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Abstract

In this paper we tackle the so-called FGF and CF problems, that are still open and, explicitly or implicitly, have been aboarded several times in the area of Module and Ring Theory. We give new partial affirmative answers to both problems.

1 Introduction and terminology

In this work we address two open questions in Module Theory to which we shall refer as the FGF problem and the CF problem, respectively:

FGF problem: Suppose R is a ring for which every finitely generated left R-module embeds in a free module. Is R a Quasi-Frobenius (QF) ring?

CF problem: Suppose that, for R as above, every cyclic left R-module embeds in a free module. Is R left Artinian?

The first problem has been explicitly aboarded many times (see [5] for a survey on the answers obtained until that time). It seems that, after [5], that problem had been forgotten until very recently in which, starting with [7], there has been a renewed interest in the subject (see [7], [12] and [14]). On the contrary, the CF problem seems to have been only implicitly tackled in the literature. However, it seems that this second problem might be a key question to ask in order to understand the FGF problem. Indeed, since for left Artinian rings the answer to the FGF problem is affirmative (see [5, Theorem 3.2]), an affirmative answer to the CF problem would automatically imply the same answer to the FGF problem.

In our present notes, we take both problems at once for the class of semiregular rings and, more restrictively, for semiperfect rings. In section 2, we prove that the answer to the FGF problem is 'yes' for semiregular rings with essential socle, whenever there is some sort of "two-sided property" on the minimal left ideals (Corollaries 2.2 and 2.3). This "two-sided property" disappears completely as an hypothesis when R is semiperfect, which entails in particular that the answer to the FGF problem is 'yes' whenever R is right perfect (Theorem 2.5).

In section 3, we try to get as much information on semiregular CF rings as possible, obtaining a list of properties under which those rings satisfy the property that every simple R-module embeds essentially in a projective module (Proposition 3.2 and Corollary 3.3). That in turns gives us a new list of affirmative answers to both proposed problems (Theorem 3.4). The most important result of the section states that every semiperfect left CF ring which is left mininjective is necessarily QF (Theorem 3.5). As a byproduct of this latter result, we get that the QF rings are precisely those for which every cyclic left (or right) module has an essential projective (pre)envelope.

We have devoted the last section of our work to study the semiregular left CF rings with the property that IR is finitely generated as a left ideal, for every minimal left ideal I of R. It turns out that, when moreover IR has a maximal right subideal and in particular when IR is also finitely generated on the right, the mentioned rings provide affirmative answers to both problems (Corollary 4.3). As an application of this latter fact, we see that semiperfect left CF K-algebras R for which $R/J(R)$ is a finitely generated K-module are already left Artinian if IR is finitely generated as a left ideal (Theorem 4.4). Going through some cardinality arguments we then prove that a countable semiperfect left CF (resp. FGF) ring is left Artinian (resp. Quasi-Frobenius) whenever one of the following two properties occurs: (a) $\bar{J}_r(R) = 0$; (b) IR is finitely generated as a right ideal, for every minimal left ideal I of R (Theorem 4.6). In particular, every countable semiperfect left CF (resp. FGF) K-algebra R with the property that R/J is a finitely generated K-module is already left Artinian (resp. Quasi-Frobenius) (Theorem 4.7).

Throughout this paper, all rings are associative with identity and all modules are unitary. We will write ${}_RM$ (resp. M_R) to indicate that M is a left R-module (resp. right R-module). In particular, ${}_RR$ and R_R will denote the canonical structures of left and right R-module in R. The lattice of submodules of ${}_RM$ will be denoted by $\mathcal{L}({}_{\mathcal{R}}\mathcal{M})$.

Let R be a ring and X a subset of ${}_RM$, the **left annihilator** of X in R will be denoted by $l_R(X)$, or simply $l(X)$ if no confusion appears. We use the notation $N \overset{*}{\hookrightarrow} M$ meaning that N is an essential submodule of ${}_RM$. The **left singular ideal** of R is the two-sided ideal $\left\{r \in R : l(r) \overset{*}{\hookrightarrow} {}_RR\right\}$ of R and will be denoted by $Z({}_RR)$.

A module M is called **finite-dimensional** when M contains no infinite independent family of non-zero submodules.

Let M be a left R-module. The **transfinite socle series** of M is defined as in [15, VIII.2] and, as there, $\overline{Soc}(M)$ denotes the largest term of that series.

