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On Borel Topologies over countable sets

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Abstract

We present some results on the Borel complexity of topologies defined over countable sets and its ideal of nowhere dense sets.

1 Introduction

In this note we will be interested in topologies over the natural numbers N (or any countable set X). We can identify every subset of N with its characteristic function, so its power set $\mathcal{P}(\mathrm{N})$ is identified with the Cantor space $2^{\mathbf{N}}$. Since every topology over N is a subset of $\mathcal{P}(\mathrm{N})$, it is clear then what we mean by saying that τ is closed, open, G_δ, Borel, analytic, etc. There are many results concerning the descriptive set theoretic properties of families of subsets of N, like ideals and filters (see [2, 4, 6, 7, 8, 10]). Every filter (and dually every ideal) has naturally associated a topology, hence those results about the existence of Borel or analytic filters (or ideals) over N immediately provide examples of topologies over N of the same (Borel, projective) complexity. These topologies are not Hausdorff, however, given a filter $\mathcal{F}$ over N by an elementary construction it is easy to define a Hausdorff topology of the same complexity as the filter $\mathcal{F}$. It is known that every G_δ filter is necessarily closed, but there are filters (and hence Hausdorff topologies) in all levels of the Borel hierarchy above the third level.

In section §1 we will present some results on Borel topologies. In particular, we will show that there are not G_δ T_1 topologies over N, but we will see that there are G_δ-complete T_0 topologies. We will show also that any regular topology is either $\mathbf{\Pi}^0_3$-hard or it has only finitely many limits points. In section §2 we present some results about the ideal of nowhere dense sets. In particular, we will show that the ideal of nowhere dense sets of any $\mathbf{\Sigma}^0_3$ Hausdorff topology is necessarily principal.

We will use the standard notions and terminology of descriptive set theory (see [3]). X will always denote a countable set. Let A, B be subsets of topological spaces Y and Z respectively, as usual $A \leq_W B$ denotes the fact that A is Wadge reducible to B, that is to say, there is a continuous function $f : Y \to Z$ such that $x \in A$ iff $f(x) \in B$. FIN denotes the ideal of finite subsets of N, $\emptyset \times$ FIN denotes the ideal over $\mathrm{N} \times \mathrm{N}$ given by $A \in \emptyset \times \mathrm{FIN}$ iff for all n, $\{i : (n, i) \in A\}$ is finite and FIN $\times\, \emptyset$ denotes the ideal given by $A \in \mathrm{FIN} \times \emptyset$ iff there is n such that $A \subseteq \{n\} \times \mathrm{N}$, where as usual we identify a natural number n with the set $\{0, \cdots, n-1\}$. $\emptyset \times$ FIN is a $\mathbf{\Pi}^0_3$-complete subset of $2^{\mathbf{N}\times\mathbf{N}}$ (see [3]). A topology τ over X is said to be **Alexandroff** if it is closed under arbitrary intersection, equivalently, if for every $x \in X$ the set $N_x = \bigcap\{V : x \in V \text{ and } V \text{ is open}\}$ is an open set. N_x is called the irreducible neighborhood of x. The following well know result characterize Alexandroff topologies in terms of partial orders.

Theorem 1.1 *A topology τ over X is Alexandroff iff there is a binary relation $\leq_\tau$ over X which is transitive and reflexive and such that $A \in \tau$ iff for every $x \in A$ we have $\{y \in X : x \leq_\tau y\} \subseteq A$. Moreover, the irreducible neighborhood of x is $\{y \in X : x \leq_\tau y\}$. Furthermore, τ is T_0 iff $\leq_\tau$ is antisymmetric. Also, $cl_\tau(A) = \bigcup_{x\in A} cl_\tau(\{x\}) = \bigcup_{x \in A}\{y \in X : y \leq_\tau x\}$. Thus $\leq_\tau$ is given by $y \leq_\tau x$ iff $y \in cl_\tau(\{x\})$.* □

The following fact is easy to verify.

Proposition 1.2 *Let $f, g : 2^X \times 2^X \to 2^X$ $h : 2^X \to 2^X$ be the functions defined by $f(A, B) = A \cap B$, $g(A, B) = A \cup B$ and $h(A) = X - A$. Then f, g and h are continuous and open. Moreover, h is an homeomorfism.* □

In particular, the descriptive set theoretic complexity of τ is the same as that of the collection of τ-closed sets. .

2 On the complexity of topologies over countable sets

In this section we will show some results about the structure of Borel topologies. We will start recalling some results from [9].

