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On a Neumann problem asymptotically linear at $-\infty$, and superlinear at $+\infty$

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Abstract

In this paper we study the existence of two solutions for the problem:

$$(\mathbb{P}) \qquad \begin{cases} -\Delta u = f(x,u) - t + h(x), & x \in \Omega \\ \dfrac{\partial u}{\partial \eta} = 0, & x \in \partial\Omega \end{cases},$$

where: Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$ ($\mathbb{N} \geq 1$), with smooth boundary $\partial\Omega$; $f : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is some continuous function, asymptotically linear at $-\infty$, and superlinear at $+\infty$; t is a real parameter and $h : \overline{\Omega} \longrightarrow \mathbb{R}$ is a function such that $\int_\Omega h = 0$

Using variational methods, we prove the existence of two solutions of $(\mathbb{P})$, for $t < 0$, and $|t|$ large enough.

Key words and phrases: asymptotically linear problem, superlinear problem, variational methods, critical point, linking condition, Palais-Smale condition.

1 Introduction

We denote by $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$ the eigenvalues of $(-\Delta; H^1(\Omega))$, where: $H^1(\Omega)$ is the usual Sobolev Space endowed with the norm:

$$||u||^2 = \int_\Omega |\nabla u|^2 + \int_\Omega u^2.$$

In problem $(\mathbb{P})$, the function f is given by:

$$(f_0) \qquad f(x,s) = -\beta s^- + c(s^+)^p,$$

where $c > 0$, and the constant β belongs to some interval contained in $(\lambda_j, \lambda_{j+1})$, for some $j \geq 1$, which will be specified later.

The problem is subcritical, since p is restricted in the usual way:

$$1 < p < \frac{N+2}{N-2} \quad \text{if } N \geq 3 \ , \quad 1 < p < \infty \quad \text{if} \quad N = 1, 2.$$

Let us fix a $j \geq 1$ and let us denote by $e_1, e_2, e_3, \cdots$ the eigenfunctions associated with the eigenvalues $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots$, such that $\int_\Omega e_i^2 = 1$, for $i = 1, 2, 3, \cdots$.

Let H_1 be the span$[e_1, \cdots, e_j]$.

We define $m = \inf\left\{\int_\Omega v^2 + \int ((e_{j+1}+v)^+)^2 \quad : \quad v \in H_1\right\}$.
It can be proved that $0 < m < 1$.
Our main result is the following theorem.

Theorem 1.1 *Assume hypothesis* (f_0), *with* $\frac{m}{m+1}\lambda_j + \frac{1}{m+1}\lambda_{j+1} < \beta < \lambda_{j+1}$. *Also suppose that* $\int_\Omega h = 0$. *Then* $(\mathbb{P})$ *has, at least two distinct solutions, for* $t < 0$ *and* $|t|$ *large enough.*

Remark 1.1 *Instead of* (f_0) *we can suppose*

$$(f_0)' \qquad f(x,s) = -\beta s^- + c(s^+)^p + W(x,s),$$

provided this last term satisfies some appropiated growth conditions.

Our work was motivated by the analogous Dirichlet problem studied by A.Micheletti and A. Pistoia [5], namely

$$(\mathbb{P}_{\mathrm{Dir}}) \qquad \begin{cases} -\Delta u = f(x,u) + h(x) - t\varphi \quad , & \Omega \\ u = 0 \quad , & \partial\Omega \quad , \end{cases}$$

where $\varphi > 0$ is an eigenfunction associated with the first eigenvalue of $(-\Delta; H_0^1(\Omega))$, and $\int_\Omega h\varphi = 0$.
We also mention that Ruf-Srikanth [6] studied $(\mathbb{P}_{\mathrm{Dir}})$ with $f(x,u) = \lambda u + (u^+)^p$,
where : $\lambda_k < \lambda < \lambda_{k+1}$ and $p > 1$ is as above. They proved that $(\mathbb{P}_{\mathrm{Dir}})$ has at least two solutions, for $t < 0$ and $|t|$ large enough. De Figueiredo [2] obtained a similar result for a larger class of nonlinearities. In these works, a solution is found directly, and the second one follows by using the Generalized Mountain Pass Theorem due to Rabinowitz. The conditions required in [2] in order to apply the Generalized Mountain Pass Theorem are:

$$(f_0'') \qquad f \in C^1 \quad , \quad f_s'(x,s) \geq -\mu > \lambda_k - \lambda \quad ,$$

and all the assumptions which are needed to get the Palais-Smale condition. In [5], A. Micheletti and A. Pistoia considered another class of nonlinearities (which do not satisfy (f_0'')) for which the result remains valid under the weaker assumption $f \in C(\overline{\Omega} \times \mathbb{R}, \mathbb{R})$. They used a slight different variational argument to obtain directly the existence of two distinct critical values for the Euler-Lagrange functional f_t associated with $(\mathbb{P}_{\mathrm{Dir}})$. In their theorem they used the following hypothesis: $f_t \in C^1$, f_t satisfies the Palais-Smale condition and some" linking condition ". We use a similar variational method, but our "linking condition "is simpler and better adapted to the geometry of f_t.