A left R-module M is **semiartinian** if every non-zero quotient module of M has non-zero socle. Thus, M is semiartinian if and only if $\overline{Soc}(M) = M$. In that case the least ordinal γ such that $M = Soc^{\gamma}(M)$ will be called the **socle length of** M and denoted by $s.l.(M)$.

The Jacobson radical of a ring R will be denoted by $J(R)$ (or simply J). The **right transfinite sequence** of powers of J is defined as follows: $J^1 = J$ and, in case J^{β} has been defined for every ordinal $\beta < \alpha$, we set $J^{\alpha} = \bigcap_{\beta<\alpha} J^{\beta}$, when α is limit, and $J^{\alpha} = J^{\alpha-1} \cdot J$, when α is non-limit. There exists a least ordinal γ such that $J^{\gamma} = J^{\alpha}$, for all ordinals $\alpha \geq \gamma$ and we write $\bar{J}_r(R) = J^{\gamma}$.

A ring R is **semiregular** when R/J is regular (in the sense of von Neumann) and idempotents of R/J can be lifted to R [11]. This is equivalent to say that every finitely presented left (or right) R -module has a projective cover.

A subset X of R is **left** (resp. **right**) **T-nilpotent** when, for every sequence $x_1, \ldots, x_n, \ldots$ of elements of X there exists $n \in \mathbb{N}$ such that $x_1 \cdots x_n = 0$ (resp. $x_n \cdots x_1 = 0$).

A ring R is **left mininjective** [12], if every homomorphism $f : I \to {}_RR$, where I is a minimal left ideal of R , extends to a homomorphism $\bar{f} : {}_RR \to {}_RR$. A **projective preevelope** of a module ${}_RM$ is a homomorphism $g : M \to P$, where P is a projective module, such that for every homomorphism $h : M \to P'$, where P' is projective, there exists a homomorphism $k : P \to P'$ such that $k \circ g = h$. When, moreover, every endomorphism $\varphi : P \to P$ such that $\varphi \circ g = g$ is an automorphism of P, we shall say that g is a **projective envelope** of M.

Following Faith's terminology [5], a ring R is **left FGF** if every finitely generated left R-module embeds in a free module. More generally, a ring R is **left CF** when every cyclic left R-module embeds in a free module.

Finally, we refer to [1] or [15] for all undefined notions used in this text.

2 Semiregular FGF rings with essential socle.

In this section we consider the FGF problem in the case that R is a semiregular ring with essential socle.

Lemma 2.1 *Let R be any ring.*

(a) If M is a left R-module then $J^\alpha \cdot Soc^\alpha(M) = 0$ for every ordinal α, where J^α is the right power of J (i.e., $J^\alpha = J^{\alpha-1} \cdot J$ if α is a non-limit ordinal).

(b) If R is a left semiartinian ring then $\bar{J}_r(R) = 0$ and $s.l.(J) < s.l.(R)$.

(c) If R is a left CF ring then R is left semiartinian if, and only if, $Soc({}_RR) \overset{*}{\hookrightarrow} {}_RR$.

Proof: (a) We use transfinite induction. The case $\alpha = 0$ is trivial. Suppose it is true for every ordinal less than α. We consider the two possible cases:

(i) $\alpha = \beta + 1$ (i.e. α is non-limit): then $J^\alpha Soc^\alpha(M) = J^\beta J Soc^{\beta+1}(M) \subseteq J^\beta Soc^\beta(M) = 0$, since $J \cdot (Soc^{\beta+1}(M)/Soc^\beta(M)) = 0$.

(ii) α is a limit ordinal: for every ordinal $\beta < \alpha$ we have $J^\alpha \cdot Soc^\beta(M) \subseteq J^\beta \cdot Soc^\beta(M) = 0$, consequently, $J^\alpha Soc^\alpha(M) = J^\alpha \cdot \left(\sum_{\beta<\alpha} Soc^\beta(M) \right) = 0$.

(b) Since R is left semiartinian, $R = \overline{Soc}({}_RR) = Soc^\gamma(R)$ for an ordinal $\gamma = s.l.({}_RR)$. Hence by part (a), $0 = J^\gamma Soc^\gamma(R) = J^\gamma R = J^\gamma$, and so, $\bar{J}_r(R) = 0$.

