Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

First Order Ordinary Differential equations with Several Bounded Separate Solutions

Antonio Tineo

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1 Introduction

Let $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that $F(t, x)$ is T-periodic in t for some $T > 0$ and

$$F(t,x) \to -\infty \quad \text{as} \quad |x| \to +\infty \quad \text{uniformly in} \quad t \in \mathbb{R}. \tag{1.1}$$

In [1], Mawhin shows the existence of $\lambda_0 \in \mathbb{R}$ such that the equation

$$x' = F(t,x) + \lambda \tag{1.2}$$

has zero, at least one or at least two T-periodic solutions according to $\lambda < \lambda_0$, $\lambda = \lambda_0$ or $\lambda > \lambda_0$; with "at least" replaced by "exactly" when $F(t, x)$ is strictly concave in x.

In this paper we replace the assumption about periodicity by:
"The restiction of F to $\mathbb{R} \times K$ is bounded for each compact subset K of $\mathbb{R}$."
and we prove the existence of $\lambda_0 \in \mathbb{R}$ such that (1.2) has zero, at least one or at least two "separate" solutions, with "at least" replaced by "exactly" when $F(t, x)$ satisfies an additional assumption concerning the concavity of $F(t, x)$ with respect to x.

More precisely, we say that the solutions $u_1, \ldots, u_N$ of (1.2) are *separete* if they are defined and bounded in $\mathbb{R}$ and

$$\inf\{|u_i(t) - u_j(t)| : t \in \mathbb{R}\} > 0 \quad \text{if} \quad i \neq j.$$

In this case, we say that (1.2) has at least N separate solutions. If in addition, (1.2) does not have $N + 1$ separate solutions, we say that (1.2) has exactly N separate solutions.

Finally, in section 4 we consider the periodic case and we complement the results in [1].

2 Coercive Systems

We begin with some notations. In the sequel, $\mathcal{C}$ denotes the space of all continuous functions $F : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that

C_1) $F(t, x) \to -\infty$ as $|x| \to +\infty$ uniformly on $t \in \mathbb{R}$.

C_2) $F(t, x)$ is bounded on $\mathbb{R} \times K$ for each compact set $K \subset \mathbb{R}$.

In order to simplify our proofs we also assume that

C_3) $F(t, x)$ is locally Lipschitz continuous in x.

However, we can show that the main results of this section remain true even if C_3) is not satisfied.

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We denote by BC the space of all bounded continuous functions $u : \mathbb{R} \to \mathbb{R}$ provided with the usual norm $\|u\|_0 = \sup\{|u(t)| : t \in \mathbb{R}\}$. Analogously we denote by BC^1 the space of all bounded and continuously differentiable functions $u : \mathbb{R} \to \mathbb{R}$ such that the derivative u' belongs to BC. Finally, we define $BC_+ = \{u \in BC : \inf(u) > 0\}$.

In the following, F, G denote two points in $\mathcal{C}$.

Proposition 2.1 *Let S_F be the set of all solutions of*

$$x' = F(t, x) \tag{2.1}$$

belonging to BC. Then S_F is a bounded subset of BC.

Proof. Let us fix $R > 0$ such that

$$F(t, x) < -1 \quad \text{if} \quad |x| > R \tag{2.2}$$

and fix $u \in S_F$. By Lemma 2.3 of [2] there exists a sequence (t_n) in $\mathbb{R}$ such that $|u(t_n)| \to \|u\|_0$ and $u'(t_n) \to 0$ as $n \to +\infty$. From this and (2.1)-(2.2), there exists $N \geq 1$ such that $|u(t_n)| \leq R$ for all $n \geq N$ and the proof follows easily.

Proposition 2.2 *If (2.1) has a bounded solution then this equation has bounded solutions Θ_F, Γ_F such that $\Theta_F \leq u \leq \Gamma_F$ for any bounded solution u of (2.1).*

Proof. Let S_F be as above and define

$$x_0 = \inf\{u(0) : u \in S_F\}, \qquad y_0 = \sup\{u(0) : u \in S_F\}.$$

Let Θ_F (resp. Γ_F) be the solution of (2.1) determined by the initial condition $\Theta_F(0) = x_0$ (resp. $\Gamma_F(0) = y_0$) and fix a sequence (u_n) in S_F such that $u_n(0) \to x_0$ (resp. $u_n(0) \to y_0$). Then $u_n(t) \to \Theta_F(t)$ (resp. $u_n(t) \to \Gamma_F(t)$) for each t in the domain of Θ_F (resp. Γ_F) and the proof follows easily from proposition 2.1.