Theorem 2.1 *([9]) Let τ be a topology over X.*

(i) $\tau \subseteq 2^X$ is closed if and only if τ is Alexandroff.

(ii) Every open topology is clopen and therefore Alexandroff.

(iii) The closure of τ in 2^X, denoted by $\overline{\tau}$, is a topology. Therefore $\overline{\tau}$ is the smallest Alexandroff topology containing τ.

(iv) τ is T_1 if and only if τ is dense in 2^X.

Proof:(Sketch) First, it is not difficult to show that if $S \subseteq 2^X$ is a closed set which is closed under finite intersections (resp. unions), then S is closed under arbitrary intersections (resp. unions). From this (iii) follows, since $\overline{\tau}$ is a closed set closed under finite intersection and unions. Also from this observation half of (i) easily follows. For the other half of (i), let A_n be a sequence of open sets in an Alexandorff topology τ converging (pointwise) to A. If $x \in A$, then N_x, the minimal neighborhood of x, is a subset of eventually every A_n and therefore a subset of A. Hence A is open. For (ii), let τ be an open topology, then $\emptyset$ and X are interior points of τ. From this, it can be shown that there is a finite set F such that F is τ-clopen and $X - F$ is discrete. From this it follows that τ is clopen. Finally, for (iv) let us suppose that τ is dense in 2^X. Let A_n be a sequence of open sets converging pointwise to $\{x\}$. Let $y \neq x$, then there is n such that $x \in A_n$ and $y \notin A_n$. Hence $\{y\}$ is closed. Conversely, suppose τ is T_1. Then the collection of τ-closed sets contains all finite sets and hence it is dense in 2^X. Since the map $A \mapsto X - A$ is an homeomorphism then τ has to be also dense. □

Examples of F_σ topologies are very easy to construct, for instance, the co-finite topology having countable many open sets is obviously F_σ (and in fact, it is F_σ-complete). Given a filter $\mathcal{F}$ over ω, then $\mathcal{F} \cup \{\emptyset\}$ is a topology, we will identify $\mathcal{F}$ with this topology. Since filters and ideals are dual objects, we will also identify an ideal with the topology associated with its dual filter. Nice examples of F_σ ideals can be found in [6]. Next we give an elementary method to construct a Hausdorff topology based on a filter, it will be used to give examples in the sequel.

Definition 2.2 *Let $\mathcal{F}$ be a filter over ω. Let $\tau_{\mathcal{F}} = \{\{\omega\} \cup A : A \in \mathcal{F}\} \cup \mathcal{P}(\omega)$.*

It is clear that if $\mathcal{F}$ is non principal then $\tau_{\mathcal{F}}$ is a Hausdorff topology over $\omega + 1$. Since the function $f : 2^\omega \to 2^{\omega+1}$ given by $f(A) = A \cup \{\omega\}$ is continuous and $A \in \mathcal{F}$ iff $f(A) \in \tau_{\mathcal{F}}$, then $\mathcal{F}$ is Wadge reducible to $\tau_{\mathcal{F}}$. Also notice that if $\mathcal{F}$ is a non trivial filter, then ω is the only limit point of $(\omega + 1, \tau_{\mathcal{F}})$. In fact, it is clear that this is a characterization of such spaces. We state this observations in the next proposition for later reference.

Proposition 2.3 *(i) For every filter $\mathcal{F}$, $\tau_{\mathcal{F}}$ is a Hausdorff topology over $\omega + 1$ and $\mathcal{F} \leq_W \tau_{\mathcal{F}}$.*

(ii) Let (X, τ) be a Hausdorff space such that $X^{(1)} = \{x_1, \cdots, x_n\}$. Then there is a partition of X in finite many clopen pieces $X_1, \cdots, X_n$ with $x_i \in X_i$ and there are non principal filters $\mathcal{F}_i$ over $X_i - \{x_i\}$ for $1 \leq i \leq n$ such that (X, τ) is homeomorphic to $\oplus_1^n (X_i, \tau_{\mathcal{F}_i}(x_i))$. In fact, the filters are given by $\mathcal{F}_i = \{A \subseteq (X_i - \{x_i\}) : A \cup \{x_i\} \in \tau\}$, thus $\mathcal{F}_i \leq_W \tau$.

