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Abstract

In this paper we study the exponential bounds and the asymptotic stability of the zero solution of a non-linear system of parabolic equations with Neumann boundary conditions. First, we write the parabolic equation as an abstract ordinary differential equation in a Hilbert space. Second, we study the linear part of this ODE and find the exponential bounds. Finally, we use the variational constant formula to prove the asymptotic stability of the zero solution of this non-linear system.

Key words. system of parabolic equations, exponential bounds, asymptotic stability.

AMS(MOS) subject classifications. primary: 34G10; secondary: 35B40.

1 Introduction

In this paper we shall study the asymptotic stability of the zero solution for the following system of parabolic equations with homogeneous Neumann boundary conditions

$$u_t = D\Delta u + A(t)u + f(t,u), \quad t \geq 0, \quad u \in I\!R^n, \tag{1}$$

$$\frac{\partial u}{\partial \eta} = 0 \quad , \quad \text{on } \partial\Omega \tag{2}$$

where $f \in C^1(I\!R \times I\!R^n)$, $A(t)$ is a continuous $n \times n$ matrix, $D = \text{diag}(d_1, d_2, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, \quad i = 1, 2, \ldots, n$ and Ω is a bounded domain in $I\!R^N (N = 1, 2, 3)$.

In order to do that; we first study the exponential bounds and the asymptotic stability of the linear system:

$$u_t = D\Delta u + A(t)u, \quad t \geq 0, \quad u \in I\!R^n, \tag{3}$$

$$\frac{\partial u}{\partial \eta} = 0 \quad , \quad \text{on } \partial\Omega \tag{4}$$

The diffusion coefficients $d_i, \quad i = 1, 2, \ldots, n$ could be of any size (**big or small**), but we shall assume that they are closed each other. More precisely, we will suppose that $|d_i - d_j|; i, j = 1, 2, \ldots, n$ are small enough.

Under this assumption, roughly speaking we prove the following statements:

S_1) If $A(t) = A + B(t)$, where A is a constant $n \times n$ matrix whose eigenvalues have all negative real parts, $\lim_{t\to\infty} \|B(t)\| = 0$ and the function $f(t,y)$ satisfies the condition

$$\lim_{\|y\|\to 0} \frac{\|f(t,y)\|}{\|y\|} = 0, \quad \text{uniformly on} \quad t, \tag{5}$$

then for some $t_0 > 0$ the solution $u = 0$ of the system (1)-(2) is uniformly asymptotically stable on $[t_0, \infty)$.

S_2) If the function f satisfies condition (5), the matrix $A(t)$ is periodic of period τ and the Floquet exponents of the system $y' = A(t)y$ have negative real parts, then the solution $u = 0$ of the system (1)-(2) is uniformly asymptotically stable.

Several mathematical models may be written as a system of reaction-diffusion of the form (1), like a models of vibration of plates(see [2]) and a Lotka-Volterra system with diffusion(see [4]).

2 Notations and Preliminaries

In this section we shall choose the space where this problem will be set.

Let $X = L^2(\Omega) = L^2(\Omega, I\!R)$ and consider the linear unbounded operator $A : D(A) \subset X \to X$ defined by $A\phi = -\Delta\phi$, where

$$D(A) = \{\phi \in H^2(\Omega, I\!R) : \frac{\partial \phi}{\partial \eta} = 0 \ \text{ on } \ \partial\Omega\}. \tag{6}$$

Since this operator is sectorial, then the fractional power space X^α associated with A can be defined. That is to say: for $\alpha \geq 0$, $X^\alpha = D(A_1^\alpha)$ endowed with the graph norm

$$\|x\|_\alpha = \|A_1^\alpha x\|, \quad x \in X^\alpha \ \text{ and } \ A_1 = A + aI, \tag{7}$$

where $\mathrm{Re}\sigma(A_1) > 0$. The norm $\|\cdot\|_\alpha$ does not depend on a (see D. Henry [5] pg 29).

Precisely we have the following situation: Let $0 = \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty$ be the eigenvalues of A each one with finite multiplicity γ_j equal to the dimension of the corresponding eigenspace. Therefore

a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A.

b) for all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j x, \tag{8}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{9}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x, \quad x \in X$.

c) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = E_1 x + \sum_{j=2}^{\infty} e^{-\lambda_j t} E_j x. \tag{10}$$

d) for $a > 0$

$$X^\alpha = D(A_1^\alpha) = \{x \in X : \sum_{j=1}^{\infty} (\lambda_j + a)^{2\alpha} \|E_j x\|^2 < \infty\},$$

and

$$A_1^\alpha x = \sum_{j=1}^{\infty} (\lambda_j + a)^\alpha E_j x. \tag{11}$$

Also, we shall use the following notation:

$$Z := L^2(\Omega, I\!R^n) = X^n = X \times \cdots \times X, \text{ and } C_n = C(\Omega, I\!R^n) = [C(\Omega)]^n,$$

with the usual norms.