If F does not satisfy C_3) then the proof of Proposition 2.2 can be obtained by a suitable application of Zorn's Lemma and Ascoli's Theorem.

Remark 2.3 *If $F(t, x)$ is T-periodic in t for some $T > 0$, then Θ_F, Γ_F are T-periodic.*

Proof. By Proposition 2.2 we have

$$\Theta_F(t) \leq \Theta_F(t + T) \quad \text{and} \quad \Theta_F(t) \leq \Theta_F(t - T); \ t \in \mathbb{R}.$$

From the last inequality we get $\Theta_F(t+T) \leq \Theta_F(t)$ and so, Θ_F is T-periodic. The rest of the proof is similar.

Remark 2.4 *Let u be a solution of (2.1) and assume that the hypothesis in Proposition 2.2 holds.*

a) If $u(t_0) > \Gamma_F(t_0)$ for some t_0, then u is defined and bounded on $[t_0, \infty)$. Moreover, if $F(t, x)$ is T-periodic in t, then $u(t) - \Gamma_F(t) \to 0$ as $t \to +\infty$.

b) If $u(t_0) < \Theta_F(t_0)$ for some t_0, then u is defined and bounded on $(-\infty, t_0]$. Moreover, if $F(t, x)$ is T-periodic in t, then $u(t) - \Theta_F(t) \to 0$ as $t \to -\infty$.

Proof. We only prove a). To this end, let us fix $R > u(t_0)$ such that $F(t, R) < 0$, then the constant function $v(t) = R$ is a supersolution of (2.1) such that $v(t_0) \geq u(t_0)$. From this $\Gamma_F(t) \leq u(t) \leq R$ for all $t \geq t_0$, t in the domain of u. In particular u is defined and bounded on $[t_0, \infty)$ and

$$\Gamma_F(t) \leq u(t) \quad \text{for all} \quad t \geq t_0. \tag{2.3}$$

Assume now that $F(t, x)$ is T-periodic in t. Since u is defined and bounded on $[t_0, \infty)$, there exists a T-periodic solution u_0 of (2.1) such that

$$u(t) - u_0(t) \to 0 \quad \text{as} \quad t \to +\infty. \tag{2.4}$$

By (2.3), $\Gamma_F \leq u_0$ and by Proposition 2.2, $u_0 = \Gamma_F$. The proof follows now from (2.4).

Proposition 2.5 *If (2.1) has a bounded solution u and $F \leq G$ then the equation*

$$x' = G(t, x) \tag{2.5}$$

has a bounded solution $v \leq u$ (resp. $v \geq u$). In particular, $\Theta_G \leq \Theta_F$ and $\Gamma_F \leq \Gamma_G$.

Proof. Let us fix $R > 0$ such that

$$G(t, R) < 0 \quad \text{and} \quad R > u(t) \quad \text{for all} \quad t \in \mathbb{R}.$$

For each integer $n \geq 1$, let v_n be the solution of (2.5) determined by the initial condition $v_n(-n) = u(-n)$; then $u(t) \leq v_n(t) \leq R$ if $t \geq n$ belongs to the domain of v_n. Note that u (resp. $w(t) \equiv R$) is a subsolution (resp. supersolution) of (2.5). From this v_n is defined on $[-n, \infty)$ and

$$u(t) \leq v_n(t) \leq R \quad \text{for all} \quad t \geq -n.$$

Since $\{v_n(0)\}$ is bounded, we can assume without loss of generality, that $v_n(0) \to x_0$ for some $x_0 \in \mathbb{R}$. Now it is easy to show that the solution v of (2.5) determined by the initial condition $v(0) = x_0$ is defined on $\mathbb{R}$ and $u(t) \leq v(t) \leq R$ for all $t \in \mathbb{R}$. That is, v is a bounded solution of (2.5) such that $u \leq v$. The rest of the proof is similar.

Let $u_1 < \cdots < u_N$ be bounded solutions of (2.1). We say that $u_1, \ldots, u_N$ are *separate* if $u_{i+1} - u_i \in BC_+$ for $i = 1, \ldots, N-1$. In this case, we say that (2.1) has (at least) N separate solutions. If in addition, (2.1) does not have $N + 1$ separate solutions, we say that (2.1) has exactly N separate solutions.

Corollary 2.6 *If $F \leq G$ and (2.1) has two separate solutions then, the same holds for (2.5).*

Proposition 2.7 *There exists $\lambda \in \mathbb{R}$ such that the system*

$$x' = F(t, x) + \lambda \tag{2.6}$$

has two separate solutions.

