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A Classification of spectral measures with the CGS-Property

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Abstract

The spectral measures defined on a σ-algebra $\mathcal{S}$ of sets with the CGS-property in a given Hilbert space H are classified as doubly infinite, simply infinite, finitely infinite and finite ones and a similar classification is given for the ordered spectral decompositions (briefly, OSDs) of H relative to such spectral measures. The main result is that a spectral measure $E(\cdot)$ is of a particular type if and only if the OSDs of H relative to $E(\cdot)$ are also of the same type. Moreover, the multiplicity set M_E of $E(\cdot)$ is described in terms of the measure sequence associated with the given OSD of H relative to $E(\cdot)$. Also is included a result on spatial isomorphism of abelian von Neumann algebras with countably decomposable commutants in terms of the multiplicity functions m_p and m_c of their canonical spectral measures.

1. INTRODUCTION

The problem of determining a complete system of unitary invariants for a self-adjoint or a normal operator on a Hilbert space H goes back to the pioneering work of Hellinger [6] in 1907. The literature on the unitary invariance problem can be classified as follows:

(a) H separable:

In 1932, Stone [16] recast the work of [4] and [6] in the set up of abstract separable Hilbert space H and extended their work to self-adjoint operators T on H. He also introduced two multiplicity functions m_p and m_c with respect to $\sigma_p(T)$ and $\sigma_c(T)$ and obtained two unitary invariance theorems (Theorems 7.7 and 7.8 of [16]), the latter in terms of m_p and m_c. Later, in 1963 Dunford and Schwartz [3] studied the problem for self-adjoint and bounded normal opearotrs T on H and the equivalence of two ordered spectral representations relative to T is a complete system of unitary invariants for T.

(b) H arbitrary, self-adjoint or normal operators on H:

In 1939, Wecken [17] studied the problem for self-ajoint operators, while in 1946 Yosida [18] studied for normal operators in terms of the von Neumann algebra generated by the range of the

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resolution of identity of the operator. Also Plesner and Rohlin [13] studied the problem for self-adjoint operators in 1946 in terms of the multiplicity functions defined on generalized Hellinger types.

(c) H arbitrary and $E(\cdot)$- a spectral measure:

Halmos [5] studied in 1951 the unitary invariance problem of an arbitrary spectral measure defined on a σ-algebra of sets Σ, and the multiplicity defined on the set of all finite (positive) measures determines the spectral measure upto unitary equivalence.In 1974 Brown extended the work of Wecken [17] to spectral measures and invoking a result of Schwartz [14], obtained his principal result (Theorem 8.4" of [1]) on unitary invariance of spectral measures $E(\cdot)$ on H.

(d) H arbitrary and operator algebras:

In 1951 Segal [15] studied the problem of unitary equivalence for abelian W^*-algebras, while Kelley [7] studied in 1952 for abelian von Neumann algebras.

Though it is clear that all these results are mutually related, as far as we know, prior to the publication of our papers, there has not been any work published in the literature which obtains all the principal results of the above mentioned authors. So, we started working upon a unified approach to deduce or generalize all the important results known on the problem of unitary invariance.

Before proceeding further, let us briefly comment on the results obtained in our papers [9 ,10,11, 12]. We use the results of Halmos [5] as basis and generalize the results of Dunford and Schwartz [3] to spectral measures with the CGS-property in [9], thereby extending the unitary invariance theorem given for self-adjoint and bounded normal operators in [3] to normal operators on separable Hilbert spaces. This also generalizes the Hellinger theory presented in [16] to such spectral measures. Also we extend in [9] the notions of the multiplicity functions m_p and m_c given in Stone [16] to spectral measures with the CGS-property on the Borel sets of a Hausdorff space.

In [10] we introduce the concepts of spectral representations such as OTSRs, BOTSRs and COBOTSRs, and obtain a few new complete sets of unitary invariants for arbitrary spectral measures in terms of the equivalence of their spectral representations. In [10] is also given an alternative proof of the unitary invariance theorem in §68 of Halmos [5] and moreover, certain results of Plesner and Rohlin [13] are generalized to spectral measures which have the generalized CGS-property or are arbitrary. Also is deduced in [10] the unitary invariance theorem (Theorem 8.4") of Brown [1], with a clear description of the cardinals mentioned in the theorem.

In [11] we use some rudiments of von Neumann algebras, and from the results of Halmos [5] we deduce the type I_n-decomposition theorem of a type I von Neumann algebra. We prove a new result in terms of the multiplicity and uniform multiplicity of projections given by Halmos [5] from which we deduce a generalization of the principal result of Yosida [18] to spectral measures. Also we deduce some of the classical results on abelian von Neumann algebras, which were first proved by Segal [15]. Finally, we show that a projection P in the von Neumann algebra W generated by the range of a spectral measure $E(\cdot)$ has uniform multiplicity n in the sense of Halmos [5] if and only if the W^*-algebra WP has uniform multiplicity n in the sense of Segal [15] and then deduce the decomposition theorem and the unitary invariance theorem of Segal [15].

Using the results of [10], we deduce in [12] the principal unitary invariance theorem of Kelley [7] and describe the Kelley multiplicity function ϕ on certain dense subset of the maximal ideal space of the abelian von Neuamann algebra in question in terms of the uniform multiplicity of projections (in the sense of Halmos [5]). Also *using Theorem 3 of the present paper*, we describe in [12] the function ϕ in terms of the multiplicity functions m_p and m_c relative to the canonical spectral measure of the abelian von Neumann algebra $\mathcal{A}$, when its commutant is countably decomposable.

In the present paper we classify spectral measures defined on a σ-algebra $\mathcal{S}$ with the CGS-property in a given Hilbert space H and also the ordered spectral decompositions (briefly, OSDs) of H relative to such spectral measures as doubly infinite, simply infinite, finitely infinite and finite ones and show that a spectral measure $E(\cdot)$ is of a particular type if and only if all the OSDs of H relative to $E(\cdot)$ are also of the same type. Moreover, the multiplicity set M_E of $E(\cdot)$ is described in terms of the measure sequence associated with the given OSD of H relative to $E(\cdot)$. We also extend the results proved for hermitian operators (resp. spectral measures) on p.143 of [13] (resp. pp. 155-156 of [1]) on separable Hilbert spaces to spectral measures with the CGS-property in H. Finally, given an abelian von Neumann algebra $\mathcal{A}$ with its commutant countably decomposable, we characterize it upto unitary equivalence in terms of the multiplicity functions m_p and m_c of the canonical spectral measure of $\mathcal{A}$. The last result is only an analogue of Theorem 7.8 of Stone [16], but is not its generalization.