For the second part, we know that γ is a non-limit ordinal, for otherwise $R = Soc^\gamma(R) = \sum_{\beta<\gamma} Soc^\beta(R)$ which implies that $R = Soc^\beta(R)$ for certain $\beta < \gamma$, a contradiction. So assume that $\gamma = \beta + 1$. Then $R/Soc^\beta(R)$ is semisimple and so $J \subseteq Soc^\beta(R)$. Hence, $s.l.(J) \leq s.l.(Soc^\beta(R)) = \beta < \beta + 1 = s.l.(R)$.

(c) We only need to show that if $Soc({}_RR) \overset{*}{\hookrightarrow} {}_RR$ then R is left semiartinian or, equivalently, that every cyclic left R-module has non-zero socle ([15, Proposition VIII.2.5]). Let C be a non-zero cyclic left R-module. Since R is left CF we can assume that C is a submodule of $R^{(m)}$ for a positive integer m. Then $Soc(C) = Soc(R^{(m)}) \cap C$ and since $Soc(R^{(m)}) \overset{*}{\hookrightarrow} R^{(m)}$, we deduce that $Soc(C) \neq 0$.■

Corollary 2.2 *Let R be a semiregular left FGF ring. The following conditions are equivalent:*

(a) $Soc({}_RR) \overset{*}{\hookrightarrow} {}_RR$ and $I \subseteq IxR$ for every minimal left ideal I of R and every $x \in R$ such that $Ix \neq 0$;

(b) R is QF.

Proof: (a)$\Rightarrow$(b) By lemma 2.1, $\bar{J}_r(R) = 0$. Let I be a minimal left ideal of R and $0 \neq x \in J(R)$. If $Ix \neq 0$ then $I \subseteq IxR \subseteq IJ$ and consequently $I \subseteq \bar{J}_r(R) = 0$, a contradiction. Hence $Soc(_RR) \cdot J = 0$. But then $l(J)$ is an essential left ideal of R, which implies that $J \subseteq Z(_RR)$. The result now follows from [14, Corollary 6].

(b)$\Rightarrow$(a) Let I be a minimal left ideal of R and $x \in R$ such that $Ix \neq 0$. Then right multiplication by x yields an isomorphism $I \xrightarrow{\phi} Ix$ which has an inverse $\phi^{-1} : Ix \to I$. The fact that R is left self-injective gives us an element $y \in R$ such that $\phi^{-1}(ax) = axy$, for every $a \in R$. Hence $I = Im\phi^{-1} = Ixy \subseteq IxR$ as desired. ■

We shall now give some examples where the hypothesis "$I \subseteq IxR$ for every minimal left ideal I of R and every $x \in R$ such that $Ix \neq 0$ " in the previous corollary is verified.

Example 2.1 *(a)* If $Tr_R(I) = IR$ for every minimal left ideal I of R. In particular, when R contains exactly one isomorphic copy of each minimal left ideal. To see this, let I be a minimal left ideal of R and $x \in R$ such that $Ix \neq 0$. Then $f : Ix \to I$ defined by $f(ax) = a$ $(a \in I)$ is an isomorphism. Hence, $I \subseteq Tr_R(Ix) = IxR$.

If R contains exactly one isomorphic copy of each minimal left ideal then for each such ideal I of R and homomorphism $f : I \to {_RR}$, we have that $f(I) = I$ and so $Tr_R(I) = I \subseteq IR$.

(b) If R is left mininjective then $Tr_R(I) = IR$, for every minimal left ideal I, by the proof of (b)$\Rightarrow$(a) in the above corollary.

(c) When every minimal left ideal of R is a two-sided ideal and, in particular, when R is commutative. Indeed, if $Ix \neq 0$ the fact that I is a two-sided ideal implies that $0 \neq Ix \subseteq I$ and, by the minimality of I, it follows that $I = Ix$ and so $I \subseteq IxR$.

Corollary 2.3 *Let R be a semiregular left FGF ring such that $Soc(_RR) \overset{*}{\hookrightarrow} {_RR}$. Each of the following assumptions forces R to be QF:*

(a) $Tr_R(I) = IR$, for every minimal left ideal I of R;

(b) R is left mininjective;

(c) Every minimal left ideal of R is two-sided.