Proof. By our assumption C_2), F is bounded on $\mathbb{R} \times [-1, 1]$ and hence, there exists $\lambda > 0$ such that

$$F(t, x) + \lambda > 0 \quad \text{if} \quad |x| \leq 1 \quad \text{and} \quad t \in \mathbb{R}.$$

Now, fix $R > 1$ such that

$$F(t, x) + \lambda < 0 \quad \text{if} \quad |x| \geq R \quad \text{and} \quad t \in \mathbb{R},$$

and define for each integer $n \geq 1$, v_n as the solution of (2.6) determined by the initial condition $v_n(-n) = 1$. By the argument in Proposition 2.5, v_n is defined on $[-n, \infty)$ and

$$1 \leq v_n(t) \leq R \quad \text{for all} \quad t \geq -n.$$

From this, (2.6) has a bounded solution v_+ such that $v_+ \geq 1$. Analogously, this equation has a bounded solution $v_- \leq -1$ and the proof is complete.

Proposition 2.8 *There exists $\lambda \in \mathbb{R}$ such that (2.6) has no bounded solutions.*

Proof. By C_1), $F(t,x)$ is bounded above and hence, there exists $\lambda < 0$ such that $F(t,x) + \lambda \leq -1$ for all $t, x \in \mathbb{R}$. The proof follows now easily.

Suppose that the partial derivative $F_x(t,x)$ is defined and continuous on $\mathbb{R} \times \mathbb{R}$. We say that a bounded solution u of (2.1) is *singular* if the linear map

$$BC^1 \to BC; \quad x \to x' - F_x(t, u(t))x$$

is not a homeomorphism onto BC.

We say that $F(t,x)$ is *locally equicontinuous in x*, if for each compact set $K \subset \mathbb{R}$ and any $\epsilon > 0$ there exists $\delta > 0$ such that

$$|F(t,x) - F(t,y)| \leq \epsilon \quad \text{if} \quad t \in \mathbb{R}; \quad x, y \in K, \quad |x-y| \leq \delta.$$

Examples.

a) If $F(t,x)$ is T-periodic in t, for same $T > 0$ then F is locally equicontinuous in x.

b) If $F(t,x) = a(t)x^N$ for same integer $N \geq 0$ and $a \in BC$, then F is locally equicontinuous in x.

c) If F, G are locally equicontinuous in x then the same holds for $F + G$.

Remark 2.9 *Suppose that $F \in \mathcal{C}$ is locally equicontinuous in x. Given a compact set K of $\mathbb{R}$ and a sequence (t_n) in $\mathbb{R}$ it is easy to show (Using C_2) and Ascoli's Theorem) the existence of a subsequence (s_n) of (t_n) and a continuous function $\varphi : K \to \mathbb{R}$ such that*

$$F(s_n, x) \to \varphi(x) \quad \text{as} \quad n \to \infty \quad \text{uniformly in} \quad K.$$

Theorem 2.10 *Let $F \in \mathcal{C}$. Then there exists $\lambda_0 = \lambda_0(F)$ in $\mathbb{R}$ with the following properties:*

a) If $\lambda \geq \lambda_0$, equation (2.6) has at least a bounded solution.

b) If $\lambda > \lambda_0$ and $F(t,x)$ is locally equicontinuous in x, then (2.6) has at least two separate solutions.

c) If $\lambda = \lambda_0$ and the partial derivative $F_x(t,x)$ is defined and continuous on $\mathbb{R} \times \mathbb{R}$, then each solution of (2.6) is singular.

e) If $\lambda < \lambda_0$, equation (2.6) has no bounded solutions.

Proof. Let us define Λ as the subset of $\mathbb{R}$ consisting of all points λ such that (2.6) has a bounded solution. By Proposition 2.7, Λ is nonempty and by Propositions 2.8 and 2.5, there exists $\lambda_1 \in \mathbb{R}$ such that (2.6) has no bounded solutions if $\lambda \leq \lambda_1$. Thus, λ_1 is an upper bound for Λ and we can define

$$\lambda_0 = \inf(\Lambda).$$

Note that, by Proposition 2.5, $(\lambda_0, \infty) \subset \Lambda \subset [\lambda_0, \infty)$.