2. PRELIMINARIES

In this section we fix notation and terminolgy and also give some definitions and results from [8,9,10] for the convenience of the reader. For other concepts and results used in the body of the paper, the reader is referred to appropriate bibliography.

H, H_1 and H_2 denote (complex) Hilbert spaces of arbitrary dimension (> 0). The closed subspace spanned by a subset $\mathcal{X}$ of a Hilbert space is denoted by $[\mathcal{X}]$. $\bigoplus M_i$ is the orthogonal direct sum of a family of mutually orthogonal closed subspaces $\{M_i\}$ of a given Hilbert space or of Hilbert spaces $\{M_i\}$.

If P is a projection in a von Neumann algebra W on H, then C_P denotes the central support of P. For $x \in H$, $[Wx] = [Ax : A \in W]$ and, sometimes, it also denotes the orthogonal projection with range $[Wx]$. We follow Dixmier [2] for the rest of termoinology and notation in von Neumann algebras.

Let $\mathcal{S}$ be a σ-algebra of subsets of a non empty set Ω. Let $E(\cdot)$ be a spectral measure on $\mathcal{S}$ with values in projections of H. For $x \in H$, $\rho_E(x)$ denotes the measure $||E(\cdot)x||^2$ on $\mathcal{S}$. Let $\Sigma(\mathcal{S})$

be the set of all finite (positive) measures on $\mathcal{S}$. For $\mu_1, \mu_2 \in \Sigma(\mathcal{S})$, we write $\mu_1 \equiv \mu_2$ if $\mu_1 \ll \mu_2$ and $\mu_2 \ll \mu_1$. Clearly, $\equiv$ is an equivalence relation on $\Sigma(\mathcal{S})$. In the sequel, $E(\cdot)$, $E_1(\cdot)$, $E_2(\cdot)$ will denote spectral measures on $\mathcal{S}$ with values in projections of H, H_1 and H_2, respectively. W will denote the von Neumann algebra generated by the range of $E(\cdot)$.

For $\mu \in \Sigma(\mathcal{S})$, the projection $C_E(\mu)$ is defined as the orthogonal projection on the closed subspace $\{x \in H : \rho_E(x) \ll \mu\}$ and it follows from [5] that $C_E(\mu) \in W$. The multiplicity $u_E(\mu)$ of $\mu \in \Sigma(\mathcal{S})$ relative to $E(\cdot)$ is defined by

$$u_E(\mu) = \min\{\text{H-multiplicity of } C_E(\nu) : 0 \neq \nu \ll \mu, \ \ \nu \in \Sigma(\mathcal{S})\}$$

if $\mu \neq 0$ and $u_E(0) = 0$, where the H-multiplicity of $C_E(\nu)$ is the multiplicity of $C_E(\nu)$ relative to $E(\cdot)$ in the sense of Halmos [5]. $\mu \in \Sigma(\mathcal{S})$ is said to have *uniform multiplicity* $u_E(\mu)$ relative to $E(\cdot)$ if $u_E(\nu) = u_E(\mu)$ for $0 \neq \nu \ll \mu, \ \ \nu \in \Sigma(\mathcal{S})$.

For $x \in H$, let $Z_E(x) = [E(\sigma)x : \sigma \in \mathcal{S}]$. Since W is the von Neumann algebra generated by the range of $E(\cdot)$, it follows that $Z_E(x) = [Wx] \in W'$, where W' is the commutant of W.

A spectral measure $E(\cdot)$ on $\mathcal{S}$ is said to have the *CGS – property* in H if there exists a countable set $\mathcal{X}$ in H such that $[E\sigma)x : x \in \mathcal{X}, \sigma \in \mathcal{S}] = H$. Then it is known from [9] that $E(\cdot)$ has the CGS-property in H if and only if H admits an ordered spectral decomposition (briefly, OSD) relative to $E(\cdot)$, where $H = \bigoplus_{j=1}^{N} Z_E(x_j)$ is called an OSD of H relative to $E(\cdot)$ if each $x_j \neq 0$, $N \in \mathbb{N} \cup \{\infty\}$ (N is called the *OSD multiplicity* of $E(\cdot)$ and we denote it by $\aleph_0$ when it is infinite) and $\rho_E(x_1) \gg \rho_E(x_2) \gg ...$ (which is called the *measure sequence* of the OSD). If $H_i = \bigoplus_{j=1}^{N_i} Z_{E_i}(x_j^{(i)})$ are OSDs of H_i relative to $E_i(\cdot)$ for $i = 1, 2$, then they are said to be equivalent if $N_1 = N_2$ and $\rho_{E_1} x_{j(1)} \equiv \rho_{E_2} x_{j(2)})$ for all j. Then $E_1(\cdot)$ and $E_2(\cdot)$ are unitarily equivalent if and only if any two OSDs of H_1 and H_2 relative to $E_1(\cdot)$ and $E_2(\cdot)$ are equivalent.

Notation 1. Let $W' = \sum \bigoplus_{n \in J} W' Q_n$ be the type I_n-direct sum decomposition of the commutant W' of W so that $W'Q_n$ is of type I_n, where the n are non zero cardinals not greater than the dimension of H. Then $\{Q_n\}_{n \in J}$ will denote these central projections in the type I_n-direct sum decomposition of W' and by M_E we shall denote $\{n : n \in J\}$. M_E is called the *multiplicity set* of $E(\cdot)$.

Notation 2. Let P be a projection in W. Then its multiplicity (resp. uniform multiplicity) in the sense of Halmos [5, pp.100-101] is referred to as its H-multiplicity (resp. UH-multiplicity) relative to $E(\cdot)$.