Example 2.2 As said above, every left mininjective ring satisfies that $Tr_R(I) = IR$, for every minimal left ideal I of R. The following example shows that the converse is false. Take an infinite field K admitting a non-epic homomorphism $\sigma : K \to K$ and $K[X, \sigma]$ the associated skew-polynomial ring (i.e., with right multiplication by scalars: $X \cdot \lambda = \lambda^\sigma \cdot X$). By taking $R = K[X, \sigma]/(X^2)$ and denoting by x the class of X in this ring, one immediately sees that R is a local ring whose unique proper left ideal is $J = Kx$. Consequently, $Tr_R(I) = IR$, for every minimal left ideal I of R. If now $\lambda \in K$ and $\phi_\lambda : J \to J$ maps x onto λx, we readily see that it is a homomorphism of left R-modules that can be extended to $_RR$ only in case $\lambda \in Im\sigma$. So R is not left mininjective.

If R is assumed to be semiperfect, then all hypothesis accompanying the assumption "$Soc(_RR) \overset{*}{\hookrightarrow} {_RR}$" can be omitted. We need first a lemma whose proof implicitly appears in [13, Lemma 11]:

Lemma 2.4 *Let R be a ring and X a finite left or right T-nilpotent subset of R. Then there exists a positive integer t such that every product of t elements of X is zero.*

Now we can prove the main result of this section.

Theorem 2.5 *Let R be a left FGF ring. The following conditions are equivalent:*

(a) R/J is semisimple and $Soc(_RR) \overset{*}{\hookrightarrow} {}_RR$;

(b) R is right perfect;

(c) R is QF.

Proof: (a)⇔(b). A left FGF (more generally, CF) ring which satisfies $Soc(_RR) \overset{*}{\hookrightarrow} {}_RR$ is left semiartinian (lemma 2.1 (c)). But it is well known that a ring is right perfect if and only if it is left semiartinian and R/J is semisimple ([15, Proposition VIII.5.1]).

(c)⇒(a) is clear.

(a),(b)⇒(c). Since the conditions we are considering are Morita-invariant, we may assume that R is basic. Let $\{e_1, \dots, e_n\}$ be a basic set of primitive idempotents of R. If none of the Re_i $(i = 1, \dots, n)$ is injective then, by [14, Proposition 4], $R = \bigoplus_{i=1}^{n} Re_i$ embeds in a finite direct sum of copies of J and so $s.l.(R) \leq s.l.(J)$. This contradicts lemma 2.1. Thus, we may assume that $\{e_1, \dots, e_n\}$ is ordered in such a way that for $i = 1, \dots, r$, Re_i embeds in the radical of a finitely generated free module while for $i = r+1, \dots, n$, Re_i is injective. Set $e = e_1 + \cdots + e_r$ and so $1 - e = e_{r+1} + \cdots + e_n$. Then there exists a monomorphism $\varphi : Re \to Re^{(k)} \bigoplus R(1-e)^{(k)}$, where k is a positive integer such that $Im\varphi \subseteq Je^{(k)} \bigoplus J(1-e)^{(k)}$. Suppose that the first component $Re \to Re^{(k)}$ of φ takes e onto the element $(x_1, \dots, x_k) \in Je^{(k)}$. Now we take the monomorphism $\varphi^{(k)} : Re^{(k)} \longrightarrow Re^{(k^2)} \bigoplus R(1-e)^{(k^2)}$ (direct sum of k copies of φ) and consider the composition

$$Re \xrightarrow{\varphi} Re^{(k)} \bigoplus R(1-e)^{(k)} \xrightarrow{\varphi^{(k)} \bigoplus 1} Re^{(k^2)} \bigoplus R(1-e)^{(k^2)} \bigoplus R(1-e)^{(k)}$$

This composition is a monomorphism whose first component $Re \to Re^{(k^2)}$ maps e onto the element $(x_{i_1} \cdot x_{i_2})_{(i_1,i_2) \in \mathbb{N}_k^2}$ of $Re^{(k^2)}$. By recursively continuing in this way we get for each $t \geq 0$ a monomorphism

$$(\varphi^{(k^t)} \bigoplus 1) \circ (\varphi^{(k^{t-1})} \bigoplus 1) \circ \cdots \circ \varphi : Re \longrightarrow Re^{(k^t)} \bigoplus R(1-e)^{(k^t + k^{t-1} + \cdots + k)}$$

whose first component $Re \to Re^{(k^t)}$ maps e onto the element $(x_{i_1} \cdots x_{i_t})_{(i_1,\dots,i_t) \in \mathbb{N}_k^t}$. The previous lemma ensures that for a large enough t, that component is zero. As a consequence, Re (and hence $_RR = Re \bigoplus R(1-e)$) embeds in a finite direct sum of copies of $R(1-e)$. Since $R(1-e)$ is injective it follows that $E(_RR)$ is projective. Thus by [10, Corollary 9], R is QF. ■

3 Semiregular CF rings.

In order to aboard the CF problem for semiregular rings, the following result is fundamental.