2 " Linking condition " and existence of two critical values

Let H a Hilbert space, which is the topological direct sum of two subespaces H_1 and H_2.

Definition 1 *Let $u_0 \in H$. The function $g : H \longrightarrow \mathbb{R}$, satisfies the " Linking condition "* **(L)** *with respect to u_0, H_1, H_2 ; if there exist $e_0 \in H_2\backslash\{0\}$, ρ_1, ρ_2 such that:*

(L)
$$\rho_1 > 2\rho_2 > 0$$
$$\sup_{u_0+\partial B_1} g < \inf_{u_0+\partial B_2} g$$

where

$$B_1 = \{u = u_1 + te_0 : u_1 \in H_1, \|u\| < \rho_1, t \geq 0\},$$
$$B_2 = \{u_2 \in H_2, \|u\| < \rho_2\}.$$

We shall use the following result:

Theorem 2.1 *Let H be as above, with* $\dim H_1 < +\infty$.
If $g \in C^1$ and satisfies the Palais-Smale condition and the " Linking condition " (**L**), *then there exist two critical values, c_0 and c_1, for g such that:*

$$\inf_{u_0+B_2} g \leq c_1 \leq \sup_{u_0+\partial B_1} g < \inf_{u_0+\partial B_2} g \leq c_0 \leq \sup_{u_0+B_1} g$$

Remark 2.1 *The proof of 2.1 is made using the deformation lemma and Brouwer's degree theory, adapting the ideas of analoguos theorem in [4].*

3 The Palais-Smale condition.

Problem (**IP**) can be written as:

(IP)
$$\begin{cases} -\Delta u = f(x,u) + h(x), & x \in \Omega \\ \dfrac{\partial u}{\partial \eta} = 0 & x \in \partial\Omega, \end{cases}$$

where $\hat{f}(x,s) = -\beta s^- + c(s^+)^p - t$.
It follows from our assumptions that

(*)
$$\lim_{s \to -\infty} [f(x,s) - \beta s] = -t \quad ,$$

and that there exist $s_0 > 0$ and $\theta \in (0, 1/2)$, such that:

$$(**) \qquad 0 < \hat{F}(x,s) \leq \theta s \hat{f}(x,s) \ , \quad \text{for} \quad s \geq s_0 \quad , \quad x \in \overline{\Omega} \ ,$$

where $\hat{F}(x,s) = \int_0^s \hat{f}(x,\tau)d\tau$.

From $(*)$ and $(**)$, it follows that the Palais-Smale condition is satisfied for the functional $f_t : H^1(\Omega) \longrightarrow \mathrm{I\!R}$, given by:

$$f_t(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega \hat{F}(x,u) - \int_\Omega hu.$$

(See Arcoya-Villegas [1], lemma 1.1).

4 Geometry of the functional f_t.

8 In this section we prove that f_t satisfies the condition (**L**). For that matter, we have to establish some technical lemmas. Here, as in the Introduction, H_1 is the span$[e_1, \cdots, e_j]$, and we define H_2 as the span$[e_{j+1}, \cdots]$.

Lemma 4.0 *Let $z \in H_2$ and $s \leq 0$.*
If $\hat{\Omega} = \{x \in \Omega : s + z(x) \leq 0\}$, then

$$\limsup_{s \to -\infty} \ meas \ \ (\Omega \backslash \tilde{\Omega}) = 0 \quad , \quad \text{uniformly for } \ ||z|| \leq const.$$

Remark 4.1 *This lemma can be proved following an idea contained in lemma 3.1 of [5].*

Lemma 4.1 *If $z \in H_2$ and $s \leq 0$, then*

$$\begin{aligned} f_t(s+z) - f_t(s) \quad \geq \quad & c_0^* . \frac{\lambda_{j+1} - \beta}{2\lambda_{j+1}} ||z||^2 - c_1 - (\int_\Omega h^2)^{1/2} . ||z|| \\ & -\omega(meas(\Omega \backslash \hat{\Omega}))||z||^{p+1} \quad , \end{aligned}$$

where c_0^* and c_1 are positive constants, and $\omega : \mathrm{I\!R}^+ \longrightarrow \mathrm{I\!R}$ is such that $\lim_{r \to 0+} \omega(r) = 0$.