Now, we define the following operator

$$\mathcal{A}_D : D(\mathcal{A}_D) \subset Z \to Z, \quad \mathcal{A}_D \psi = -D\Delta\psi = DA\psi, \tag{12}$$

where

$$D(\mathcal{A}_D) = \{\phi \in H^2(\Omega, I\!R^n) : \frac{\partial \phi}{\partial \eta} = 0 \text{ on } \partial\Omega\}.$$

Therefore, $\mathcal{A}_D$ is a sectorial operator and the fractional power space Z^α associated with $\mathcal{A}_D$ is given by

$$Z^\alpha = D(\mathcal{A}_{D1}^\alpha) = X^\alpha \times \cdots \times X^\alpha = [X^\alpha]^n. \tag{13}$$

endowed with the graph norm

$$\|z\|_\alpha = \|\mathcal{A}_{D1}^\alpha z\|, \quad z \in Z^\alpha \text{ and } \mathcal{A}_{D1} = \mathcal{A}_D + aI, \tag{14}$$

where

$$a > 0, \quad \mathcal{A}_{D1}^\alpha z = \sum_{j=1}^{\infty} D^\alpha (\lambda_j + a)^\alpha P_j z, \qquad D^\alpha = \operatorname{diag}(d_1^\alpha, d_2^\alpha, \cdots, d_n^\alpha), \tag{15}$$

and $P_j = \operatorname{diag}(E_j, E_j, \cdots, E_j)$ is an $n \times n$ matrix.

The C_o−semigroup $\{e^{-\mathcal{A}_D t}\}_{t \geq 0}$ generated by $-\mathcal{A}_D$ is given as follow

$$e^{-\mathcal{A}_D t} z = P_1 z + \sum_{j=2}^{\infty} e^{-\lambda_j D t} P_j z, \quad z \in Z. \tag{16}$$

Clearly, $\{P_j\}$ is a family of orthogonal projections in Z which is complete. So,

$$z = \sum_{j=1}^{\infty} P_j z, \quad \|z\|^2 = \sum_{j=1}^{\infty} \|P_j z\|^2 \text{ and } \|z\|_\alpha^2 = \sum_{j=1}^{\infty} \|P_j z\|_\alpha^2. \tag{17}$$

From (16) it follows that there exists a constant $M > 0$ such that for all $z \in Z^\alpha$

$$\begin{aligned} \|e^{-\mathcal{A}_D t} z\|_\alpha &\le M\|z\|_\alpha, \quad t \ge 0, & (18)\\ \|e^{-\mathcal{A}_D t} z\|_\alpha &\le M t^{-\alpha}\|z\|, \quad t > 0. & (19)\end{aligned}$$

From Theorem 1.6.1 in D. Henry [5] it follows that for $\frac{3}{4} < \alpha \le 1$ the following inclusions

$$Z^\alpha \subset C(\Omega, \mathbb{R}^n) \quad \text{and} \quad Z^\alpha \subset L^p(\Omega, \mathbb{R}^n), \quad p \ge 2, \tag{20}$$

are continuous.

2.1 Setting the Problem

Now, the systems (1)-(2) and (3)-(4) can be written in an abstract way on Z as follow:

$$\begin{aligned} z' &= -\mathcal{A}_D z + \mathcal{A}(t) z + f^e(t, z), \quad z(t_0) = z_0 \quad t \ge t_0 > 0. & (21)\\ z' &= -\mathcal{A}_D z + \mathcal{A}(t) z, \quad t \ge 0. & (22)\end{aligned}$$

Where $\mathcal{A}(t)z(x) = A(t)z(x)$ and the function $f^e : \mathbb{R} \times Z^\alpha \to Z$ is given by:

$$f^e(t, z)(x) = f(t, z(x)), \quad x \in \Omega. \tag{23}$$

To show that equation (21) is well posed in Z^α we have to prove the following lemma.