Let us fix a sequence $\lambda_1 > \lambda_2 > \ldots$ in (λ_0, ∞) converging to λ_0 and define $F_n(t,x) = \lambda_n + F(t,x)$, $u_n = \Theta_{F_n}$, $v_n = \Gamma_{F_n}$. By Proposition 2.5, $u_1 \leq \cdots \leq u_n \leq v_n \leq \cdots \leq v_1$ and hence, (2.6) has a bounded solution for $\lambda = \lambda_0$. Thus, $\Lambda = [\lambda_0, \infty)$.

Let us fix $\lambda > \lambda_0$ and a bounded solution u_0 of

$$x' = F(t,x) + \lambda_0. \tag{2.7}$$

By Proposition 2.5, (2.6) has a bounded solution $v_1 \geq u_0$. If $v_1(t_0) = u_0(t_0)$ for some to then $u'(t_0) < v_1'(t_0)$ and hence $v_1 < u$ on $(t_0 - \epsilon, t_0)$ for some $\epsilon > 0$. This contradiction proves that $v_1 - u_0 > 0$.

Claim If F is locally equicontinuous in x, then $v_1 - u_0 \in BC_+$. To show this define $\omega = v_1 - u_0$ and suppose on the contrary that $\inf(\omega) = 0$. By Lemma 2.3 of [2] there exists a sequence $\{t_n\}$ in $\mathbb{R}$ such that $\omega(t_n) \to 0$ and $\omega'(t_n) \to 0$ as $n \to +\infty$. Now, let us fix a compact set K of $\mathbb{R}$ contraining $v_1(\mathbb{R})$ and $u_0(\mathbb{R})$. By Remark 2.9, we can assume the existence of a continuous function $\varphi : K \to \mathbb{R}$ such that

$$F(t_n, x) \to \varphi(x) \quad \text{as} \quad n \to +\infty \quad \text{uniformly on} \quad K. \tag{2.8}$$

On thé other hand, since u_0, v_1 are bounded and $\omega(t_n) \to 0$, we can assume without lass of generality the existence of a $x_0 \in \mathbb{R}$ such that

$$u_0(t_n) \to x_0, \qquad v_1(t_n) \to x_0 \quad \text{as} \quad N \to +\infty.$$

From this and (2.8),

$$F(t_n, u_0(t_n)) - F(t_n, v_1(t_n)) \to \varphi(x_0) - \varphi(x_0) = 0$$

and hence $\lambda = \lambda_0$, since $\omega'(t_n) \to 0$. This contradiction proves the claim.

Similarly, if $\lambda > \lambda_0$ and $F(t,x)$ is locally equicontinuous in x, then (2.6) has a bounded solution v_0 such that $u_0 - v_0 \in BC_+$. Thus, v_1, v_0 are separate solutions of (2.6).

Finally, assume that $F_x(t,x)$ is defined and continuous in $\mathbb{R} \times \mathbb{R}$ and suppose that u_0 is a bounded solution of (2.7) which is not singular. If we define $\mathcal{F} : BC^1 \to BC$, $\mathcal{F}(x) = x' - F(t,x) - \lambda_0$; then the Frechet derivative $\mathcal{F}'(u_0) : BC^1 \to BC$, is a linear homeomorphism into BC, and by the Inverse Function Theorem there exists $\epsilon > 0$ such that the equation

$$x' = F(t,x) + \lambda_0 - \epsilon$$

has a bounded solution. Therefore, $\lambda_0 - \epsilon \in \Lambda$ and this contradiction ends the proof.

Theorem 2.10 improves theorem 1 of [1].

In the next result we study the continuity of the number $\lambda_0(F)$ (given by Theorem 2.10) with respect to F.

Theorem 2.11 *Let $\{F_n\}$ be a sequence on $\mathcal{C}$ and let $F \in \mathcal{C}$. If*

$$F_n(t,x) \to F(t,x) \quad \textit{as} \quad n \to \infty \quad \textit{uniformly on} \quad R \times K$$

for each compact subset K of $\mathbb{R}$ then, given $\epsilon > 0$ there exists an integer $N \geq 1$ such that

$$\lambda_0(F_n) - \lambda_0(F) \leq \epsilon \quad \textit{for all} \quad n \geq N.$$

Further, if there exist positive real numbers δ, R such that

$$F_n(t,x) \leq -\delta \quad \textit{for} \quad |x| > R, \quad n \in N, \quad t \in \mathbb{R}, \tag{2.9}$$

then $\lambda_0(F_n) \to \lambda_0(F)$.