As shown in [8], a projection P' in W' is abelian if and only if it is a row in the sense of Halmos [5] and the column generated by a projection in W' is the same as its central support. Thus Theorem 64.4 of [5] can be reformulated as follows:

PROPOSITION 1. *A non zero projection F in the von Neumann algebra W generated by the range of $E(\cdot)$ has UH-multiplicity n relative to $E(\cdot)$ if and only if there exists an orthogonal family $\{E'_\alpha\}_{\alpha \in J}$ of abelian projections in W' such that card.$J = n$, $C_{E'_\alpha} = F$ for each $\alpha \in J$ and*

$\sum_{\alpha\in J} E'_\alpha = F$; *in other words, if and only if $W'F$ is of type I_n or, equivalently, if and only if* $0 \neq F \leq Q_n$.

3. SOME LEMMAS

LEMMA 1. *Let P be a countably decomposable non zero projection in W. Then P has UH-multiplicity $N \leq \aleph_0$ if and only if there exists an OSD $PH = \bigoplus_1^N Z_E(x_i)$ of PH with $\rho_E(x_1) \equiv \rho_E(x_2) \equiv \ldots$. Then $C_E(\rho_E(x_i)) = P$ for all i.*

Proof. Suppose P has UH-multiplicity $N \leq \aleph_0$. Then by Proposition 1 there exists an orthogonal family $\{P'_j\}_{j\in J}$ of abelian projections in W' such that card.$J = N$, $C_{P'_j} = P$ for each $j \in J$ and $P = \sum_{j\in J} P'_j$. Let $J = \{1, 2, ..., N\}$ if $N \in \mathbb{N}$ and let $J = \{1, 2, ...\}$ if $N = \aleph_0$. Since P is countably decomposable in W and P'_j has its central support P, by Theorem 58.3 of Halmos [5] there exists a vector $x_j \in P'_j H$ such that $P = C_{P'_j} = C_{[Wx_j]}$. As $[Wx_j] = [WP'_j x_j] = P'_j[Wx_j]$, we have $[Wx_j] \leq P'_j$. Consequently, as P'_j is abelian, by the discussion on p.123 of [2] we have $[Wx_j] = C_{[Wx_j]} P'_j = PP'_j = P'_j$. Thus there exists $x_j \in P'_j H$ such that $P'_j = [Wx_j]$, $j \in J$. Therefore, $PH = \bigoplus_{j\in J}[Wx_j] = \bigoplus_{j\in J} Z_E(x_j)$. Moreover, by Theorem 66.2 of [5], $C_E(\rho_E(x_j)) = C_{[Wx_j]} = C_{P'_j} = P$ for all j. Consequently, by Theorem 65.2 of [5], $\rho_E(x_j) \equiv \rho_E(x_{j'})$ for all $j, j' \in J$. Hence the condition is necessary.

Conversely, if such an OSD $PH = \bigoplus_1^N Z_E(x_j)$ exists, then clearly $N \leq \aleph_0$. As $\rho_E(x_1) \equiv \rho_E(x_2) \equiv \ldots$, by Theorem 66.2 of [5] it follows that $C_{[Wx_1]} = C_{[Wx_2]} = \ldots = Q$ (say). Then $P = \sum_1^N [Wx_j] \leq Q$. On the other hand, as $P \in W$, we also have $[Wx_j] \leq C_{[Wx_j]} \leq P$ for all j. Thus $Q = P$. Since $[Wx_j] = Z_E(x_j)$ is abelian in W' by Theorem 60.2 of [5], from Proposition 1 it follows that P has UH-multiplicity N.

This completes the proof of the lemma.

LEMMA 2. *Suppose $E(\cdot)$ has the CGS-property in H. Then $Q_n = 0$ for $n > \aleph_0$. Let $M_E \cap \mathbb{N} = \{n_p\}_{p=1}^k$, where $k \in \mathbb{N} \cup \{\infty\}$. Then:*

(i) There exist vectors $x_{n_p}^{(j)}$, with $||x_{n_p}^{(j)}|| = 1$ for $j = 1, 2, ..., n_p$, in $Q_{n_p}H$ such that

$$Q_{n_p}H = \bigoplus_{j=1}^{n_p} Z_E(x_{n_p}^{(j)}) \text{ and } \rho_E(x_{n_p}^{(1)}) \equiv \rho_E(x_{n_p}^{(2)}) \equiv \ldots \equiv \rho_E(x_{n_p}^{(n_p)}) \quad (1)$$

for $p = 1, 2, ..., k$ (resp. for $p = 1, 2, ...$) if k is finite (resp. if $k = \infty$). Then $C_E(\rho_E(x_{n_p}^{(j)})) = Q_{n_p}$ for $j = 1, 2, ..., n_p$ and for $p = 1, 2, ..., k$ (resp. for $p = 1, 2, ...$) if k is finite (resp. if $k = \infty$).

(ii) If $\aleph_0 \in M_E$, then there exist vectors $x_{\aleph_0}^{(j)}$, with $||x_{\aleph_0}^{(j)}|| = 1$ for $j \in \mathbb{N}$, in $Q_{\aleph_0}H$ such that

$$Q_{\aleph_0}H = \bigoplus_1^{\infty} Z_E(x_{\aleph_0}^{(j)}) \text{ and } \rho_E(x_{\aleph_0}^{(1)}) \equiv \rho_E(x_{\aleph_0}^{(2)}) \equiv \ldots \quad (2)$$

Then $C_E(\rho_E(x^{(j)}_{\aleph_0})) = Q_{\aleph_0}$, *for* $j \in \mathbb{N}$.

(iii) Let

$$x_j = \sum_{p=1}^{k'} \frac{1}{n_p} x^{(j)}_{n_p} + x^{(j)}_{\aleph_0},\ j \in \mathbb{N}$$

where $k' = \infty$ *if* M_E *is an infinite set,* $k' = k$ *if* $M_E \cap \mathbb{N} = \{n_1 < n_2 < \ldots < n_k\}$, $x^{(j)}_{n_p} = 0$ *for* $j > n_p$, $j \in \mathbb{N}$ *and* $x^{(j)}_{\aleph_0}$ *is omitted if* $\aleph_0 \notin M_E$. *Then*

$$H = \bigoplus_{1}^{N} Z_E(x_j)$$

is an OSD of H *relative to* $E(\cdot)$ *and the OSD-multiplicity* $N = \aleph_0$ *if* M_E *is an infinite set or if* $\aleph_0 \in M_E$ *and* $N = n_k$ *if* $M_E = \{n_1 < n_2 < \ldots < n_k\}$.