Lemma 2.1 *The function f^e given in (23) is locally Hölder continuous in t and locally Lipschitz in z. i.e., given an interval $[a, b]$ and a ball $B_r^\alpha(0)$ in Z^α there exists $\theta > 0$ and $L > 0$ such that*

$$\|f^e(t, z_1) - f^e(s, z_2)\| \le L(|t - s|^\theta + \|z_1 - z_2\|_\alpha), \quad \|z_1\|_\alpha, \|z_2\|_\alpha \le r, \quad t, s \in [a, b].$$

Proof Since $f \in C^1(\mathbb{R} \times \mathbb{R}^n)$, then for each interval $[a, b]$ and a ball $B_\rho(0) \subset \mathbb{R}^n$ there exist constants $k > 0$ and $M(\rho) > 0$ such that

$$\|f(t, x) - f(s, y)\| \le k|t - s| + M(\rho)\|x - y\| \text{ if } \|x\|, \|y\| \le \rho, \quad t, s \in [a, b].$$

From the continuous inclusion $Z^\alpha \subset C_n$ there exists $l > 1$ such that

$$\sup_{x \in \Omega} \|z(x)\|_{\mathbb{R}^n} \le l\|z\|_\alpha, \quad z \in Z^\alpha.$$

Now, let $B_r^\alpha(0)$ be a ball in Z^α. Then putting $\rho = lr$ we get that

$$\|f(t, z_1(x)) - f(s, z_2(x))\| \le k|t - s| + M(lr)\|z_1(x) - z_2(x)\|, \quad x \in \Omega,$$

if $\|z_1\|_\alpha, \|z_2\|_\alpha \le r$ and $t, s \in [a, b]$.

Therefore, if $\|z_1\|_\alpha, \|z_2\|_\alpha \in B_r^\alpha(0)$ and $t, s \in [a, b]$, then

$$\|f^e(t, z_1) - f^e(s, z_2)\| \le k\mu(\Omega)^{1/2}|t - s| + M(lr)\|z_1 - z_2\|,$$

where $\mu(\Omega)$ denote the Lebesgue measure of Ω.

Now, from the continuous inclusion $Z^\alpha \subset L^2(\Omega, I\!R^n)$ there exists a constant $R > 0$ such that

$$\|z\|_{L^2} \leq R\|z\|_\alpha, \quad z \in Z^\alpha.$$

Hence, if $\|z_1\|_\alpha, \|z_2\|_\alpha \in B_r^\alpha(0)$ and $t, s \in [a, b]$, then

$$\|f^e(t, z_1) - f^e(s, z_2)\| \leq k\mu(\Omega)^{1/2}|t - s| + RM(lr)\|z_1 - z_2\|_\alpha.$$

We complete the proof by putting $\theta = 1$ and $L = \max\{k\mu(\Omega)^{1/2}, RM\}$. □

The following proposition can be proved in the same way as the foregoing lemma.

Proposition 2.1 *If the function $f(t, y)$ satisfies the condition (5), then for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\|f^e(t, z)\| \leq \epsilon\|z\|_\alpha, \quad \text{if} \quad \|z\|_\alpha \leq \delta. \tag{24}$$

From now on, we will suppose that $\frac{3}{4} < \alpha < 1$.

3 Main Theorems

Now, we are ready to formulate the main results of this paper, which are statements S_1) and S_2) of the Introduction.

Theorem 3.1 *Suppose the function $f(t, y)$ satisfies the condition (5) and $A(t) = A + B(t)$ with*

$$a = max\{Re\rho : \rho \in \sigma(A)\} < 0 \quad \text{and} \quad \lim_{t\to\infty} \|B(t)\| = 0. \tag{25}$$

Then the following holds:
A) if $D = diag(d, d, \cdots, d) = dI$, then for some $t_0 > 0$ the solution $z = 0$ of the equation (21) is uniformly asymptotically stable.
B) if $D = dI + diag(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ with ϵ_i small enough, then for some $t_0 > 0$ the solution $z = 0$ of the equation (21) is uniformly asymptotically stable.

Theorem 3.2 *Suppose the function $f(t, y)$ satisfies the condition (5), the matrix $A(t)$ is periodic of period τ and the Floquet exponents of the system $y' = A(t)y$ have negative real parts. Then the following holds:*
A) if $D = diag(d, d, \cdots, d) = dI$, then the solution $z = 0$ of the equation (21) is uniformly asymptotically stable.
B) if $D = dI + diag(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ with ϵ_i small enough, then the solution $z = 0$ of the equation (21) is uniformly asymptotically stable.

3.1 Proof of Theorem 3.1

In this section we shall assume that $A(t) = A + B(t)$, so the operator $\mathcal{A}(t)$ can be written as follows

$$\mathcal{A}(t) = \mathcal{A} + \mathcal{B}(t), \quad \mathcal{A}z(x) = Az(x), \quad \mathcal{B}(t)z(x) = B(t)z(x).$$

Before we prove Theorem 3.1, we shall give some lemmas.