Lemma 3.1 *Let R be a semiregular left CF ring. If U_0 is a cyclic uniform left R-module then one of the following conditions hold:*

(a) U_0 embeds essentially in a projective module;

(b) There exists a sequence $x_1, \ldots, x_n, \ldots$ of elements of $J(R)$ and a left ideal $\overline{U_0}$ of R such that

$$U_0 \cong \overline{U_0} \subseteq l(x_1 \cdots x_n) = l(x_1 \cdots x_{n+1}) \text{ for every } n \geq 1$$

Proof: If U_0 does not embed essentially in a projective module, then we can easily adapt the proof of Theorem 7 in [14] to get condition (b).■

In the next proposition we give a list of conditions over a ring R for which part (b) of the previous lemma fails when U is a minimal left ideal of R:

Proposition 3.2 *Let R be a ring satisfying one of the following conditions:*

(a) J is left T-nilpotent;

(b) R is left mininjective;

(c) $\bar{J}_r(R) = 0$ and $I \subseteq IxR$ for every minimal left ideal I of R and every $x \in R$ such that $Ix \neq 0$;

(d) $\bar{J}_r(R) = 0$ and IR is finitely generated as a left ideal of R, for every minimal left ideal I of R;

(e) IR is semiartinian as a right R-module, for every minimal left ideal $I$of R.

Then for every minimal left ideal I of R and every sequence $x_1, \ldots, x_n, \ldots$ of elements of $J(R)$ there exists a positive integer k such that $Ix_1 \cdots x_k = 0$.

Proof: (a) is clear and (e) is an immediate consequence of [15, Proposition VIII.2.6].

(b) We will prove that $Soc(_RR) \cdot J = 0$. Let I be a minimal left ideal of R and $0 \neq x \in J(R)$. If $Ix \neq 0$ then $f : Ix \to I$ defined by $f(ax) = a$ $(a \in I)$ is an isomorphism. Then f extends to a homomorphism $\bar{f} : R \to R$ because R is left mininjective. Since $\bar{f}$ is right multiplication by an element $r \in R$, for every $a \in I$ we have that $a = f(ax) = \bar{f}(ax) = axr$ and so $a(1 - xr) = 0$. But $x \in J(R)$, so that $1 - xr$ is invertible and consequently $a = 0$. This shows that $I = 0$, a contradiction. Hence $Ix = 0$ for every minimal left ideal I of R and $x \in J(R)$, which implies that $Soc(_RR) \cdot J = 0$.

(c) In this case we also have $Soc(_RR) \cdot J = 0$ (see proof of corollary 2.2).

(d) Suppose by contradiction that I is a minimal left ideal of R and $x_1, \ldots, x_n, \ldots$ is a sequence of elements of $J(R)$ such that $Ix_1 \cdots x_n \neq 0$ for every $n \geq 1$. Set $I_i = Ix_1 \cdots x_i$ for every $i \in \mathbb{N}$ $(I_0 = I)$. Since IR finitely generated as a left ideal of R, there exists $m \geq 1$ such that $I_{m+1} \subseteq I_0 + \cdots + I_m$. If $0 \neq a \in I_{m+1}$ then we can write $a = a_0 + \cdots + a_m$ where $a_i \in I_i$ $(i = 0, \ldots, m)$. Let $k = min\{j \in \{0, \ldots, m\} : a_j \neq 0\}$. Then $a = a_k + \cdots + a_m$ and so $0 \neq a_k = -a_{k+1} - \cdots - a_m + a \in I_{k+1} + \cdots + I_m + I_{m+1}$. Thus $I_k = Ra_k \subseteq I_{k+1} + \cdots + I_m + I_{m+1} \subseteq I_kJ$ which implies $I_k \subseteq \bar{J}_r(R)$. This contradicts our assumption $\bar{J}_r(R) = 0$. ■