Since every G_δ filter is necessarily closed, then 2.3 does not provide examples of G_δ topologies. In fact the situation is quite different. We show next that there are no non discrete T_1 topologies over $\mathbf{N}$ that are G_δ as subsets of $2^{\mathbf{N}}$ and later we give an example of a G_δ-complete T_0 topology. First we recall that $2^{\mathbf{N}}$ has a group structure: If $A, B \in 2^{\mathbf{N}}$ then put $A + B = A \triangle B$. Then $(2^{\mathbf{N}}, +)$ is a Polish group (i.e, it is a topological group such that its topology is separable and completely metrizable). $2^{\mathbf{N}}$ it is the countable product of the group $\{0,1\}$ with addition module 2. The following fact is known (for instance, see I.9.6 of [3]), its proof is based on a Baire category argument.

Theorem 2.4 *Let G be a Polish group and N a subgroup of G. If N is a G_δ subset of G then N is closed.* □

Theorem 2.5 *Let G be a dense G_δ subset of $2^{\mathbf{N}}$, if G is closed under finite unions and intersection then $G = 2^{\mathbf{N}}$. In particular, if τ is a T_1 topology over $\mathbf{N}$ and $\tau \subseteq 2^{\mathbf{N}}$ is G_δ, then τ is the discrete topology.*

Proof: Let G be closed under finite unions and intersections. Let $clopen(G) = \{A \in 2^{\mathbf{N}} : A, A^c \in G\}$, then $clopen(G)$ is a subgroup of the cantor group $2^{\mathbf{N}}$. Since G is G_δ then $clopen(G) = G \cap \{\mathbf{N} - A : A \in G\}$ is also G_δ (since $A \mapsto \mathbf{N} - A$ is an homeomorphism). Thus by 2.4 $clopen(G)$ is closed. But G is dense, hence by the Baire category theorem, $clopen(G)$ is also dense, therefore $G = 2^{\mathbf{N}}$. The last claim follows from 2.1(iv). □.

There are some simple Δ_2^0 topologies over N (i.e., they are both G_δ and F_σ). For instance, let $X = \omega+1$ with the usual order and τ be the corresponding Alexandroff topology. Let $\tau' = \tau - \{\{\omega\}\}$. Then it is easy to check that $\overline{\tau'} = \tau$ and also that τ' is Δ_2^0, i.e., it is both F_σ and G_δ. Next example shows that there are true G_δ topologies. But first a general result.

Proposition 2.6 *Let τ be an Alexandroff topology over a countable set X and let $D(\tau) = \{A \in \tau : A$ is τ-dense$\}$. Then $D(\tau)$ is G_δ in 2^X and therefore every Alexandroff topology contains the G_δ topology given by $D(\tau) \cup \{\emptyset\}$.*

Proof: It is straightforward to check that $A \in D(\tau)$ iff for all $x \in X$ there is $y \in A$ such that $x \leq_\tau y$, where $\leq_\tau$ is the order given by 1.1. □

In general, the topology given by the previous proposition is not a true G_δ set. For instance, let $<_\tau$ be defined in $\mathbf{Z}$ by $i <_\tau j$ if $j = 2n+1$ & $(i = 2n$ or $i = 2n+2)$. Let τ be the Alexandroff topology given by $<_\tau$. Notice that $\{2n+1\}$ is τ-open and therefore an open set V is τ-dense iff for all n, $2n+1 \in A$. Hence $D(\tau)$ is closed.

Example 2.7 *A G_δ-complete T_0 topology on a countable set.*
Let $X = 2^{<\omega}$, so X consists of all binary sequences. Let $\preceq$ be the usual extension order on sequences. Let τ be the Alexandroff topology over X given by $\preceq$. For each $s \in 2^{<\omega}$ the irreducible neighborhood of s is $N_s = \{t \in 2^{<\omega} : s \preceq t\}$. We claim that $D(\tau)$ is G_δ-complete. Let $\tau' = D(\tau) \cup \{\emptyset\}$, then τ' is clearly a topology. Since τ is T_0 and τ' is dense in τ, then τ' is also T_0 and it is a G_δ-complete subset of $2^{2^{<\omega}}$. So it remains to prove the claim, for that end, we will show some simple facts that will simplify the arguments.