Lemma 3.1 *Let $a = max\{Re\rho : \rho \in \sigma(A)\}$ and $T(t,s)$ the evolution operator generated by the equation (22). Then for all $b > a$ there exist constants $R > 0$ and $k > 0$ depending on A and b such that*

$$\|T(t,s)z\|_\alpha \leq R \quad \|z\|_\alpha exp\left(b(t-s) + k\int_s^t \|B(\tau)\|d\tau\right), t \geq s \tag{26}$$

$$\|T(t,s)z\|_\alpha \leq R \quad \|z\|(t-s)^{-\alpha} exp\left(b(t-s) + k\int_s^t \|B(\tau)\|d\tau\right), t > s. \tag{27}$$

Proof Since the operators $\mathcal{A}(t)$ and $-\mathcal{A}_d$ commute, then the evolution operator corresponding to the equation (22) is given by:

$$T(t,s) = e^{-\mathcal{A}_d(t-s)}U(t)U^{-1}(s) \in L(Z^\alpha, Z), \tag{28}$$

where $e^{-\mathcal{A}_D t}$ is the strongly continuous semigroup generated by $-\mathcal{A}_D$ and $U(t)$ is the fundamental matrix of the linear system of ODEs $\dot{x}(t) = A(t)x(t)$. i.e.,

$$\begin{cases} \dot{U}(t) = & A(t)U(t), \\ U(0) = & I. \end{cases} \tag{29}$$

Therefore, from (18) and (19) we obtain that

$$\begin{aligned} \|T(t,s)z\|_\alpha &\leq M\|z\|_\alpha \|U(t)U^{-1}(s)\|, \quad t \geq s, \\ \|T(t,s)z\|_\alpha &\leq M(t-s)^{-\alpha}\|z\|\|U(t)U^{-1}(s)\|, \quad t > s. \end{aligned}$$

To complete the proof, it is enough to show that

$$\|U(t)U^{-1}(s)\| \leq kexp\left(b(t-s) + k\int_s^t \|B(\tau)\|d\tau\right), \quad t \geq s. \tag{30}$$

In fact; if we put $\Psi(t,s) = U(t)U^{-1}(s)$, then

$$\Psi(t,s)y = e^{A(t-s)}y + \int_s^t e^{A(t-\tau)}B(\tau)\Psi(\tau,s)yds.$$

From the Jordan form of A we get that for $b > a$ there exists $k > 0$ such that

$$\|e^{A(t-s)}\| \leq ke^{b(t-s)}, \quad t \geq s.$$

Then

$$\|\Psi(t,s)y\| \leq ke^{b(t-s)}\|y\| + k\int_s^t e^{b(t-\tau)}\|B(\tau)\|\|\Psi(\tau,s)\|\|y\|d\tau.$$

Therefore

$$\|\Psi(t,s)\| \leq ke^{b(t-s)} + k\int_s^t e^{b(t-\tau)}\|B(\tau)\|\|\Psi(\tau,s)\|d\tau.$$

Applying the Gronwall's Lemma we get that

$$\|\Psi(t,s)y\| \leq kexp\left(b(t-s) + k\int_s^t \|B(\tau)\|d\tau\right), \quad t \geq s.$$

□

Corolary 3.1 *If* $a = \quad max\{Re\rho : \rho \in \sigma(A)\} < 0$ *and*

$$bt + k\int_0^t \|B(s)\|ds \to -\infty, \quad as \quad t \to +\infty,$$

then the zero solution of the equation (22) is Asymptotically stable.

Corolary 3.2 *The operator* $-\mathcal{A}_d + \mathcal{A}$ *generates a strongly continuous semi-group* $\{e^{(-\mathcal{A}_d+\mathcal{A})t}\}_{t\geq 0}$ *which satisfies the following estimates:*

$$\|e^{(-\mathcal{A}_d+\mathcal{A})(t-s)}z\|_\alpha \quad \leq R \quad \|z\|_\alpha e^{b(t-s)}, \quad t \geq s \tag{31}$$

$$\|e^{(-\mathcal{A}_d+\mathcal{A})(t-s)}z\|_\alpha \quad \leq R \quad \|z\|(t-s)^{-\alpha}e^{b(t-s)}, \quad t > s. \tag{32}$$

Proof of Theorem 3.1 part A).
Now, we are ready to prove part A) of Theorem 3.1. From proposition 2.1 and the assumption on $B(t)$, for $\epsilon > 0$ small enough there exist $\delta > 0$ such that

$$\|f^e(t,z)\| \leq \epsilon\|z\|_\alpha, \text{ if } \|z\|_\alpha \leq \delta, \quad t \geq 0,$$

$$\epsilon RL\int_0^\infty s^{-\alpha}e^{(b+\beta')s}ds < \frac{1}{4},$$

and

$$\|\mathcal{B}(t)\| = \|B(t)\| < \epsilon, \quad t \geq t_0.$$

Where $\max\{Re\rho : \rho \in \sigma(A)\} < b < 0$, $0 < \beta' < -b$, $L \geq 1$ is a constant given by the **continuous inclusion** $Z^\alpha \subset Z$ and t_0 is big enough.