(iv) If $M_E \cap \mathbb{N} = \emptyset$, *then* $M_E = \{\aleph_0\}$, $Q_{\aleph_0} = I$ *and (2) gives an OSD of* H *relative to* $E(\cdot)$.

Proof. Since $E(\cdot)$ has the CGS-property in H, there exists a countable set $\mathcal{X}$ in H such that $[E(\sigma)x : x \in \mathcal{X}, \sigma \in \mathcal{S}] = H$. Thus $[Wx : x \in \mathcal{X}] = H$ and hence W' is countably decomposable. If $n \in M_E$ and $n > \aleph_0$, then $Q_n = 0$. For, otherwise, by Proposition 1 there would exist an orthogonal family of abelian projections $\{E'_\alpha\}_{\alpha \in J}$ in W' such that card.$J = n$, $C_{E'_\alpha} = Q_n$ for each $\alpha \in J$ and $Q_n = \sum_{\alpha \in J} E'_\alpha$. This contradicts the hypothesis that W' is countably decomposable. Thus, if $n \in M_E$, then $n \leq \aleph_0$. Let $M_E \cap \mathbb{N} = \{n_j\}_1^k$, $k \in \mathbb{N} \cup \{\infty\}$.

(i) and (ii) are now immediate from the fact that Q_n, $n \in M_E$, are countably decomposable in W with $n \leq \aleph_0$ and from Lemma 1.

(iii) Suppose $M_E = \{n_p\}_{p=1}^{\infty} \cup \{\aleph_0\}$. With $x^{(j)}_{n_p}$ and $x^{(j)}_{\aleph_0}$ as in (1) and (2) of the lemma, let us define $x^{(j)}_{n_p} = 0$ for $j > n_p$, $j \in \mathbb{N}$. Let

$$x_j = \sum_{p=1}^{\infty} \frac{1}{n_p} x^{(j)}_{n_p} + x^{(j)}_{\aleph_0},\ j \in \mathbb{N}.$$

Since $\aleph_0 \in M_E$ or since M_E is infinite, $x_j \neq 0$ for all $j \in \mathbb{N}$. Foı $\sigma, \delta \in \mathcal{S}$ and $j \neq j'$, we have

$$(E(\sigma)x_j, E(\delta)x_{j'}) = \sum_{p=1}^{\infty} \frac{1}{n_p^2}(x^{(j)}_{n_p}, E(\sigma \cap \delta)x^{(j')}_{n_p}) + (x^{(j)}_{\aleph_0}, E(\sigma \cap \delta)x^{(j')}_{\aleph_0}) = 0$$

since $Q_n Q_{n'} = 0$ for $n, n' \in M_E$ with $n \neq n'$ and $Z_E(x^{(j)}_n) \perp Z_E(x^{(j')}_n)$ for $n \in M_E$. Consequently, $\{Z_E(x_j)\}_{j=1}^{\infty}$ is an orthogonal family of non zero subspaces of H.

We shall show that $\rho_E(x_j) \gg \rho_E(x_{j+1})$. Choose p_0 such that $n_{p_0} < j \leq n_{p_0+1}$, where we take $n_0 = 0$. Then $x^{(j)}_{n_p} = x^{(j+1)}_{n_p} = 0$ for $p = 1, 2, \ldots, p_0$. Thus

$$x_j = \sum_{p=p_0+1}^{\infty} \frac{1}{n_p} x^{(j)}_{n_p} + x^{(j)}_{\aleph_0}$$

and

$$x_{j+1} = \sum_{p=p_0+1}^{\infty} \frac{1}{n_p} x_{n_p}^{(j+1)} + x_{\aleph_0}^{(j+1)}$$

where $x_{n_{p_0+1}}^{(j+1)} = 0$ if $n_{p_0+1} < j+1$, which is the case when $j = n_{p_0+1}$. Suppose $\rho_E(x_j)(\sigma) = 0$. Then

$$||E(\sigma)x_j||^2 = \sum_{p=p_0+1}^{\infty} \frac{1}{n_p^2} ||E(\sigma)x_{n_p}^{(j)}||^2 + ||E(\sigma)x_{\aleph_0}^{(j)}||^2 = 0$$

and hence $\rho_E(x_{n_p}^{(j)})(\sigma) = 0$ for $p \geq p_0 + 1$ and $\rho_E(x_{\aleph_0}^{(j)})(\sigma) = 0$. Note that in (1) we have $\rho_E(x_{n_p}^{(j)}) \equiv \rho_E(x_{n_p}^{(j+1)})$ if $j+1 \leq n_p$ and by definition, $\rho_E(x_{n_p}^{(j)}) \gg \rho_E(x_{n_p}^{(j+1)}) = 0$ if $j+1 > n_p$. Moreover, in (2) we have $\rho_E(x_{\aleph_0}^{(j)}) \equiv \rho_E(x_{\aleph_0}^{(j+1)})$ for all $j \in \mathbb{N}$. Therefore, we conclude that $\rho_E(x_{j+1})(\sigma) = 0$. Thus we have

$$\rho_E(x_1) \gg \rho_E(x_2) \gg \ldots.$$

Finally we assert that $H = \bigoplus_1^\infty Z_E(x_j)$. For, otherwise, let $\bigoplus_1^\infty Z_E(x_j) = K \neq H$. Let $y \in H \ominus K$, $y \neq 0$. Then there would exist n_{p_0} such that $y_{n_{p_0}} = Q_{n_{p_0}} y \neq 0$ or $Q_{\aleph_0} y \neq 0$ since $\sum_{n \in M_E} Q_n = I$. We shall show that this is impossible. Suppose $y_{n_{p_0}} = Q_{n_{p_0}} \neq 0$.