Lemma 4.2 *If $s \leq 0$ and $t = s\beta$, then there exists $c_2 > 0$ such that*
$\sup_{v \in H_1} f_t(s+v) \quad \leq \quad f_t(s) + c_2.$

Remark 4.2 *In the proof of lemmas 4.1 and 4.2 we use the variational inequalities:*

$$\int_\Omega \|\nabla v\|^2 \leq \lambda_j \int_\Omega v^2 \quad \textit{for} \quad v \in H_1$$

$$\int_\Omega \|\nabla z\|^2 \geq \lambda_{j+1} \int_\Omega z^2 \quad \textit{for} \quad z \in H_2.$$

8 From Lemmas 4.0 , 4.1 and 4.2, it follows easily:

Lemma 4.3 *With the same hypothesis of Lemmas* 4.1 *and* 4.2, *there exist* $R_1 > 0$ *and* $t_0 < 0$, *such that:*

$$\inf\{f_t(s+z) : z \in H_2, \ \ \|z\| = R_1\} > \sup_{v \in H_1} f_t(s+v)$$

for $s\beta = t \leq t_0$.

Before the next lemma we will define an auxiliary function, as follows.
First, we take ϵ, such that:

(1) $$0 < \epsilon < \beta - \frac{m}{m+1}\lambda_j - \frac{1}{m+1}\lambda_{j+1}.$$

Then, we choose α such that:

(2) $$0 < \frac{\lambda_{j+1} + \epsilon - \beta}{\alpha - \beta} < m.$$

Definition 2 *Let* $Q_0 : H^1(\Omega) \longrightarrow \mathbb{R}$, *given by*

$$Q_0(u) = \int_\Omega |\nabla u|^2 + \epsilon \int_\Omega u^2 - \alpha \int_\Omega (u^+)^2 - \beta \int_\Omega (u^-)^2.$$

From (1) and (2) we can prove that

(3) $Q_0(v + e_{j+1}) < 0$, for all $v \in H_1$.

(3)' Let $M = \{u \in H^1(\Omega) : Q_0'(u).v = 0,$ for all $v \in H_1\}$.
One can prove the following fact:

(4) Given $u_2 \in H_2$, there exists a unique $u_1 \in H_1$ such that $u_2 + u_1 \in M$.

Remark 4.3 *A proof of* (4) *can be made applyng the theorem of Minty* (*See* [3]) *to the function* $P_1 \circ T \circ (-Q_0)'(\cdot + u_2)$, *where* $P_1 : H_1 \oplus H_2 \longrightarrow H_1$ *is the canonical projection,* $T : (H^1(\Omega))^* \longrightarrow H^1(\Omega)$ *is the mapping given by the Riesz's Representation Theorem.*

From (4) *we conclude that there exists a mapping* $\gamma_0 : H_2 \longrightarrow H_1$ *such that*

(5) $$u_2 + \gamma_0(u_2) \in M, \quad \textit{for each} \quad u_2 \in H_2.$$

In lemma (4.4) *below we consider the element* $u^* = e_{j+1} + \gamma_0(e_{j+1})$. *In particular, using* (3),(5) *and* (3)', *we have that:*

(6) $$Q_0(u^*) < 0 \quad \textit{and} \quad Q_0'(u^*).v = 0 \ , \quad \textit{for all} \quad v \in H_1.$$

Lemma 4.4 *Assuming $s \leq 0$ and u^* as above, it follows that:*

$$\lim f_t(s + \sigma u^* + v) = -\infty, \; 8$$

as $\|\sigma u^* + v\| \longrightarrow +\infty$, *where $\sigma \geq 0$ and $v \in H_1$.*

Remark 4.4 *Associated with Q_0, there are the expressions:*

$$\Gamma(u) = \frac{1}{2}\alpha(u^+)^2 + \frac{1}{2}\beta(u^-)^2 \quad \text{and} \quad \gamma(u) = \alpha u^+ - \beta u^-,$$

which satisfy the following inequalities:

$$\text{(7)} \qquad \frac{\min\{\alpha,\beta\}}{2}(u-\tilde{u})^2 \leq \Gamma(u) - \Gamma(\tilde{u}) - \gamma(u)(u-\tilde{u})^2 \leq \frac{\max\{\alpha,\beta\}}{2}(u-\tilde{u})^2$$

for all $u, \tilde{u} \in H^1(\Omega)$.

