)) Let $E \in \mathcal{B}(T)$ (resp. $E \in \mathcal{B}_c(T)$, $E \in \mathcal{B}_o(T)$). Then by hypothesis, $m(E)$ (resp. $m_c(E)$, $m_o(E)$) belongs to X. Thus (ii) (resp. (iii), (iv)) holds.

(xxxiv) $\Rightarrow$(viii) Let U be an open Baire set. Then by § 14, Chapter III of Dinculeanu [2], there exists an increasing sequence K_n of compact G_δs such that $U = \cup_1^\infty K_n$. Then by Urysohn's lemma we can choose continuous functions g_n with compact support such that $g_n \nearrow \chi_U$. Thus χ_E belongs to the first Baire class and is bounded. Then by hypothesis, $m_o(U) \in X$. Thus (viii) holds.

(i)$\Rightarrow$(xxxv) If u is weakly compact, then by Proposition 2, u^{**} has range in X. Since the bounded scalar functions of the first Baire class belong to $C_o^{**}(T)$, (xxxv) holds.

(xxxv)$\Rightarrow$(viii) By Proposition 5(v), $u^{**}(\chi_U) = m(U)$ for $U \in \mathcal{U}_o$. As observed in the proof of (xxxiv) $\Rightarrow$ (viii), χ_U is bounded and belongs to the first Baire class. Hence, by hypothesis, $m(U) \in X$. Thus (viii) holds.

This completes the proof of the theorem.

Remark 2. As in [14], the strict Dunford-Pettis property of $C_o(T)$ is an immediate consequence of the above theorem and the proof of the latter is not based on this property unlike the proof of Theorem 6 of Grothendieck [6]. Theorem 5.3 of Thomas [16] is also deducible from the above theorem by the same argument used in the proof of Theorem 13 in [14], where our proof is direct and does not use the technique of reduction to metrizable compact case.

Remark 3. All these 35 characterizations are given in [14] in Theorems 2-9. Some of the proofs given here are the same as those in [14], but for the sake of completenss we have given the proof of the equivalence of all the assertions. The proof of Theorems 2 and 9 of [14] is the same as that given here for the corresponding assertions. Also the proof of the equivalence of (ii), (vii) and (xi) of Theorem 3 of [14] is the same as that of the equivalence of (ii), (viii) and (xii) of the above theorem. But the proof of the equivalence of (i) and (viii) in the above theorem is different from that given in the proof of Theorem 3 of [14]. The present proof is also different from the proof of Theorems 4 and 5 and that of the equivalence among the first three assertions of Theorem 6 of [14]. Also the proof of Theorems 7 and 8 of [14] is different from the present one (except for the proof of (xxx)$\Rightarrow$ (xxix) and the details of this implication are suppressed in [14]).

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