The initial value problem 21 can be written as follow

$$z' = (-\mathcal{A}_d + \mathcal{A})z + \mathcal{B}(t)z + f^e(t,z), \quad z(t_0) = z_0 \quad t \geq t_0 > 0. \tag{33}$$

Then, from Theorem 7.1.4 in [5], for all $T > t_0$ we have the following:

A continuous function $z(\cdot) : (t_0, T) \to Z^\alpha$ is solution of the integral equation

$$z(t) = e^{(-\mathcal{A}_d+\mathcal{A})(t-t_0)}z_0 + \int_{t_0}^t e^{(-\mathcal{A}_d+\mathcal{A})(t-s)}[\mathcal{B}(s)z(s) + f^e(s,z(s))]ds, \quad t \in (t_0, T] \tag{34}$$

if and only if $z(\cdot)$ is a solution of (33).

Now, let $z(t, t_0, z_0)$ be the solution of (33) starting in z_0 at $t = t_0$ with $\|z_0\|_\alpha < \frac{\delta}{2RL}$. Then $\|z(t)\|_\alpha \leq \delta$ on some interval $[t_0, t_1)$. As long as $\|z(t)\|_\alpha$ remains less than δ we get the following:

$$\begin{aligned}
\|z(t)\|_\alpha &= \|e^{(-\mathcal{A}_d+\mathcal{A})(t-t_0)}z_0 + \int_{t_0}^{t} e^{(-\mathcal{A}_d+\mathcal{A})(t-s)}(\mathcal{B}(s)z(s) + f^e(s, z(s)))ds\|_\alpha \\
&\leq Re^{-\beta'(t-t_0)}\|z_0\|_\alpha + R\int_{t_0}^{t}(t-s)^{-\alpha}e^{b(t-s)}\|f^e(s, z(s))\|ds \\
&+ R\int_{t_0}^{t}(t-s)^{-\alpha}e^{b(t-s)}\|B(s)\|\|z(s)\|ds \\
&\leq RLe^{-\beta' t}\|z_0\|_\alpha + 2\epsilon RL\int_{t_0}^{t}(t-s)^{-\alpha}e^{b(t-s)}\|z(s)\|_\alpha ds \\
&\leq \frac{\delta}{4} + \delta 2\epsilon RL\int_{t_0}^{t}(t-s)^{-\alpha}e^{b(t-s)}ds < \delta.
\end{aligned}$$

If $\|z(t)\|_\alpha < \delta$ on $[t_0, t_1)$ with t_1 been maximal, then either $t_1 = \infty$ or $\|z(t_1)\|_\alpha = \delta$. But the second case contradicts this computation. Therefore, the solution remains in the ball $B^\alpha(0, \delta)$ of center zero and radio δ in Z^α for $t \geq t_0$.

If we put $u(t) = \sup\{\|z(s)\|_\alpha e^{\beta'(t-s)} : t_0 \leq s \leq t\}$, then

$$\begin{aligned}
\|z(t)\|_\alpha e^{\beta'(t-t_0)} &\leq RL\|z_0\|_\alpha + 2\epsilon RL\int_{t_0}^{t}(t-s)^{-\alpha}e^{(b+\beta')(t-s)}ds\,u(t) \\
&\leq RL\|z_0\|_\alpha + \frac{1}{2}u(t).
\end{aligned}$$

So

$$u(t) \leq RL\|z_0\|_\alpha + \frac{1}{2}u(t).$$

Then $u(t) \leq 2RL\|z_0\|_\alpha$. Therefore

$$\|z(t)\|_\alpha \leq 2RL\|z_0\|_\alpha e^{-\beta'(t-t_0)}, \quad t \geq t_0.$$

This complete the proof of Theorem 3.1 part A).

Proof of Theorem 3.1 part B).

In this case $D = dI + E$ with $E = \text{diag}(\epsilon_1, \epsilon_2, \cdots, \epsilon_n)$ and ϵ_i been small enough. The problem (21) can be written as follows

$$z' = (-\mathcal{A}_D + \mathcal{A})z + \mathcal{B}(t)z + f^e(t, z), \quad z(t_0) = z_0 \quad t \geq t_0 > 0. \tag{35}$$

Here $-\mathcal{A}_D = -\mathcal{A}_d - \mathcal{A}_E$, with $-\mathcal{A}_d\phi = d\Delta\phi$ and $\mathcal{A}_E\phi = E\Delta\phi$.

Then the proof of part B) of this Theorem follows from the following lemma, in the same way as part A).