Proof of Lemma 4.4

Taking into account that $\beta < \alpha$ and using (7), we obtain:

$$\text{(8)} \qquad Q_0(s + \sigma u^* + v) - Q_0(\sigma u^*) \leq \int_\Omega |\nabla v|^2 + \epsilon \int_\Omega v^2 - \beta \int_\Omega v^2 + Q_0'(\sigma u^*).(s+v).$$

Now, in the expression of $f_t(s + \sigma u^* + v)$ we apply (8) and the facts:

$$\int_\Omega |\nabla v|^2 \leq \lambda_j \int_\Omega v^2 \quad , \quad Q_0(\sigma u^*) = \sigma^2 Q_0(u^*) \quad , \quad Q_0'(u^*).(s+v) = 0.$$

So, we arrive at

$$\begin{aligned} f_t(s + \sigma u^* + v) \;\; &\leq \;\; \tfrac{1}{2}\sigma^2 Q_0(u^*) + \frac{\lambda_j + \epsilon - \beta}{2(\lambda_j + \epsilon)} \left[\int_\Omega |\nabla v|^2 + \epsilon \int_\Omega v^2 \right] \\ &+ \;\; \frac{\epsilon}{2} \int_\Omega s^2 + \epsilon \int_\Omega sv - \beta \int_\Omega sv - \sigma \int_\Omega hu^* - \int_\Omega hv \\ &+ \;\; ts|\Omega| + t\sigma \int_\Omega u^* + t \int_\Omega v + \text{const.} \end{aligned}$$

Since $Q_0(u^*) < 0$ and $\lambda_j + \epsilon < \beta$, the Lemma 4.4 follows. ■

Lemma 4.5 *Assuming the same hypothesis as in Lemmas 4.1 and 4.2, it follows that, for t negative and small enough, the functional f_t satisfies condition* (**L**), *with respect to* $u_0 = \dfrac{t}{\beta} = s$, $H_1 = span[e_1, \cdots, e_j]$, $H_2 = span[e_{j+1}, \cdots]$.

Proof.
From Lemma 4.3, there exist $\rho_2 > 0$ and $t_1 < 0$, such that

$$\text{(9)} \qquad \inf\{f_t(s+z) \;:\; z \in H_2 \;,\; \|z\| = \rho_2\} > \sup_{v \in H_1} f_t(s+v) \quad \text{,for } t = s\beta \leq t_1.$$

On the other hand, applyng Lemma 4.4 we can choose $\rho_1 > 2\rho_2$ such that:

(10) $$\sup\{f_t(s+\sigma u^* + v) \ : \ \sigma \geq 0, \ v \in H_1, \ \|\sigma u^* + v\| = \rho_1\} \leq \sup_{v \in H_1} f_t(s+v).$$

Then, from (9) and (10) it follows that:

$$\sup\{f_t(s+\sigma u^* + v) \ : \ \sigma \geq 0, \ v \in H_1, \ \|\sigma u^* + v\| = \rho_1\} < \inf\{f_t(s+z) \ : z \in H_2 \ , \ \|z\| = \rho_2\}.$$

Hence, f_t satisfies condition (**L**) with respect to:

$$u_0 = s = \frac{t}{\beta} \quad , \quad H_1 = \text{span}[e_1, \cdots, e_j] \quad , \quad H_2 = \text{span}[e_{j+1}, \cdots] \ ,$$

taking $e_0 = e_{j+1}$, $B_1 = \{u \in H_1 \oplus \mathbb{R}^+ e_0 \ : \ \|u\| < \rho_1\}$ and $B_2 = \{u \in H_2 \ : \ \|u\| < \rho_2\}$.

Remark 4.5 *. The functional f_t then satisfies the Palais-Smale condition and, for $t < 0$ small enough, also satisfies condition* (**L**)*. On the other hand, $f_t \in C^1(H^1(\Omega), \mathbb{R})$, with $f_t'(u).v = \int_\Omega \nabla u.\nabla v - \int_\Omega f(x,u)v - \int_\Omega hv + t\int_\Omega v$. Hence, the theorem 2.1 can be applied to obtain our theorem 1.1.*

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