Claim 1: Let $T \subseteq 2^{<\omega}$, then T is τ-closed iff T is a tree.
Proof: Since τ is an Alexandroff topology, then by 1.1 $cl_\tau(\{s\}) = \{t \in 2^{<\omega} : t \preceq s\}$ and T is τ-closed iff $cl_\tau(\{s\}) \subseteq T$ for all $s \in T$. □

Claim 2: Let T be a binary tree, as usual $[T]$ denotes the set of (infinite) branches of T. Then T is τ-closed-nowhere-dense iff $[T]$ is nowhere dense in $2^{\mathbf{N}}$.
Proof: It is easy to check that for every τ-closed set T and every $s \in 2^{<\omega}$, $U_s = \{\alpha \in 2^{\mathbf{N}} : s \prec \alpha\} \subseteq [T]$ iff $N_s = \{t \in 2^{<\omega} : s \preceq t\} \subseteq T$. □

The following is a well known fact (see [3], pag 27): Let $\varphi : \mathcal{K}(2^{\mathbf{N}}) \mapsto 2^{2^{<\omega}}$ given by $\varphi(K) = \{s \in 2^{<\omega} : U_s \cap K \neq \emptyset\}$, that is to say $\varphi(K) = \{s \in 2^{<\omega} : \exists \alpha \in K \; s \prec \alpha\}$. Then φ is 1-1, continuous and $K = [\varphi(K)]$. In fact, φ is an homeomorphism of $\mathcal{K}(2^{\mathbf{N}})$ onto the set of binary pruned trees.

Since the collection of nowhere dense closed subsets of $2^{\mathbf{N}}$ is G_δ-complete (see [5]), then from the facts above we conclude that $\{F \subseteq 2^{<\omega} : F$ is τ-closed-nowhere-dense set$\}$ is also G_δ-complete. Finally let us observe that $D(\tau) = \{V \subseteq 2^{<\omega} : 2^{<\omega} - V$ is τ-closed-nowhere-dense$\}$ and therefore (by 1.2) $D(\tau)$ is G_δ-complete. (end of example 2.7)□

Next we will show some simple facts about the complexity of a topology generated by closed, F_σ or analytic bases.

Proposition 2.8 *(i) Every topology over a X with a F_σ base (or subbase) is $\mathbf{\Pi}^0_3$. In particular, every second countable topology over X is $\mathbf{\Pi}^0_3$.*

(ii) Every topology over X with a $\mathbf{\Sigma}^1_1$ bases is $\mathbf{\Sigma}^1_1$.

(iii) Let τ be a Hausdorff topology over X with a F_σ base. If $X^{(1)}$ is finite, then τ is F_σ.

Proof: Let $\mathcal{B}$ be a base for τ, then we have

$$A \in \tau \iff \forall x\, [\, x \in A \to \exists\, B \in \mathcal{B}\, (\, x \in B \;\&\; B \subseteq A\,)\,] \tag{1}$$

(i) If $\mathcal{B}$ is F_σ, then from (1) it follows that τ is $\mathbf{\Pi}^0_3$. If $\mathcal{S}$ is a F_σ sub-base for τ then by 1.2 the base generated by $\mathcal{S}$ is also F_σ. (ii) clearly follows from (1). (iii) follows from 2.3(ii). Since the filters $\mathcal{F}_i$ given there are clearly generated by a F_σ set and therefore they must be F_σ. Hence τ is F_σ. □

Remark: There are topologies over X such that $X^{(1)}$ is finite but τ is not F_σ (and of course τ has not a F_σ base). For instance, let $\mathcal{F}$ be a filter over ω which is not F_σ (for example, the dual filter of $\emptyset \times \mathrm{FIN}$). Then $\tau_{\mathcal{F}}$ (2.2) is not F_σ (actually, it is $\mathbf{\Pi}^0_3$-complete), but $X^{(1)} = \{\omega\}$.

The next theorem says that under some conditions a Hausdorff topology is at least $\mathbf{\Pi}^0_3$.

Theorem 2.9 *Let (X,τ) be a regular space with τ Borel, then one of the following holds:*

(i) $X^{(1)}$ is finite and therefore there are non principal filters $\mathcal{F}_i$, $1 \le i \le n$, over ω such that (X,τ) is homeomorphic to $\oplus_1^n\ (\omega+1,\ \tau_{\mathcal{F}_i}(\omega)\ )$. Moreover, $\mathcal{F}_i \le_W \tau$.

(ii) $\rho \le_W \tau$, where ρ denotes the topology of $\oplus_1^\infty\ (\omega$, co-finite). In particular, τ is $\mathbf{\Pi}^0_3$-hard.