Proof. Let us fix a bounded solution u_0 of (2.7). Given $\epsilon > 0$ we define $\lambda = \lambda_0(F) + \epsilon$ and we fix $R_0 > \sup\{u_0(t) : t \in \mathbb{R}\}$ such that

$$F(t, R_0) + \lambda \leq -1 \quad \text{for all} \quad t \in \mathbb{R}. \tag{2.10}$$

Now, we fix a compact subset K of $\mathbb{R}$ such that $R_0, u_0(t) \in int(K)$ for all $t \in \mathbb{R}$. By our assumption, there exists an integer $N \geq 1$ such that

$$F_n(t,x) + \lambda \geq F(t,x) + \lambda_0 \quad \text{if} \quad |x| \leq R_0, \quad n \geq N, \quad t \in \mathbb{R}$$

and by (2.10), we can suppose that

$$F_n(t, R_0) + \lambda < 0 \quad \text{for all} \quad n \geq N \quad \text{and} \quad t \in \mathbb{R}.$$

Using the argument in Proposition 2.5 we show that the equation

$$x' = F_n(t, x) + \lambda \tag{2.11}$$

has a bounded solution for $n \geq N$, and hence $\lambda \geq \lambda_0(F_n)$ for all $n \geq N$. Thus, the proof of our first assertion is complete.

Assume now that (2.9) holds. Fix $\epsilon > 0$ and define $\lambda = \lambda_0(F) - \epsilon$.

Claim There exists an integer $N \geq 1$ such that (2.11) has no bounded solutions for all $n \geq N$. To show this, assume on the contrary, that there exists a subsequence $\{G_k\}$ of $\{F_n\}$ such that the equation

$$x' = G_k(t, x) + \lambda$$

has a bounded solution u_k for all $k \in \mathbb{N}$. Using (2.9) and the argument in Proposition 2.1, we get

$$||u_k||_0 \leq R \quad \text{for all} \quad k \in \mathbb{N}.$$

On the other hand,

$$G_k(t, x) \to F(t, x) \quad \text{as} \quad k \to \infty \quad \text{uniformly in} \quad \mathbb{R} \times [-R - 1, R + 1]$$

and now it is easy to show that (2.6) has a bounded solution. This contradicts Theorem 2.10 and the proof of the claim is complete.

By the above claim, there exists $N \geq 1$ such that $\lambda_0(F_n) \geq \lambda$ for $n \geq N$ and so, $\lambda_0(F_n) - \lambda_0(F) \geq -\epsilon$ for $n \geq N$. Thus, the proof is complete.

3 Concave Systems

In this section we give a version of Theorem 2 of [1] for non periodic systems.

Theorem 3.1 *Suppose that for each $R, \epsilon > 0$ there exists a continuous function $b : \mathbb{R} \to [0, \infty)$ such that*

$$F(t, (1 - \mu)x + \mu y) \geq (1 - \mu)F(t, x) + \mu F(t, y) + b(t)\mu(1 - \mu) \quad \textit{if}$$

$$|x - y| \geq \epsilon, \quad |x| \leq R, \quad |y| \leq R, \quad \mu \in [0, 1], \quad t \in \mathbb{R}. \tag{3.1}$$

$$\int_0^{\infty} b(t)dt = \int_{-\infty}^{0} b(t)dt = +\infty. \tag{3.2}$$

If $u_0 < u < u_1$ are bounded solutions of (2.1) and u_0, u_1 are separate then

$$u_1(t) - u(t) \to 0 \quad \textit{as} \quad t \to +\infty \quad \textit{and}$$
$$u(t) - u_0(t) \to 0 \quad \textit{as} \quad t \to -\infty.$$

Proof. Let us define $\epsilon = \inf\{u_1(t) - u_0(t) : t \in \mathbb{R}\}$ and fix $R > 0$ such that $|u_i(t)| \leq R$ for $t \in \mathbb{R}$ and $i = 0, 1$. Take a continuous function $b : \mathbb{R} \to [0, \infty)$ satisfying (3.1)-(3.2) and define

$$v(t) = \frac{u(t) - u_0(t)}{u_1(t) - u_0(t)}.$$

It is easy to show that $v(t) \in (0,1)$, $u = (1-v)u_0 + vu_1$, and

$$v' = \left[\frac{F(t,u) - F(t,u_0)}{u - u_0} - \frac{F(t,u_1) - F(t,u_0)}{u_1 - u_0}\right],$$

from this and (3.1)

$$v' \geq a(t)v(1-v) \tag{3.3}$$

where $a = (u_1 - u_0)b$. Note that

$$\int_0^\infty a(t)dt = \int_{-\infty}^0 a(t)dt = +\infty$$

since (3.2) holds and $u_1 - u_0 \in BC_+$.