Lemma 3.2 *If $\|E\| = \epsilon$ is small enough, then there exist constants $R > 0$ such that the strongly continuous semigroup $\{e^{(-\mathcal{A}_D+\mathcal{A})t}\}_{t\geq 0}$ generated by $-\mathcal{A}_D + \mathcal{A}$ satisfies the following estimates:*

$$\|e^{(-\mathcal{A}_D+\mathcal{A})(t-s)}z\|_\alpha \leq R\ \|z\|_\alpha e^{\frac{b}{2}(t-s)}, \quad t \geq s \tag{36}$$

$$\|e^{(-\mathcal{A}_D+\mathcal{A})(t-s)}z\|_\alpha \leq R\ \|z\|(t-s)^{-\alpha}e^{\frac{b}{2}(t-s)}, \quad t > s, \tag{37}$$

where $max\{Re\lambda : \lambda \in \sigma(A)\} < b < 0$.

Proof In this case the operators $\mathcal{A}(t)$ and $-\mathcal{A}_D$ do not commute, then we can look the equation

$$z' = (-\mathcal{A}_D + \mathcal{A})z = (-\mathcal{A}_d + \mathcal{A})z - \mathcal{A}_E z, \quad t > 0, \tag{38}$$

as an **unbounded perturbation**($-\mathcal{A}_E$ is an unbounded operator) of the equation

$$z' = (-\mathcal{A}_d + \mathcal{A})z, \quad t > 0. \tag{39}$$

Let us define:

$$\Psi(t,s) = e^{-(-\mathcal{A}_D+\mathcal{A})(t-s)} \text{ and } T(t,s) = e^{-(-\mathcal{A}_d+\mathcal{A})(t-s)}, \quad t \geq s.$$

Then using the orthogonal projections $\{P_j\}_{j\geq 1}$ given by (16) we get that

$$\Psi(t,s)z = \sum_{j=1}^{\infty} \Psi_j(t,s)P_j z, \tag{40}$$

$$T(t,s)z = \sum_{j=1}^{\infty} T_j(t,s)P_j z. \tag{41}$$

Therefore, $\Psi_j(t,s)$ and $T_j(t,s)$ are the evolution operators for the following systems of ODEs:

$$y' = (-\lambda_j dI + A)y - \lambda_j E y, \quad y \in \mathrm{Ran}(P_j), \quad j = 1, 2, \ldots, \tag{42}$$

$$y' = (-\lambda_j dI + A)y, \quad y \in \mathrm{Ran}(P_j), \quad j = 1, 2, \ldots. \tag{43}$$

On the other hand, using the formula (16) and $e^{-\mathcal{A}_d t}P_j z = e^{-\lambda_j d t}P_j z$ we get the following estimates

$$\|e^{-\mathcal{A}_d t}P_j z\|_\alpha \leq M\|P_j z\|_\alpha e^{-\lambda_j d(t-s)}, \quad t \geq s, \tag{44}$$

$$\|e^{-\mathcal{A}_d t}P_j z\|_\alpha \leq M(t-s)^{-\alpha}\|P_j z\| e^{-\lambda_j d(t-s)}, \quad t > s. \tag{45}$$

From (43) we get that

$$T_j(t,s)y = e^{-\lambda_j d(t-s)} e^{A(t-s)} y, \quad y \in \ \mathrm{Ran}(P_j), \quad j = 1, 2, \ldots.$$

Therefore

$$\|T_j(t,s)y\|_\alpha \leq R\|y\|_\alpha e^{(-\lambda_j d + b)(t-s)}, \quad t \geq s, \tag{46}$$

$$\|T_j(t,s)y\|_\alpha \leq R(t-s)^{-\alpha}\|y\| e^{(-\lambda_j d + b)(t-s)}, \quad t > s. \tag{47}$$

Now, the operator $\Psi_j(t,s)$ is given by the variational constant formula:

$$\Psi_j(t,s)y = T_j(t,s)y + \int_s^t T_j(t,\tau)(-\lambda_j E)\Psi_j(\tau,s)y\,d\tau, \quad t \geq s. \tag{48}$$

This implies that

$$\|\Psi_j(t,s)y\|_\alpha \leq R\|y\|_\alpha e^{(-\lambda_j d+b)(t-s)} + \int_s^t R\epsilon\lambda_j e^{(-\lambda_j d+b)(t-\tau)}\|\Psi_j(\tau,s)y\|_\alpha d\tau.$$

Now, putting $u(t) = e^{(\lambda_j d-b)(t-s)}\|\Psi_j(t,s)y\|_\alpha$ we get

$$u(t) \le R\|y\|_\alpha + \int_s^t R\epsilon\lambda_j u(\tau)d\tau.$$

Hence, applying Gronwall's Lemma we get that

$$u(t) \le R\|y\|_\alpha e^{R\epsilon\lambda_j(t-s)}, \quad t \ge s.$$

So

$$\|\Psi_j(t,s)y\|_\alpha \le R\|y\|_\alpha e^{((R\epsilon-d)\lambda_j+b)(t-s)}, \quad t \ge s.$$