Corollary 2.10 *The topology of the rationals is Π^0_3-complete.* □

Proof: We will need the following easy facts:

Lemma 2.11 *Let Y be a countable set, $\{P_i\}_i$ a partition of Y and $\{V_i\}_i$ pairwise disjoint nonempty subsets of X. Let $f_i : 2^{P_i} \to 2^{V_i}$ be continuous function. Define $f : 2^Y \to 2^X$ by $f(A) = \bigcup_i f_i(A \cap P_i)$. Then f is continuous.* □

Lemma 2.12 *Let ρ be the topology of $\oplus_1^\infty$ (ω, co-finite), the free sum of ω copies of ω with the co-finite topology. Then ρ is $\mathbf{\Pi}^0_3$-complete.*

Proof: Consider the function $f : 2^{\omega\times\omega} \to 2^{\omega\times\omega}$ defined by $f(A) = \{(n,2m) : (n,m) \in A\}$. Then f is continuous and $A \in \emptyset \times \mathrm{FIN}$ iff $f(A)$ is ρ-closed. Since $\emptyset \times \mathrm{FIN}$ is $\mathbf{\Pi}^0_3$-complete then so is ρ □

The following result is the main ingredient for the proof of 2.9

Lemma 2.13 *Let τ be a T_1 Borel topology. Suppose there is an infinite collection of pairwise disjoint non-discrete τ-open sets. Let ρ be topology over $\omega\times\omega$ defined in 2.12. Then $\rho \le_W \tau$, that is to say, there is a continuous function $f : 2^{\omega\times\omega} \to 2^X$ such that A is ρ-closed if and only if $f(A)$ is τ-closed. In particular, τ is $\mathbf{\Pi}^0_3$-hard.*

Proof: Let V_i be a pairwise disjoint τ-open sets such that for each i, $V_i \cap X^{(1)} \neq \emptyset$. Since $\tau|V_i$ is T_1 and non discrete then by 2.5 $\tau|V_i$ is not G_δ. Since $\tau|V_i$ is clearly Borel then by a result of Wadge $\tau|V_i$ is $\mathbf{\Sigma}^0_2$-hard (see, for instance, theorem II.22.10 in [3]). Let $f_i : 2^{\mathbf{N}} \to 2^{V_i}$ be a continuous function such that A is closed in the co-finite topology iff $f_i(A)$ is closed in V_i. Let $K = cl_\tau(\bigcup_i V_i) - \bigcup_i V_i$, then K is τ-closed. We will identify ω with $\{i\} \times \omega$, so f_i can be seen as a function from $2^{\{i\}\times\omega}$. Let $f : 2^{\omega\times\omega} \to 2^X$ be defined by

$$f(A) = K \cup \bigcup_i f_i(A \cap (\{i\} \times \omega))$$

then by 2.11 it is easy to check that f is continuous.

We claim that A is ρ-closed iff $f(A)$ is τ-closed: (a) Suppose A is ρ-closed, then for every i, $A\cap(\{i\}\times\omega$ is co-finite-closed, hence $f_i(A\cap(\{i\}\times\omega))$ is closed in V_i. Let $y \in cl_\tau(f(A))$, we will show that $y \in f(A)$. If $y \in K$ then there is nothing to show. Suppose $y \in \bigcup_i V_i$, and let i be such that $y \in V_i$. Let $W = V_i - f_i(A \cap (\{i\} \times \omega))$, then W is τ-open. Notice that $f(A) \cap V_i = f_i(A \cap (\{i\} \times \omega))$, so if $y \notin f(A)$, then clearly $y \in W$, but this is a contradiction since $y \in cl_\tau(f(A))$. (b) Suppose A is not ρ-closed, then there is i such that $\{n : (i,n) \in A\}$ is not co-finite-closed, and hence $f_i(A \cap (\{i\} \times \omega))$ is not closed in V_i. Since $f(A) \cap V_i = f_i(A \cap (\{i\} \times \omega))$ then $f(A)$ can not be τ-closed. □

We will show next, some conditions where the previous proposition can be applied. Notice that ω with the co-finite topology has no isolated points but there are no disjoint open sets, so we will work with Hausdorff spaces.

Lemma 2.14 *Let (X, τ) be a Hausdorff space such that $X^{(1)}$ is infinite. Then any of the following conditions implies that there is an infinite collection of pairwise disjoint non-discrete τ-open sets.*

(i) $X^{(2)} \neq \emptyset$.

(ii) (X, τ) *is regular.*

Proof: (i) Suppose $X^{(2)} \neq \emptyset$. Let $x \in X^{(2)}$ and $y_1 \neq x$ with $y_1 \in X^{(1)}$. Let W and V_1 be disjoint open sets containing x and y_1 respectively. Then $W \cap X^{(1)} \neq \emptyset$. Let $y_2 \in W$ be a limit point. We can repeat the construction inside W and find V_2 with $y_2 \in V_2$. In this way we construct a sequence of limit points $\{y_n\}$ and pairwise disjoint open sets $\{V_n\}$ with $y_n \in V_n$.