Integrating (3.3) over $[0,t]$, for $t > 0$, we obtain,

$$1 > v(t) > \frac{v(0)e^{\int_0^t a(s)ds}}{1 - v(0) + v(0)e^{\int_0^t a(s)ds}} \to 1 \text{ as } t \to +\infty.$$

Thus, $v(t) \to 1$ as $t \to +\infty$ and hence, $u_1(t) - u(t) \to 0$ as $t \to +\infty$. The rest of the proof is similar.

Corollary 3.2 *Under the assumptions in Theorem 3.1, equation (2.1) has at most two separate solutions. Moreover if $u_0 < u_1$ are separate solutions of (2.1) and $u \neq u_0, u_1$ is a bounded solution of this equation, then $u_0 < u < u_1$.*

Proof. The first assertion is clear. Assume now that u is a bounded solution of (2.1) such that $u < u_0 < u_1$, then u_1, u are separate and by Theorem 3.1, $u_1(t) - u_0(t) \to 0$ as $t \to +\infty$. Similarly, we get a contradiction if we assume the existence of a bounded solution of (2.1) such that $u_0 < u_1 < u$. Thus, the proof is complete.

Remark 3.3 *Suppose that $F(t,x)$ is T-periodic in t and that the partial derivative $F_x(t,x)$ is defined and continuous in $\mathbb{R} \times \mathbb{R}$. If $F(t,x)$ is strictly concave in x then, for each $R, \epsilon > 0$ there exists a positive constant function b satisfying (3.1).*

Proof. Assume on the contrary the existence of $R, \epsilon > 0$ and sequences $\mu_n \in (0,1), t_n \in [0,T]$, $|x_n| \leq R$, $|y_n| \leq R$, $|x_n - y_n| \geq \epsilon$ such that

$$F(t_n, (1-\mu_n)x_n + \mu_n y_n) < (1-\mu_n)F(t_n, x_n) + \mu_n F(t_n, y_n) + \frac{1}{n}\mu_n(1-\mu_n). \tag{3.4}$$

Without lost of generality we can suppose that

$$\mu_n \to \mu, \quad t_n \to \tau, \quad x_n \to x \quad \text{and} \quad y_n \to y.$$

Note that $x \neq y$ since $|x - y| \geq \epsilon$.

If $\mu \in (0,1)$, then by (3.5),

$$F(\tau, (1-\mu)x + \mu y) \leq (1-\mu)F(\tau, x) + \mu F(\tau, y)$$

which contradicts the fact that $F(t,x)$ is structly concave in x. Thus $\mu \in \{0,1\}$.

Assume $\mu = 0$. By the Mean Value Theorem, there exists $\xi_n \in (x_n, (1-\mu_n)x_n + \mu_n y_n)$ such that

$$F(t_n, (1-\mu_n)x_n + \mu_n y_n) - F(t_n, x_n) = \mu_n(y_n - x_n)F_x(t_n, \xi_n)$$

and by (3.5),

$$(y_n - x_n)F_x(t_n, \xi_n) < F(t_n, y_n) - F(t_n, x_n) + \frac{1}{n}(1 - \mu_n).$$

Letting $n \to +\infty$ we obtain

$$(y - x)F_x(\tau, x) \leq F(\tau, y) - F(\tau, x).$$

(Note that $\xi_n \to x$ since $\mu_n \to 0$), which contradicts the fact that $F(\tau, x)$ is strictly concave in x.
Analogously, we obtain a contradiction if $\mu = 1$, and the proof is complete.

Let $a \in BC$. As in [3], we define the lower average of a by

$$A_L(a) = \lim_{r \to +\infty} \inf_{t-s \geq r} \frac{1}{t-s} \int_s^t a(\tau) d\tau.$$

Remark 3.4 *Let $a \in BC$ be nonnegative. It is easy to show that the linear operators $L_\pm : BC^1 \to BC$; $L_\pm(x) = x' \pm a_x$; are homeomorphisms onto BC if and only if $A_L(a) > 0$. In this case, for each $b \in BC$ we have,*

$$L_+^{-1}(b) = -\int_t^\infty b(s) \exp\left(-\int_t^s a(\tau)d\tau\right) ds,$$

$$L_-^{-1}(b) = \int_{-\infty}^t b(s) \exp\left(-\int_s^t a(\tau)d\tau\right) ds.$$

Proposition 3.5 *If $a \in BC$ is nonnegative and $A_L(a) > 0$, then there exists $\delta > 0$ such that the equation*

$$y' = a(t)y(1 - y) - \delta \tag{3.5}$$

has two separate solutions.