Corollary 3.3 *Let R be a semiregular left CF ring. If R satisfies one of the conditions in proposition 3.2, then every simple left R -module embeds essentially in a projective module.*

Proof: It is an immediate consequence of lemma 3.1 and proposition 3.2.■

Example 3.1 Neither of the classes of rings satisfying (c) or (d) above is contained in the other. Indeed, if K is a field and V is an infinite-dimensional K-vector space, then $R = End_K(V)$ satisfies (c) but not (d) (see example 2.1 (b)). On the contrary, if R is the K-algebra with basis $\{e_1, e_2, x\}$, whose multiplication extends by linearity the rules

$$e_ie_j = \delta_{ij}e_j$$
$$xe_1 = 0 = e_2x$$
$$xe_2 = x = e_1x$$

then $I = Re_1$ is a minimal (projective) left ideal, $x \in J(R)$ and $Ix \neq 0$, thus implying that $I \not\subseteq IxR$.

Last corollary yields some new partial affirmative answers to the CF problem and so to the FGF problem.

Theorem 3.4 *Let R be a semiperfect left CF (resp. FGF) ring satisfying one of the conditions (a)-(e) of proposition 3.2. Then R is left artinian (resp. QF). In particular, every left perfect left CF ring is left artinian.*

Proof: By corollary 3.3, every simple left R -module embeds essentially in a projective module. Now the proof of Lemma 3.1 in [9] tells us that R is left finitely cogenerated. But, when R is left CF (resp. FGF), the latter condition is equivalent to R being left artinian (resp. QF).■

The part of the above theorem concerning condition (b) in Proposition 3.2 can be considerably improved via the next result, which extends Corollary 2.2 of [6]:

Theorem 3.5 *A ring R is QF if, and only if, it is semiperfect left CF and left mininjective.*

Proof: By Theorem 3.4, R is left artinian. Let $\{e_1, \ldots, e_m\}$ be a basic set of primitive idempotents of R. Since each simple embeds essentially in a projective module, $S_i = Soc(Re_i) \overset{*}{\hookrightarrow} Re_i$ $(i = 1, \ldots, m)$, where $\{S_1, \ldots, S_m\}$ is a representative set of simple left R-modules ([9, proof of Lemma 3.1]).

Suppose that $R \cong Re_1^{(k_1)} \oplus \cdots \oplus Re_m^{(k_m)}$ and so $Soc(_RR) \cong S_1^{(k_1)} \oplus \cdots \oplus S_m^{(k_m)}$. Since R is left mininjective, for each i, $S_i \hookrightarrow Re_i$ is a projective preenvelope of S_i. It follows from [17, Proposition 1.2.16] that $Soc(_RR) \hookrightarrow {}_RR$ or, more generally, that $Soc(P) \hookrightarrow P$ is a projective preenvelope for every projective module $_RP$.

We will show that if L is a cyclic left R-module and K is a submodule of L, then every homomorphism $f : K \to R$ extends to a homomorphism $\bar{f} : L \to R$. If K is semisimple, bearing in mind that R is left CF, we construct a diagram as follows:

$$\begin{array}{ccccc} K & \hookrightarrow & L & \hookrightarrow & F \\ & {}_s\searrow\nwarrow_p & & \nearrow_j & \\ f\downarrow & & Soc(F) & & \\ & & & & \\ R & & & & \end{array}$$

where F is free, $p \circ s = id_K$ and j is the canonical inclusion. Then, by our above remark, f extends to a homomorphism $\hat{f} : F \to R$ which induces by restriction a homomorphism $\bar{f} : L \to R$ that extends f.

In the general case in which K is an arbitrary submodule of L, we use induction on $c(L)$ (the composition length of L). This is obvious if $c(L) = 1$ since L is simple. Assume it holds for every cyclic left R -module of composition length less than $c(L)$ and let $p_K : K \to K/SocK$ and $p_L : L \to L/SocL$ be the canonical projections. We first consider the case $f(SocK) = 0$. Then there exist a homomorphism $\hat{f} : K/SocK \to R$ completing the diagram

$$\begin{array}{ccccc} K & & \stackrel{j}{\hookrightarrow} & & L \\ & \searrow^f & & & \\ p_K\downarrow & & R & & \downarrow^{p_L} \\ & \nearrow_{\hat{f}} & & \nwarrow^h & \\ K/SocK & & \stackrel{\bar{j}}{\hookrightarrow} & & L/SocL \end{array}$$

Since $L/SocL$ is cyclic of composition length less than $c(L)$, by induction, $\hat{f}$ extends to a homomorphism $h : L/SocL \to R$. Hence $h \circ p_L \circ j = h \circ \bar{j} \circ p_K = \hat{f} \circ p_K = f$ and so $h \circ p_L : L \to R$ is an extension of f.