(ii) If τ is zero-dimensional (i.e., it admits a base of clopen sets), $X^{(1)}$ is infinite and $X^{(2)} = \emptyset$, then such family of open sets exists. In fact, we can define by induction a collection $\{W_x : x \in X^{(1)}\}$ of pairwise τ-clopen sets with $x \in W_x$. If τ is regular, $X^{(1)}$ is infinite and $X^{(2)} = \emptyset$, then τ is zero-dimensional. In fact, let $x \in X^{(1)}$ and V open such that $x \in V$ and $X^{(1)} \cap V = \{x\}$. Then by regularity, there is $W \subseteq V$ open such that $x \in W$ and $cl_\tau(W) \subseteq V$. Then $cl_\tau(W) \cap X^{(1)} = \{x\}$, thus W is clopen. □

Now 2.9 follows from 2.3(ii), 2.14 and 2.13. (□ 2.9)

Example 2.15 *A Σ^1_1-complete zero dimensional, perfect topology with a Borel subbase.*

Let $X = \mathbf{Z}^{<\omega}$ and let $\prec$ be the usual partial order over sequences given by extension. Let $<_l$ be defined as follows: if $s \prec t$ then $s <_l t$, and if s and t are not $\prec$-comparable then put $s <_l t$ if s is less than t in the lexicographic order over $\mathbf{Z}^{<\omega}$. Let τ_l be the order topology over $\mathbf{Z}^{<\omega}$ given by $<_l$. Since $<_l$ is isomorphic to the order of the rationals, then $(\mathbf{Z}^{<\omega}, \tau_l)$ is homeomorphic to the topology of the rationals, hence τ_l is $\mathbf{\Pi}^0_3$ and zero dimensional. For every $\alpha \in \mathbf{Z}^\omega$ let $S_\alpha = \{s \in \mathbf{Z}^{<\omega} : \text{for all } i < lh(s),\, \alpha(i) \leq s(i)\}$. Notice that each S_α is τ_l-closed-nowhere-dense. Let $\mathcal{F}$ be the filter generated by the S_α with $\alpha \in \mathbf{Z}^\omega$. Let τ be the topology generated by $\mathcal{F} \cup \tau_l$ (i.e. τ is the supremum of τ_l and the topology associated with $\mathcal{F}$). Since the collection $\{S_\alpha : \alpha \in \mathbf{N}^\mathbf{N}\}$ is a Borel set, then τ has a Borel subbase. Notice that for every s, t and α, if $(s,t) \cap S_\alpha$ is not empty then it is infinite. So τ has no isolated points. Since each S_α is now τ-clopen and τ_l is zero dimensional then it is easy to check that τ_l is also zero dimensional. We claim that τ is $\mathbf{\Sigma}^1_1$-complete, it is clearly $\mathbf{\Sigma}^1_1$. It was shown in [10] that $\mathcal{F}$ is $\mathbf{\Sigma}^1_1$-complete and a similar argument also works in our case. For every tree T on $\mathbf{Z}$ we define $f(T)$ by

$$f(T) = \{s \in \mathbf{Z}^{<\omega} : \text{there is } t \in T \text{ with } lh(s) = lh(t) \text{ and such that for all } i < lh(s),\, t(i) \leq s(i)\}$$

It is easy to verify that f is continuous function from the collection of trees over $\mathbf{Z}$ (which is a closed subset of $2^{\mathbf{Z}^{<\omega}}$) into $2^{\mathbf{Z}^{<\omega}}$.

We will show first that if T is well-founded then $f(T)$ is not τ-open. Let T be a well-founded tree and towards a contradiction let us suppose that there are $s, t \in \mathbf{Z}^{<\omega}$ and $\alpha \in \mathbf{Z}^\omega$ such that $(s,t) \cap S_\alpha \neq \emptyset$ and $(s,t) \cap S_\alpha \subseteq f(T)$, where $(s,t) = \{u \in \mathbf{Z}^{<\omega} : s <_l u <_l t\}$. Let u be such that $s <_l u <_l t$ and $u \in S_\alpha$. Let $n = lh(u)$ and α_u be defined by $\alpha_u(i) = u(i)$ if $i < n$ and $\alpha_u(i) = \alpha(i)$ otherwise. Then it is easy to see that for all m, $\alpha|m \in (s,t) \cap S_\alpha$ and therefore $\alpha|m \in f(T)$. Consider $T' = \{t \in T : \text{for all } i < lh(t),\, t(i) \leq \alpha_u(i)\}$. By definition of $f(T)$ and the fact that $\alpha|m \in f(T)$ for all m we conclude that T' is a finitely branching infinite tree, therefore by Koning's lemma T' is not well-founded, which is a contradiction.