As $Q_{n_{p_0}} Z_E(x_j) = Q_{n_{p_0}}[Wx_j] \subset [Wx_j] = Z_E(x_j)$, it follows that $Q_{n_{p_0}} y \perp K$. Therefore,

$$0 = (y_{n_{p_0}}, E(\sigma)x_j) = (Q_{n_{p_0}} y_{n_{p_0}}, E(\sigma)x_j) = (y_{n_{p_0}}, Q_{n_{p_0}} E(\sigma)x_j) = (y_{n_{p_0}}, E(\sigma)\frac{1}{n_{p_0}} x_{n_{p_0}}^{(j)})$$

for $j = 1, 2, ..., p_0$ and for $\sigma \in \mathcal{S}$. Hence $y_{n_{p_0}} \perp Z_E(x_{n_{p_0}}^{(j)})$ for $j = 1, 2, ..., n_{p_0}$. Consequently, $y_{n_{p_0}} \perp \bigoplus_{j=1}^{n_{p_0}} Z_E(x_{n_{p_0}}^{(j)}) = Q_{n_{p_0}} H$ so that $y_{n_{p_0}} = 0$. This contradiction proves that $Q_{n_p} y = 0$ for all $p \in \mathbb{N}$. Similarly, $Q_{\aleph_0} y = 0$ and hence $y = 0$. Thus

$$H = \bigoplus_1^\infty Z_E(x_j)$$

is an OSD of H relative to $E(\cdot)$. Consequently, the OSD-multiplicity of $E(\cdot)$ is $\aleph_0$.

When $M_E \cap \mathbb{N}$ is an infinite set with $\aleph_0 \notin M_E$, then in the above definition of the vectors x_j we have to suppress the term $x_{\aleph_0}^{(j)}$ and the rest of the argument remains the same and shows that

$$H = \bigoplus_1^\infty Z_E(x_j)$$

is an OSD of H relative to $E(\cdot)$. Hence the OSD-multiplicity of $E(\cdot)$ is $\aleph_0$. When $M_E = \{n_1 < n_2 < \ldots < n_k\} \cup \{\aleph_0\}$, the argument is similar to the case discussed above with obivious modifications and again the OSD-multiplicity of $E(\cdot)$ is $\aleph_0$. Finally, when $M_E = \{n_1 < n_2 < \ldots < n_k\}$, it can similarly be shown that

$$H = \bigoplus_1^{n_k} Z_E(x_j)$$

is an OSD of H relative to $E(\cdot)$ and hence the OSD-multiplicity of $E(\cdot)$ is n_k.

(iv) Since $H \neq \{0\}$, $M_E \neq \emptyset$. Moreover, $Q_n = 0$ for $n > \aleph_0$ as $E(\cdot)$ has CGS-property and by hypothesis, $Q_n = 0$ for $n \in \mathbb{N}$. Therefore, $M_E = \{\aleph_0\}$. Then by Proposition 1 we have $Q_{\aleph_0} = I$. Thus (2) gives an OSD of H relative to $E(\cdot)$.

This completes the proof of the lemma.

Remark 1. The above lemma provides an alternate proof of Theorem 3.7 of [9].

COROLLARY 1. *If $E(\cdot)$ has the CGS-property in H, then its OSD-multiplicity is $\aleph_0$ if $\aleph_0 \in M_E$ or if M_E is an infinite set.*

4. A CLASSIFICATION OF SPECTRAL MEASURES WITH THE CGS-PROPERTY

We introduce four types of OSDs of H relative to spectral measures on $\mathcal{S}$ with the CGS-property in H and call them doubly infinite, simply infinite, finitely infinite and finite OSDs, respectively. Similarly, considering the multiplicity set M_E of the spectral measures $E(\cdot)$ on $\mathcal{S}$ with the CGS-property in H, we clasify them into four types and call them doubly infinite, simply infinite, finitely infinite and finite spectral measures, respectively. The purpose of this section is to show that a spectral measure $E(\cdot)$ on $\mathcal{S}$ with the CGS-property in H is doubly infinite (resp. simply infinite, finitely infinite, finite) if and only if the same is true for any OSD of H relative to $E(\cdot)$. Moreover, given an OSD $H = \bigoplus_{i=1}^{N} Z_E(x_i)$ relative to a spectral measure $E(\cdot)$, we describe M_E and $\{Q_n\}_{n\in M_E}$ in terms of $\{\rho_E(x_i)\}_{i=1}^{N}$.

Notation 3. In the sequel, given a spectral measure $E(\cdot)$ on $\mathcal{S}$ with the CGS-property in H, $\{x_j\}_j$ will denote the set of vectors defined in Lemma 2(iii). For $n \in M_E$, μ_{Q_n} denotes the measure in $\Sigma(\mathcal{S})$ for which $C_E(\mu_{Q_n}) = Q_n$. (Note that such a measure μ_{Q_n} exists by Lemma 1 and is unique upto equivalence by Theorem 65.2 of [5].)

DEFINITION 1. Let $H = \bigoplus_1^\infty Z_E(w_i)$ be an OSD of H relative to $E(\cdot)$. Then we say that the OSD is

(i) *doubly infinite* if there exists an infinite subsequence $\{n_k\}_1^\infty$ of $\mathbb{N}$ such that

$$\rho_E(w_{n_k}) \underset{\not\equiv}{\gg} \rho_E(w_{n_{k+1}}) \qquad (3)$$

and if there exists $\nu \in \Sigma(\mathcal{S})$ with $C_E(\nu) \neq 0$ such that

$$\rho_E(w_j) \underset{\not\equiv}{\gg} \nu \quad \text{for } j \in \mathbb{N}; \quad (4)$$

(ii) *simply infinite* if there exists an infinite subsequence $\{n_k\}_1^\infty$ of $\mathbb{N}$ such that (3) holds and if there does not exist any $\nu \in \Sigma(\mathcal{S})$ with $C_E(\nu) \neq 0$ such that (4) holds for all $j \in \mathbb{N}$;

(iii) *finitely infinite* if there exists $n_0 \in \mathbb{N}$ such that

$$\rho_E(w_n) \equiv \rho_E(w_{n_0}) \qquad (5)$$

for all $n \geq n_0$.