Proof. Let us define $\mathcal{F} : BC^1 \to BC$ by $\mathcal{F}(y) = ay(1 - y) - y'$. Then, the Frechet derivatives $\mathcal{F}'(0), \mathcal{F}'(1)$ are linear homeomorphisms onto BC and by the Inverse Function Theorem there exists $\delta > 0$ such that (3.5) has bounded solutions v_0, v_1 such that $||v_0|| < \frac{1}{4}$ and $||1 - v_1||_0 < \frac{1}{4}$. It is clear that v_0, v_1 are separate and the proof is complete.

Remark 3.6 *Let v be a bounded solution of (3.5) where $\delta > 0$, $a \in BC$, $a \geq 0$ and $A_L(a) > 0$. Then, $0 < \inf(v) \leq \sup(v) < 1$.*

Proof. By Lemma 2.3 of [2] there exists a sequence (t_n) in $\mathbb{R}$ such that

$$v(t_n) \to \inf(v) \quad \text{and} \quad v'(t_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

On the other hand $\{a(t_n)\}$ is bounded and so we can assume that $a(t_n) \to \alpha$ for some $\alpha \geq 0$. Form this

$$\alpha \inf(v)(1 - \inf(v)) = \delta > 0$$

and hence $\alpha > 0$. Consequently, $\inf(v) > 0$.
Analogously, $\sup(v) < 1$ and the proof is complete.

Theorem 3.7 *Let $F \in C$ and suppose that:*

i) *$F(t,x)$ is locally equicontinuous in x.*

ii) *For each $R, \epsilon > 0$ there exists $b \in BC$ nonnegative satisfying (3.1) such that $A_L(b) > 0$.*
Then there exists $\lambda_0 = \lambda_0(F)$ with the following properties:

a) *If $\lambda < \lambda_0$, then (2.6) has no bounded solutions.*

b) *if $\lambda > \lambda_0$, then (2.6) has exactly two separate solutions.*

c) *If $\lambda = \lambda_0$ and the partial derivative $F_x(t,x)$ is defined and continuous in $\mathbb{R}$, then (2.6) has exactly a separate solution.*

Proof. Let λ_0 be given by Theorem 2.10. Obviously, a) is satisfied and by Corollary 3.2, b) is also satisfied. Thos show c), assume in the contrary that (2.7) has two separate solutions $u_0 < u_1$ and fix $R, \epsilon > 0$ such that $u_1(t) - u_0(t) \geq \epsilon$, $|u_0(t)| \leq R$, $|u_1(t)| \leq R$ for all $t \in \mathbb{R}$. Fix also $b \in BC$ nonnegative satisfying (3.1) such that $A_L(b) > 0$ and define $a = b(u_1 - u_0)$. Since $u_1 - u_0 \in BC_+$, then $a \in BC$ is nonnegative and $A_L(a) > 0$. Thus, by Proposition 3.4 and Remark 3.5, there exists $\delta > 0$ such that (3.5) has separate solutions v_0, v_1 and $0 < v_0 < v_1 < 1$.

Let us define

$$\omega_i = (1 - v_i)u_0 + v_i u_1 = u_0 + v_i(u_1 - u_0)$$

then, using (3.1) we obtain

$$\omega_i' \leq F_i(t, \omega_i) + \lambda_0 - \eta$$

where $\eta = \delta \ \inf(u_1 - u_0)$. Note also that

$$u_i' F_i(t, u_i) + \lambda_0 > F_i(t, u_i) + \lambda_0 - \eta$$

and that $u_0 < \omega_0 < \omega_1 < u_1$. Thus, the equation

$$x' = F(t,x) + \lambda_0 - \eta$$

has bounded solutions u_o^*, u_1^* such that $u_0 \leq u_0^* \leq \omega_0$ and $\omega_1 \leq u_1^* \leq u_1$. This contradicts part a) and the proof is complete.

Remark 3.8 *Theorem 2 of [1] and Theorem 3.7 above agree on the class of all $F \in C$ such that*

1) *$F(t,x)$ is T-periodic in t for some $T > 0$.*

2) *$F(t,x)$ is strictly concave in x.*

3) *The partial derivative $F_x(t,x)$ is defined and continuous in $\mathbb{R} \times \mathbb{R}$.*

4 Periodic Case

In this section we assume that $F(t,x)$ is T-periodic in the time t for some $T > 0$, and we complement the results in [1].