On the other hand, if T is not well founded and $\alpha \in [T]$ then it is easy to check that $S_\alpha \subseteq f(T)$ and therefore $f(T) \in \mathcal{F}$ and thus $f(T)$ is τ-open. This finishes the proof that τ is $\mathbf{\Sigma}^1_1$-complete.

3 Ideals of nowhere dense sets over countable sets.

If τ is a topology over X, we will denote by $ND(\tau)$ the collection of τ-nowhere dense sets, i.e. those subsets $A \subseteq X$ such that $cl_\tau(A)$ has empty interior. We will show some simple facts about the problem of representing ideals of subsets of X as the nowhere dense sets with respect to a topology over X. This problem has been studied in [1]. Let $\mathcal{I}$ be an ideal over ω containing all singletons. Then the dual filter (together with $\emptyset$) is a T_1 (but not Hausdorff) topology such that its nowhere dense sets are exactly the sets in $\mathcal{I}$. Here we are interested in the following question: given a Borel (Analytic) ideal $\mathcal{I}$ over X, which are the possible complexities for a topology τ such that $\mathcal{I} = ND(\tau)$? It is known that there is no Hausdorff topology τ such that $ND(\tau) = \text{FIN}$ (see [1]). We will show next that this result extends to F_σ ideals. The following result generalizes Lemma 3.2 of [1].

Proposition 3.1 *Let $\mathcal{I}$ be a F_σ ideal with FIN $\subseteq \mathcal{I}$ and τ a topology such that $\mathcal{I} \cap \tau = \{\emptyset\}$ and FIN $\subseteq ND(\tau)$. If there exists an infinite family $\{V_n\}$ of nonempty disjoint open sets, then there is $A \in ND(\tau) - \mathcal{I}$.*

Proof: Let $\mathcal{I} = \bigcup_n F_n$ with each F_n closed. We can assume w.l.o.g. that each F_n is hereditary and $F_n \subseteq F_{n+1}$. First, we claim that for every nonempty $V \in \tau$, $\{n :$ there is a finite set $K \in F_n$ with $K \subseteq V$ and $K \notin F_m$, for $m < n\}$ is infinite. In fact, otherwise there is n such that every finite subset of V belongs to F_n. But as F_n is closed, then $V \in F_n$ and therefore $V \in \mathcal{I}$, which is a contradiction. Let $\{V_n\}$ be a infinite collection of pairwise disjoint nonempty open sets and K_n be a finite subset of V_n such that $K_n \notin F_m$ for $m < n$. Let $A = \bigcup_n K_n$, since each F_n is hereditary then $A \notin F_n$, i.e. $A \notin \mathcal{I}$. On the other hand, $A \in ND(\tau)$ because every finite set is τ-nowhere dense and $A \cap V_n$ is finite. □

It is well known that in every Hausdorff space there is an infinity family of nonempty disjoint open sets, therefore we have the following result which generalizes theorem 3.4 of [1].

Theorem 3.2 *If τ is a Hausdorff topology such that FIN$\subseteq ND(\tau)$, then $ND(\tau)$ is not F_σ.*

The next observation is that if (X, τ) is a scattered, then $ND(\tau)$ is principal.

Proposition 3.3 *Let (X, τ) be topological space such that for some α, $X^{(\alpha)} = \emptyset$. Then $ND(\tau)$ is a principal ideal and therefore closed in 2^X, in fact, $ND(\tau) = \mathcal{P}(X^{(1)})$.*

Proof: It is clear that $ND(\tau) \subseteq \mathcal{P}(X^{(1)})$. On the other hand, it suffices to show that $X^{(1)}$ is nowhere dense, equivalently, that the collection of isolated points is open dense. Let V a non empty open set and $x \in V$ with $rank(x) \neq 0$ (i.e. x is not isolated) we will show that there is $y \in V$ with $rank(y) < rank(x)$, therefore there must be $y \in V$ with y isolated. Let $rank(x) = \beta$, i.e. $x \in X^{(\beta)}$ but $x \notin X^{(\beta+1)}$. Hence there is an open set W with $x \in W$ such that $W \cap X^{(\beta)} = \{x\}$. Since x is a limit point, then there is $y \in W \cap V$ with $y \neq x$. Thus $y \notin X^{(\beta)}$, i.e. $rank(y) < \beta$. □