An OSD $H = \bigoplus_1^N Z_E(w_i)$ of H relative to $E(\cdot)$ is said to be *finite* if $N \in \mathbb{N}$.

Given an OSD of H relative to $E(\cdot)$, obviously it belongs to one and only one of the above types. Moreover, in virtue of Theorem 3.11 of [9], all the OSDs of H relative to a given spectral measure $E(\cdot)$ belong to the same type.

DEFINITION 2. If $E(\cdot)$ has the CGS-property in H, then $E(\cdot)$ is said to be a *doubly infinite* (resp. *simply infinite, finitely infinite, finite*) spectral measure if its multiplicity set M_E is an infinite set with $\aleph_0 \in M_E$ (resp. an infinite set with $\aleph_0 \notin M_E$, a finite set with $\aleph_0 \in M_E$, a finite set of natural numbers).

Given a spectral measure $E(\cdot)$ on $\mathcal{S}$ with the CGS-property in H, clearly $E(\cdot)$ belongs to one and only one of the above types.

THEOREM 1. *For the spectral measure $E(\cdot)$ with the CGS-property in H the following assertions hold:*

(i) $\rho_E(x_j) \equiv \bigvee\{\mu_{Q_n} : n \in M_E,\ n \geq j\}$ *if* $x_j \neq 0$.

(ii) $C_E(\rho_E(x_1)) = I$.

(iii) $C_E(\rho_E(x_j)) = I - \sum_{n \in M_E, n<j} Q_n$ *if* $x_j \neq 0$.

Consequently, if $H = \bigoplus_1^N Z_E(w_i)$, $N \in \mathbb{N} \cup \{\infty\}$, *is an OSD of H relative to $E(\cdot)$, then (i)- (iii) hold for w_j and w_1 in stead of x_j and x_1.*

Proof. Suppose $M_E = \{n_k\}_1^\infty \cup \{\aleph_0\}$. The other cases of M_E can similarly be dealt with. By the definition of x_j in Lemma 2(iii) we have

$$x_j = \sum_1^\infty \frac{1}{n_p} x_{n_p}^{(j)} + x_{\aleph_0}^{(j)}, \quad j \in \mathbb{N}$$

and each $x_j \neq 0$.

By Lemma 2(i), $Q_{n_p}H = \bigoplus_1^{n_p} Z_E(x_{n_p}^{(j)})$ is an OSD with $C_E(\rho_E(x_{n_p}^{(1)})) = C_E(\rho_E(x_{n_p}^{(2)})) = \ldots = C_E(\rho_E(x_{n_p}^{(n_p)})) = Q_{n_p}$. Since $C_E(\mu_{Q_{n_p}}) = Q_{n_p}$, by Theorem 65.2 of [5] we have $\rho_E(x_{n_p}^{(j)}) \equiv \mu_{Q_{n_p}}$ for $1 \leq j \leq n_p$. Similarly, $\rho_E(x_{\aleph_0}^{(j)}) \equiv \mu_{Q_{\aleph_0}}$, for $j \in \mathbb{N}$.

As $Q_n Q_m = 0$ for $n \neq m$, $n, m \in M_E$, by Theorem 65.1 of [5] it follows that $\{\rho_E(x_{n_p}^{(j)})\}_{p=1}^\infty \cup \{\rho_E(x_{\aleph_0}^{(j)})\}$ is an orthogonal family of measures in $\Sigma(\mathcal{S})$ for each $j \in \mathbb{N}$. Besides,

$$\rho_E(x_j) = \sum_1^\infty \frac{1}{n_p^2} \rho_E(x_{n_p}^{(j)}) + \rho_E(x_{\aleph_0}^{(j)}), \quad j \in \mathbb{N}.$$

Let $n_0 = 0$. If $n_{p_0} < j \leq n_{p_0+1}$, $x_{n_p}^{(j)} = 0$ for $p = 1, 2, \ldots, n_{p_0}$ and hence

$$\rho_E(x_j) \equiv \sum_{p=p_0+1}^\infty \frac{1}{n_p^2} \rho_E(x_{n_p}^{(j)}) + \rho_E(x_{\aleph_0}^{(j)}), \text{ for } n_{p_0} < j \leq n_{p_0+1}.$$

Consequently, by the discussion on p.79 of [5], we have

$$\begin{aligned}\rho_E(x_j) &\equiv (\bigvee_{p=p_0+1}^{\infty} \rho_E(x_{n_p}^{(j)})) \bigvee \rho_E(x_{\aleph_0}^{(j)}) \\ &\equiv \bigvee\{\mu_{Q_n} : n \in M_E, n \geq j\} \qquad (6)\end{aligned}$$

Thus (i) holds.

By Theorem 66.5 of [5] and by (6) we have

$$C_E(\rho_E(x_1)) = \sum_{n\in M_E} C_E(\mu_{Q_n}) = \sum_{n\in M_E} Q_n = I;$$

and

$$C_E(\rho_E(x_j)) = \sum_{n\in M_E, n\geq j} Q_n = I - \sum_{n\in M_E, n<j} Q_n.$$

Thus (ii) and (iii) hold.

The last part follows from the previous parts by Theorem 3.11 of [9].

This completes the proof of the theorem.

THEOREM 2. *For a spectral measure $E(\cdot)$ with the CGS-property in H the following assertions hold:*

(i) If $M_E = \{n_1 < n_2 < ...\} \cup \{\aleph_0\}$, then

(a) $\rho_E(x_{n_i}) \underset{\not\equiv}{\gg} \rho_E(x_j) \equiv \rho_E(x_{n_{i+1}})$ for $n_i < j \leq n_{i+1}, i = 0, 1, 2, ...,$ where $n_0 = 0$ and the term corresponding to n_0 is omitted; and

(b) there exists $\nu \in \Sigma(S)$ such that $C_E(\nu) \neq 0$ and $\rho_E(x_j) \underset{\not\equiv}{\gg} \nu$, for all $j \in \mathbb{N}$.