Given $x, \lambda \in \mathbb{R}$ we denote by $u(t,x,\lambda)$ the solution of (2.6) determined by the initial condition $u(0,x,\lambda) = x$. We define

$$\mathcal{D} = \{(x,\lambda) \in \mathbb{R} \times \mathbb{R} : u(\cdot, x, \lambda) \quad \text{is defined in} \quad [0,T]\} \quad \text{and} \quad \pi : \mathcal{D} \to \mathbb{R}$$

by $\pi(x,\lambda) = u(T,x,\lambda)$.

Theorem 4.1 *Let $\lambda_0 = \lambda_0(F)$ be given by Theorem 2.10, then $\pi(x, \lambda_0) - x \leq 0$ if $(x, \lambda_0) \in \mathcal{D}$.*

Proof. Let $\Theta_{F+\lambda_0}, \Gamma_{F+\lambda_0}$ be given by Proposition 2.2. By Remark 2.3, we know that $\Theta_{F+\lambda_0}, \Gamma_{F+\lambda_0}$ are T-periodic. Define $\underline{x} = \Theta_{F+\lambda_0}(0)$ and $\overline{x} = \Gamma_{F+\lambda_0}(0)$, by Remark 2.4 we know that

$$\pi(x, \lambda_0) < x \quad \text{if either} \quad x < \underline{x} \text{ or } x > \overline{x}.$$

Assume now that our result is false, then there exist $y_0 < z_0$ such that

$$\pi(y_0, \lambda_0) = y_0, \quad \pi(x, \lambda_0) > x \quad \text{if} \quad x \in (y_0, z_0), \quad \pi(z_0, \lambda_0) = z_0.$$

Fix $x_* \in (y_0, z_0)$. Since $\pi(x_*, \lambda_0) > x_*$ there exists $\delta > 0$ such that

$$\pi(x_*, \lambda) > x_* \quad \text{if} \quad (\lambda - \lambda_0) < \delta.$$

Without loos of generality, we can assume that $(y_0, \lambda), (z_0, \lambda) \in \mathcal{D}$ if $|\lambda - \lambda_0| < \delta$. On the other'hand, if $\lambda \in (\lambda_0 - \delta, \lambda_0)$ we have $\pi(y_0, \lambda) < \pi(y_0, \lambda_0) = y_0$, and hence, $\pi(x_\lambda, \lambda) = x_\lambda$, for some $x_\lambda \in (y_0, x_*)$. That is, (2.6) has a T-periodic solution if $\lambda \in (\lambda_0 - \delta, \lambda_0)$. This contradicts the definition of λ_0 and the proof is complete.

Corollary 4.2 *Suppose that the partial derivatives $F_x(t, x)$, $F_{xx}(t, x)$ are defined and continuous in $\mathbb{R} \times \mathbb{R}$. If λ_0 is given by Theorem 2.10 and u is a bounded solution of (2.6) then*

$$\int_0^T F_x(t, u(t))dt = 0$$

and

$$\int_0^T F_{xx}(t, u(t)) \exp\left(\int_0^t F_x(s, u(s)ds\right) dt \leq 0.$$

Proof. Let us write $x_0 = u(0)$. By Theorem 4.1 we have $\pi_x(x_0, \lambda_0) = 1$ and $\pi_{xx}(x_0, \lambda_0) \leq 0$, and the proof follows easily.

Corollary 4.3 *Let λ_0 be given by Theorem 2.10 and suppose that (2.7) only has a finite number N of T-periodic solutions, then there exists $\delta > 0$ such that (2.6) has at least $2N$ T-periodic solutions for all $\lambda \in (\lambda_0, \lambda_0 + \delta)$.*

Proof. Let $x_1, \ldots, x_N$ be the fixed points of $\pi(\cdot, \lambda_0)$ and fix and open interval U_i of $\mathbb{R}$ containing x_i such that $U_i \cap U_j = \emptyset$ for $i \neq j$. Since $\pi(x, \lambda) > \pi(x, \lambda_0)$ if (x, λ), $(x, \lambda_0) \in \mathcal{D}$ and $\lambda > \lambda_0$, it is easy to show that hence exists $\delta > 0$ such that $\pi(\cdot, \lambda)$ has two fixed points in U_i for all $\lambda \in (\lambda_0, \lambda_0 + \delta)$, $i = 1, \ldots, N$. So, the proof is complete.

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Departamento de Matemáticas. Facultad de Ciencias. Universidad de Los Andes. Mérida 5101. Venezuela.
e-mail address atineo@ciens.ula.ve