If $f(SocK) \neq 0$ then, since $Soc(K)$ is a semisimple submodule of L, the homomorphism $f\,|_{SocK}$: $SocK \to R$ extends to a homomorphism $w : L \to R$. Consequently, $(w\,|_K - f)(SocK) = 0$ and, as we have just seen, $w\,|_K - f$ can be extended to a homomorphism $u : L \to R$. Thus, $u\,|_K = w\,|_K - f$ and so $f = (w - u)\,|_K$.

It follows that R is left self-injective and hence, by [2] or [16], R is QF.■

In recent years, after the appearance of [4], it has been a very usual task to identify the rings for which a wide class of modules have an envelope in a significative class of modules (e.g. flat, projectives, etc.). By [7, Corollary 3.5], the rings for which every finitely generated left R-module has an essential projective envelope (i.e. a projective envelope that is an essential monomorphism) are precisely the QF. The following result tells us that it is enough to have that condition only for the cyclic modules.

Corollary 3.6 *Let R be any ring. R is QF if, and only if, every cyclic left (resp. right) R-module has an essential projective (pre)envelope.*

Proof: If every cyclic left R-module has an essential projective preenvelope then, by [7, Corollary 3.3], R is left artinian and, by [9, proof of Lemma 3.1], $Soc(Re_i) \stackrel{*}{\hookrightarrow} Re_i$ is a projective preenvelope for every $i = 1, \ldots, m$, where $\{e_1, \ldots, e_m\}$ is a basic set of primitive idempotents of R. Thus by [17, Proposition 1.2.16], $Soc(_RR) \overset{*}{\hookrightarrow} {_RR}$ is a projective preenvelope. Hence for every semisimple left ideal I of R and homomorphism $f : I \to {_RR}$, it follows from the diagram

$$\begin{array}{ccccc} I & & \hookrightarrow & & R \\ & {}_s\searrow\nwarrow_p & & \nearrow_j & \\ {}^f\downarrow & & \mathrm{Soc(R)} & & \\ & & & & \\ R & & & & \end{array}$$

where $p \circ s = id_K$ and j is the canonical inclusion, that f extends to a homomorphism $\bar{f} : R \to R$. In particular, R is left mininjective and, by theorem 3.5, R is QF.■

4 Semiperfect CF rings with IR finitely generated on the left, for every minimal left ideal I.

Theorem 3.4 tells us that the rings of the title of this section are left artinian whenever $\bar{J}_r(R) = 0$. We shall see that this second hypothesis can be sometimes omitted or replaced by another one in order to get artinianity. First of all, we shall see that the class of rings in question contains all the semiregular left CF rings with socle finitely generated on the left.

Proposition 4.1 *If R is a semiregular left CF ring such that $Soc(_RR)$ is finitely generated as a left ideal, then R is semiperfect.*

Proof: First of all observe that, since R is left CF, $Soc(_RR)$ is finitely generated if and only if $Soc(R/I)$ is finitely generated, for every left ideal I of R. By [3, Lemma], that means that R/I is finite-dimensional, for every left ideal I of R. In particular, R/J is finite-dimensional as a left R- or R/J -module. But it is well-known that a regular left finite-dimensional ring is semisimple (see, e.g., [8, 3.B, exercise 14]). ■

Next we see what the precise obstacle is for our class of rings not to be included in that of the left artinian ones.

Proposition 4.2 *Let R be a semiperfect left CF ring such that IR is finitely generated as a left ideal, for every minimal left ideal I of R. If R is not left artinian, then there is a minimal left ideal I' of R such that $I'R = I'J$.*

Proof: If R is not left artinian then, by [9, proof of Lemma 3.1], there exists a minimal left ideal I of R that does not embed essentially in a projective module. Hence, by lemma 3.1, there exists a sequence $x_1, \ldots, x_n, \ldots$ in $J(R)$ such that $Ix_1 \cdots x_n \neq 0$ for all $n \geq 1$. But then, since IR is finitely generated as a left ideal of R, we can find, as in the proof of proposition 3.2 part (d), a minimal left ideal $I' = Ix_1 \cdots x_k$ of R such that $I' \subseteq I'J$. Consequently, $I'R = I'J$. ■

As a consequence, we immediately get a new partial affirmative answer to the CF (resp. FGF) problem.