(ii) If $M_E = \{n_1 < n_2 < ...\}$, then (i)(a) holds and there does not exist any $\nu \in \Sigma(S)$ with $C_E(\nu) \neq 0$ such that

$$\rho_E(x_j) \underset{\not\equiv}{\gg} \nu \text{ for all } j \in \mathbb{N}.$$

(iii) If $M_E = \{n_1 < n_2 < ... < n_k\} \cup \{\aleph_0\}$, then (i)(a) holds for $i = 0, 1, 2, ..., k-1$ and

$$\rho_E(x_j) \equiv \mu_{Q_{\aleph_0}}, \text{ for all } j > n_k.$$

If $M_E = \{\aleph_0\}$, then $Q_{\aleph_0} = I$ and

$$\rho_E(x_j) \equiv \mu_{Q_{\aleph_0}}, \text{ for all } j \in \mathbb{N}.$$

(iv) If $M_E = \{n_1 < n_2 < \ldots < n_k\}$, then (i)(a) holds for $i = 0, 1, 2, \ldots, k-1$.

Proof.
(i) Let $M_E = \{n_j\}_1^\infty \cup \{\aleph_0\}$. Then, as observed in the proof of Lemma 2(iii), $x_j \neq 0$ for all $j \in \mathbb{N}$. Moreover, for $n_p < j \leq n_{p+1}$, by Theorem 1(i) we have

$$\rho_E(x_{n_p}) \underset{\not\equiv}{\gg} \rho_E(x_j) \equiv \rho_E(x_{n_{p+1}})$$

for $p = 0, 1, 2, \ldots$, where $n_0 = 0$ and the term corresponding to n_0 is omitted. Thus (i)(a) holds. Since $\mu_{Q_{\aleph_0}} \neq 0$, it follows from Theorem 1(i) that $\rho_E(x_j) \underset{\not\equiv}{\gg} \mu_{Q_{\aleph_0}}$, for $j \in \mathbb{N}$. Besides, $C_E(\mu_{Q_{\aleph_0}}) = Q_{\aleph_0} \neq 0$. Let $\nu = \mu_{Q_{\aleph_0}}$. Then (i)(b) holds.

(ii) If $M_E = \{n_j\}_{j=1}^\infty$, clearly the argument in the proof of (i)(a) holds here verbatim and hence (i)(a) holds in this case too. If there exists $\nu \in \Sigma(\mathcal{S})$ with $C_E(\nu) \neq 0$ such that $\rho_E(x_j) \underset{\not\equiv}{\gg} \nu$ for all $j \in \mathbb{N}$, then by Theorem 66.3 of [5] and by Theorem 1(iii) we have

$$C_E(\rho_E(x_j)) = (I - \sum_{n \in M_E, n < j} Q_n) \geq C_E(\nu) \neq 0$$

for all $j \in \mathbb{N}$. Consequently,

$$0 = (I - \sum_{n \in M_E} Q_n) = \bigwedge_{j \in \mathbb{N}} (I - \sum_{n \in M_E, n < j} Q_n) \geq C_E(\nu) \neq 0$$

which is absurd. Hence (ii) holds.

(iii) Suppose $M_E = \{n_j\}_{j=1}^k \cup \{\aleph_0\}$. From the definition of x_j in Lemma 2(iii) it is clear that $x_j = x_{\aleph_0}^{(j)}$ for $j > n_k$. Hence by Lemma 2(ii) we have $\rho_E(x_j) = \rho_E(x_{\aleph_0}^{(j)}) = \mu_{Q_{\aleph_0}}$ for all $j > n_k$. As in the case of (i), by Theorem 1(i) we also have

$$\rho_E(x_{n_i}) \underset{\not\equiv}{\gg} \rho_E(x_j) \equiv \rho_E(x_{n_{i+1}})$$

for $n_i < j \leq n_{i+1}$, $i = 0, 1, 2, \ldots, k-1$ with $n_0 = 0$ and the term corresponding to n_0 being omitted. When $M_E = \{\aleph_0\}$, $M_E \cap \mathbb{N} = \emptyset$ and hence $x_j = x_{\aleph_0}^{(j)}$ and $\rho_E(x_j) = \rho_E(x_{\aleph_0}^{(j)}) \equiv \mu_{Q_{\aleph_0}}$ for all $j \in \mathbb{N}$. Then by Theorem 1(ii), $Q_{\aleph_0} = I$.

(iv) Noting that $x_j = 0$ for $j > n_k$, we observe that the proof of (i)(a) is applicable here and hence (iv) holds.

This completes the proof of the theorem.

The following theorem which is based on Theorem 2 not only gives a description of the multiplicity set of $E(\cdot)$ in terms of the measure sequence of an OSD of H relative to $E(\cdot)$ but also expresses the reciprocal relationship between the classifications of OSDs and those of spectral measures with the CGS-property in H.

THEOREM 3. *Let $E(\cdot)$ have the CGS-property in H. If $E(\cdot)$ is doubly infinite (resp. simply infinite, finitely infinite, finite), then the same is true for evey OSD of H relative to $E(\cdot)$. Conversely, suppose*

$$H = \bigoplus_{1}^{N} Z_E(w_i), \quad N \in \mathbb{N} \cup \{\infty\} \quad (7)$$

is an OSD of H relative to $E(\cdot)$. Let $\{n_i\}_{i=1}^{k}$ be a subsequence of $\mathbb{N}$ such that

$$\rho_E(w_{n_i}) \underset{\not\equiv}{\gg} \rho_E(w_j) \equiv \rho_E(w_{n_{i+1}}), \quad n_i < j \le n_{i+1}, (8)$$

for $i = 0, 1, 2, ..., k-1$ if k is finite and for $i = 0, 1, 2, ...,$ if $k = \infty$, where $n_0 = 0$, and the term corresponding to n_0 is omitted. Moreover, if k is finite and $N = \infty$, then (8) is replaced by

$$\rho_E(w_{n_k}) \underset{\not\equiv}{\gg} \rho_E(w_j) \equiv \rho_E(w_{n_k+1}) \text{ for all } j \ge n_k + 1 \quad (9)$$

and by

$$\rho_E(w_{n_i}) \underset{\not\equiv}{\gg} \rho_E(w_j) \equiv \rho_E(w_{n_{i+1}}) \text{ for } n_i < j \le n_{i+1} \text{ and for } i = 0, 1, 2, ..., k-1 \quad (10)$$

where $n_0 = 0$ and the terem corresponding to n_0 is omitted.