If we take $\epsilon < \frac{d}{R}$, then $(R\epsilon - d)\lambda_j < 0$. Using (48) again we get

$$\begin{aligned}\|\Psi_j(t,s)y\|_\alpha &\le R(t-s)^{-\alpha}\|y\|e^{(-\lambda_j+b)(t-s)} \\ &+ \int_s^t R\epsilon\lambda_j(t-\tau)^{-\alpha}e^{(-\lambda_j d+b)(t-\tau)}\|\Psi_j(\tau,s)y\|_\alpha d\tau.\end{aligned}$$

Now, putting $u(t) = (t-s)^\alpha e^{(\lambda_j d-b/2)(t-s)}\|\Psi_j(t,s)y\|_\alpha$ we get

$$u(t) \le R\|y\| + R\epsilon\lambda_j\int_s^t \left(\frac{t-s}{\tau-s}\right)^\alpha e^{\frac{b}{2}(t-\tau)}u(\tau)d\tau.$$

Applying Gronwall's Lemma we get

$$u(t) \le R\|y\|\exp\{\epsilon R\lambda_j(t-s)\int_s^t(\tau-s)^{-\alpha}e^{\frac{b}{2}(\tau-s)}d\tau\}.$$

Since $\frac{3}{4} < \alpha < 1$, then there exists a constant $H > 0$ such that

$$\int_s^t(\tau-s)^{-\alpha}e^{\frac{b}{2}(\tau-s)}d\tau \le H, \quad t > s \ge 0.$$

Therefore

$$u(t) \le R\|y\|e^{\epsilon RH\lambda_j(t-s)}.$$

Hence

$$\|\Psi(t,s)y\|_\alpha \le R\|y\|(t-s)^{-\alpha}e^{(\lambda_j(\epsilon RH-d)+\frac{b}{2})(t-s)}, \quad t > s.$$

Now, if we take also $\epsilon < \frac{d}{RH}$, then $\lambda_j(\epsilon RH - d) < 0$.

Therefore, we get the following estimates:

$$\|\Psi_j(t,s)y\|_\alpha \le R\|y\|_\alpha e^{\frac{b}{2}(t-s)}, \quad t \ge s, \tag{49}$$

$$\|\Psi(t,s)y\|_\alpha \le R\|y\|(t-s)^{-\alpha}e^{\frac{b}{2}(t-s)}, \quad t > s. \tag{50}$$

Now, from (40) we get that

$$\begin{aligned}\|\Psi(t,s)z\|_\alpha^2 &= \sum_{j=1}^\infty \|\Psi_j(t,s)P_jz\|_\alpha^2 \\ &\le \sum_{j=1}^\infty R^2e^{b(t-s)}\|P_jz\|_\alpha^2 \\ &= R^2e^{b(t-s)}\sum_{j=1}^\infty\|P_jz\|_\alpha^2 = R^2e^{b(t-s)}\|z\|_\alpha^2.\end{aligned}$$

So

$$\|\Psi(t,s)z\|_\alpha \le R\|z\|_\alpha e^{\frac{b}{2}(t-s)}, \quad t \ge s.$$

In the same way we get that

$$\|\Psi(t,s)z\|_\alpha \le R(t-s)^{-\alpha}\|z\|_\alpha e^{\frac{b}{2}(t-s)}, \quad t > s.$$

□

4 Proof of Theorem 3.2

Since $A(t)$ is periodic of period τ, it is well known that the fundamental matrix $U(t)$ of the system $x' = A(t)x$ can be written as follows

$$U(t) = N(t)e^{tL}, \quad N(t+\tau) = N(t), \quad t \in I\!R, \tag{51}$$

where $N(t)$ is a continuous matrix and L is a constant matrix.

Definition 4.1 *The eigenvalues of the matrix L given by (51) are called the Floquet exponentes of the system $x' = A(t)x$.*

Proof of part A).
Since the operators $-\mathcal{A}_d$ and $\mathcal{A}(t)$ commute, then the evolution operator $T(t,s)$ associated to equation (22) is given by:

$$T(t,s)z = e^{-\mathcal{A}_d(t-s)}U(t)U^{-1}(s)z = N(t)e^{-\mathcal{A}_d(t-s)}e^{L(t-s)}N^{-1}(s)z. \tag{52}$$

If $a = \max\{\text{Re}\lambda : \lambda \in \sigma(L)\} < b < 0$, then there exists a constant $l > 0$ such that

$$\|U(t)U^{-1}(s)\| \le le^{b(t-s)}, \quad t \ge s.$$

Hence, using (18) and (19) we get

$$\|T(t,s)z\|_\alpha \ge R\|z\|_\alpha e^{b(t-s)}, \quad t \ge s, \tag{53}$$

$$\|T(t,s)z\|_\alpha \ge R\|z\|(t-s)^{-\alpha}e^{b(t-s)}, \quad t > s. \tag{54}$$