Theorem 3.4 *Let τ be a $\mathbf{\Sigma}^0_3$ Hausdorff topology over X, then $ND(\tau)$ is principal (hence closed).*

Proof: We claim that $X^{(2)} = \emptyset$, otherwise by 2.13 together with 2.14 τ would be $\mathbf{\Pi}^0_3$-hard, which is a contradiction. Hence by 3.3 $ND(\tau)$ is principal. □

This result implies that in order to represent non principal ideals with Hausdorff topologies we must look for topologies as least as complicated as $\mathbf{\Pi}^0_3$. We will focus next on Alexandroff topologies.

Proposition 3.5 *Let τ be a second countable topology, then $ND(\tau)$ is $\mathbf{\Pi}^0_3$.*

Proof: Let $\{V_n\}$ be a countable base for τ. Then

$$A \in ND(\tau) \text{ if and only if } \forall n\, \exists x\, (x \in V_n \;\&\; x \notin cl_\tau(A))$$

Now,

$$x \in cl_\tau(A) \text{ if and only if } \forall n\, (x \in V_n \Rightarrow V_n \cap A \neq \emptyset)$$

This is clearly a G_δ relation, and therefore $ND(\tau)$ is $\mathbf{\Pi}^0_3$. □

Proposition 3.6 *(i) Let $A \subseteq X$, then there is a T_0 Alexandroff topology τ over X such that $ND(\tau) = \mathcal{P}(A)$.*

(ii) There is a T_0 Alexandroff topology τ over ω such that $ND(\tau) = FIN$.

(iii) There is a T_0 Alexandroff topology τ over $\omega \times \omega$ such that $ND(\tau) = FIN \times \emptyset$.

(iv) There is a T_0 Alexandroff topology τ over $\omega \times \omega$ such that $ND(\tau) = \emptyset \times FIN$.

Proof: We will define for each case a partial order $\leq_\tau$ and the topology will be given by 1.1.

(i) Let $\leq_\tau$ be defined by $x <_\tau y$ for all $x \in A$ and $y \notin A$.

(ii) Let $\leq_\tau$ be the usual order over ω.

(iii) Let $\leq_\tau$ be defined over $\omega \times \omega$ as follows: $(n,m) <_\tau (n',m')$ if $n < n'$ and $(n,m) <_\tau (n,m')$ if $m' < m$, so the order of $\{n\} \times \omega$ is the reversed order of ω. In other words, we have put a copy of ω^* for each element of ω. This is a total order without a maximal point, hence a set is nowhere dense iff it is bounded. From this the result easily follows.

(iv) Let $\leq_\tau$ be defined over $\omega \times \omega$ as follows: $(n,m) \leq_\tau (n,m')$ if $m \leq m'$. Notice that we have ω copies of the usual order of the natural numbers and hence for every n, $\{n\} \times \omega$ is τ-clopen. Thus A is τ-nowhere-dense iff for every n, $A \cap (\{n\} \times \omega)$ is nowhere dense in $\{n\} \times \omega$ iff for every n, $A \cap (\{n\} \times \omega)$ is finite. □

Remarks: Since every Alexandroff topology is second countable, then its nowhere dense sets form a $\mathbf{\Pi}^0_3$ ideal. The previous proposition suggests the following question: which $\mathbf{\Pi}^0_3$ ideals $\mathcal{I}$ over $\mathbf{N}$ admit an Alexandroff topology τ such that $ND(\tau) = \mathcal{I}$? It is easy to check using 1.1 and 2.1 that the collection of Alexandroff topologies is a closed subset of the hyperspace $\mathcal{K}(2^{\mathbf{N}})$. Now, if we fix a $\mathbf{\Pi}^0_3$ ideal $\mathcal{I}$ over $\mathbf{N}$ then $Top(\mathcal{I}) = \{\tau : \tau \text{ is Alexandroff and } ND(\tau) = \mathcal{I}\}$ is a $\mathbf{\Pi}^1_1$ subset of $\mathcal{K}(2^{\mathbf{N}})$. Notice that the equivalence relation saying that two Alexandroff topologies have the same ideal of nowhere dense sets is a $\mathbf{\Pi}^1_1$ equivalence relation. Is $Top(\text{FIN})$ Borel?

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