Corollary 4.3 *Let R be a semiperfect left CF (resp. FGF) ring such that IR is finitely generated as a left ideal, for every minimal left ideal I of R. If, in addition, IR contains a maximal right subideal, for every such I, then R is left artinian (resp. QF). In particular, every semiperfect left CF (resp. FGF) ring such that IR is finitely generated as a left and as a right ideal, for every minimal left ideal I of R, is left artinian (resp. QF).*

There is a particular instance in which the conditions of the above corollary hold that is very interesting in itself.

Theorem 4.4 *Let K be a commutative ring and R be a K-algebra such that $R/J(R)$ is finitely generated as a K-module. If R is semiperfect left CF (resp. FGF) and IR is finitely generated as a left ideal, for every minimal left ideal I of R, then R is left artinian (resp. QF). In the particular case when K is a field, R is moreover finite-dimensional.*

Proof: We shall see that if I is a minimal left ideal then IR is also finitely generated on the right and so Corollary 4.3 will apply. For every minimal left ideal I of R we know that IR is finitely generated as a left ideal of R. But then IR is a finitely generated left R/J-module and, since R/J is a finitely generated K-module, IR is also finitely generated as a K-module. But every K-generating set of IR is also a generating set of IR as a right R-module and so, IR is a finitely generated right R-module. In case K is a field, if $J^n = 0$ then $dim_K R = \sum_{k=0}^{n-1} dim_K \left(J^k/J^{k+1}\right)$ and, since each J^k/J^{k+1} is a finitely generated left R/J-module, it is also of finite dimension over K. Consequently, R is finite-dimensional over K.■

The final part will be devoted to apply the results of the first part of the section to the CF (resp. FGF) problem for countable rings and algebras.

Lemma 4.5 *Let R be a left CF ring. Then $|\mathcal{L}(_{\mathcal{R}}\mathcal{R})| \leq |R|$.*

Proof: Since R is left CF, given any left ideal I of R there exists a finite subset X of R such that $I = l(X)$. Thus we can define an injective map $\Phi : \mathcal{L}(_{\mathcal{R}}\mathcal{R}) \rightarrow \mathcal{FP}(\mathcal{R})$, where $\mathcal{FP}(\mathcal{R})$ is the set of finite parts of R, by $\Phi(I) = X$ and consequently, $|\mathcal{L}(_{\mathcal{R}}\mathcal{R})| \leq |\mathcal{FP}(\mathcal{R})| = |R|$.■

Remark 4.1 *Although we've not been able to use it in order to get new answers to the CF (resp. FGF) problem, as an extra information for the reader, we have proved (with considerably more difficulties than the above lemma) that if R is a semiperfect left CF ring such that $J^\omega = 0$, then only one of the following conditions can occur:*

(a) $|R/J| \leq \aleph_0$ *and* $|R| \leq 2^{\aleph_0}$;

(b) $|R| = |R/J| > \aleph_0$.

Our two main results concerning the CF (resp. FGF) problem for countable rings and algebras are now available.

Theorem 4.6 *Let R be a semiregular countable left CF (resp. FGF) ring. Each of the following assumptions forces R to be left artinian (resp. QF):*

(a) $\bar{J}_r(R) = 0$;

(b) IR contains a maximal right subideal, for every minimal left ideal I of R. In particular, when such an IR is finitely generated on the right.

Proof: By lemma 4.5, $\mathcal{L}({}_{\mathcal{R}}\mathcal{R})$ is a countable set and so the lattice of left subideals of IR is also countable. But, since IR is semisimple as a left module, the latter implies that IR is a direct sum of finitely many minimal left ideals. Hence IR is finitely generated as a left ideal and the result follows from Theorem 3.4 and Corollary 4.3. ■

Theorem 4.7 *Let K be a commutative ring and R a countable K-algebra such that $R/J(R)$ is finitely generated as a K-module. If R is semiperfect left CF (resp. FGF), then R is left artinian (resp. QF). In case K is a field, R is moreover finite-dimensional.*

Proof: Straightforward consequence of Theorem 4.4 and the proof of Theorem 4.6.■

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