Now, from Theorem 7.14 in [5] we have that the solution $z(\cdot) : (t_0,\infty) \to Z^\alpha$ of (21) is given by

$$z(t) = T(t,t_0)z_0 + \int_{t_0}^{t} T(t,s)f^e(s,z(s))ds, \quad t \in [t_0,+\infty). \tag{55}$$

From here, the result follows easier than the proof of Theorem 3.1 part A).
Proof of Theorem 3.2 part B).
In this case $D = dI + E$ with $E = \text{diag}(\epsilon_1,\epsilon_2,\cdots,\epsilon_n)$ and ϵ_i been small enough. The problem (21) can be written as follows

$$z' = (-\mathcal{A}_d + \mathcal{A}(t))z - \mathcal{A}_E z + f^e(t,z), \quad z(t_0) = z_0 \quad t \ge t_0 > 0. \tag{56}$$

Then the proof of part B), follows from the following lemma, in the same way as part A).

Lemma 4.1 *If* $||E|| = \epsilon$ *is small enough, then there exists a constants* $R > 0$ *such that the evolution operator* $\Psi(t,s)$ *generated by the equation*

$$z' = (-\mathcal{A}_d + \mathcal{A}(t))z, \quad t > 0, \tag{57}$$

satisfies the following estimates:

$$\begin{aligned} ||\Psi(t,s)z||_\alpha &\leq R \quad ||z||_\alpha e^{\frac{b}{2}(t-s)}, \quad t \geq s \quad (58)\\ ||\Psi(t,s)z||_\alpha &\leq R \quad ||z||(t-s)^{-\alpha} e^{\frac{b}{2}(t-s)}, \quad t > s, \quad (59)\end{aligned}$$

where $max\{Re\lambda : \lambda \in \sigma(L)\} < b < 0$.

Proof We can look the equation

$$z' = (-\mathcal{A}_D + \mathcal{A}(t))z = (-\mathcal{A}_d + \mathcal{A}(t))z - \mathcal{A}_E z, \quad t > 0, \tag{60}$$

as an **unbounded perturbation** ($-\mathcal{A}_E$ is an unbounded operator) of the equation

$$z' = (-\mathcal{A}_d + \mathcal{A}(t))z, \quad t > 0. \tag{61}$$

If $T(t,s)$ is the evolution operator generated by the equation (61), then using the orthogonal projections $\{P_j\}_{j\geq 1}$ given by (16) we get that

$$\begin{aligned} \Psi(t,s)z &= \sum_{j=1}^{\infty} \Psi_j(t,s)P_j z, \quad (62)\\ T(t,s)z &= \sum_{j=1}^{\infty} T_j(t,s)P_j z. \quad (63)\end{aligned}$$

Therefore, $\Psi_j(t,s)$ and $T_j(t,s)$ are the evolution operators for the following systems of ODEs:

$$\begin{aligned} y' &= [-\lambda_j dI + A(t)]y - \lambda_j E y, \quad y \in \mathrm{Ran}(P_j), \quad j = 1,2,\ldots, \quad (64)\\ y' &= [-\lambda_j dI + A(t)]y, \quad y \in \mathrm{Ran}(P_j), \quad j = 1,2,\ldots. \quad (65)\end{aligned}$$

On the other hand, using the formula (16) and that $e^{-\mathcal{A}_d t}P_j z = e^{-\lambda_j d t}P_j z$ we get the following estimates

$$\begin{aligned} ||e^{-\mathcal{A}_d t}P_j z||_\alpha &\leq M||P_j z||_\alpha e^{-\lambda_j d(t-s)}, \quad t \geq s, \quad (66)\\ ||e^{-\mathcal{A}_d t}P_j z||_\alpha &\leq M(t-s)^{-\alpha}||P_j z||e^{-\lambda_j d(t-s)}, \quad t \geq s. \quad (67)\end{aligned}$$

Since $A(t)$ is periodic, we get that

$$T_j(t,s)y = e^{-\lambda_j d(t-s)}N(t)e^{L(t-s)}N^{-1}(s)y, \quad y \in \quad \mathrm{Ran}(P_j), \quad j = 1,2,\ldots.$$

Therefore

$$\begin{aligned} ||T_j(t,s)y||_\alpha &\leq R||y||_\alpha e^{(-\lambda_j d + b)(t-s)}, \quad t \geq s,\\ ||T_j(t,s)y||_\alpha &\leq R(t-s)^{-\alpha}||y||e^{(-\lambda_j d + b)(t-s)}, \quad t \geq s.\end{aligned}$$

From here, the remainder of the proof follows in the same way as Lemma 3.2.

□

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