If such a sequence $\{n_i\}_1^k$ does not exist and $N = \infty$, then instead of (9) and (10) we have

$$\rho_E(w_1) \equiv \rho_E(w_2) \equiv .... \quad (11)$$

Then the following hold:

(i) $M_E \cap \mathbb{N} = \{n_i\}_1^k$ when such a sequence exists and on the contrary, $M_E = \{\aleph_0\}$ if $N = \infty$ and $M_E = \{N\}$ if $N \in \mathbb{N}$.

(ii) The OSD (7) is simply infinite if and only if k is infinite and $\bigwedge_{i=1}^{\infty} C_E(\rho_E(x_{n_i})) = 0$. In that case, $M_E = \{n_i\}_{i=1}^{\infty}$.

(iii) The OSD (7) is doubly infinite if and only if k is infinite and $\bigwedge_{i=1}^{\infty} C_E(\rho_E(x_{n_i})) = Q$ (say) $\neq 0$. In that case, $M_E = \{n_i\}_{i=1}^{\infty} \cup \{\aleph_0\}$ and $Q_{\aleph_0} = Q$.

(iv) The OSD (8) is finitely infinite if and only if N is infinite and k is finite or $M_E = \{\aleph_0\}$. In the former case, $M_E = \{n_i\}_{i=1}^{k} \cup \{\aleph_0\}$ and $Q_{\aleph_0} = C_E(\rho_E(w_{n_k+1}))$. In the latter case, $Q_{\aleph_0} = I = C_E(\rho_E(w_1))$.

(v) If N is finite, then either $M_E = \{n_i\}_{i=1}^{k}$ with $n_k = N$ or $M_E = \{N\}$.

(vi) *(a) In case (ii)*

$$Q_{n_i} = C_E(\rho_E(w_{n_i})) - C_E(\rho_E(w_{n_{i+1}})) \quad (12)$$

for $i \in \mathbb{N}$.

(b) In case (iii), (12) holds for $i \in \mathbb{N}$ *and* $Q_{\aleph_0} = Q$.

(c) In case (iv), (12) holds for $i = 1, 2, ..., k-1$, $Q_{n_k} = C_E(\rho_E(w_{n_k})) - C_E(\rho_E(w_{n_k+1}))$ *and* $Q_{\aleph_0} = C_E(\rho_E(w_{n_k+1}))$ *when* $M_E \neq \{\aleph_0\}$. *If* $M_E = \{\aleph_0\}$, *then* $Q_{\aleph_0} = C_E(\rho_E(w_1)) = I$.

(d) In case (v), (12) holds for $i = 1, 2, ..., k-1$ *and* $Q_{n_k} = C_E(\rho_E(w_{n_k}))$ *when* $M_E = \{n_i\}_1^k$ *and* $Q_N = I = C_E(\rho_E(w_i))$ *for* $i = 1, 2, ..., N$ *when* $M_E = \{N\}$.

Consequently, if the OSD (7) is doubly infinite (resp. simply infinite, finitely infinite, finite) then the same is true for every OSD of H *relative to* $E(\cdot)$ *and for the spectral measure* $E(\cdot)$.

Proof. The first part follows from Theorem 2 and from Theorem 3.11 of [9]. For the OSD (7) let $\{n_i\}_{i=1}^k$ be given as in the theorem. By the second part of Theorem 1 and by Theorem 1(iii) we have:

$$C_E(\rho_E(w_j)) = I - \sum_{n \in M_E, n<j} Q_n. \quad (13)$$

(i) Let $M_E \cap \mathbb{N} = \{p_i\}_1^{k'}$ where $k' \in \mathbb{N} \cup \{\infty\}$. Then by Theorem 3.11 of [9] and by Theorem 2 we have

$$\rho_E(w_{p_i}) \underset{\not\equiv}{\gg} \rho_E(w_j) \equiv \rho_E(w_{p_{i+1}}) \text{ for } p_i < j \leq p_{i+1}$$

and $i = 0, 1, 2, ..., k'-1$ if $k' < \infty$ and $i = 0, 1, 2, ...$ if $k' = \infty$, where $p_0 = 0$ and the term corresponding to p_0 is omitted. Then from the hypothesis it follows that $k' = k$ and $p_i = n_i$ for each i. Thus $M_E \cap \mathbb{N} = \{n_i\}_1^k$. When such a sequence does not exist and $N = \infty$ (resp. and $N \in \mathbb{N}$)

$$\rho_E(w_1) \equiv \rho_E(w_2) \equiv ... \equiv \rho_E(w_j) \equiv ...$$

for $j \in \mathbb{N}$ (resp. for $1 \leq j \leq N$) and hence (7) implies that $I = \sum_1^N E_i'$ where $E_i' = Z_E(w_i)$, $1 \leq i \leq N$, $i \in \mathbb{N}$, are mutually orthogonal abelian projections in W' by Theorem 60.2 of [5]. Moreover, by Theorem 62.2 of [5] $C_{E_i'} = I$ for all such i and hence by Proposition 1 we conclude that I has UH-multiplicity $\aleph_0$ (resp. N). Thus $\aleph_0 \in M_E$ and $Q_{\aleph_0} = I$ (resp. $N \in M_E$ and $Q_N = I$). Moreover, again by Proposition 1, we conclude that $M_E = \{\aleph_0\}$ (resp. $M_E = \{N\}$).

(ii) Let the OSD (7) be simply infinite. Then $k = \infty$ and there does not exist any $\nu \in \Sigma(S)$ with $C_E(\nu) \neq 0$ such that $\rho_E(w_j) \gg \nu$ for all $j \in \mathbb{N}$. If possible, let $\bigwedge_1^\infty C_E(\rho_E(w_j)) = Q \neq 0$. Then for each non zero vector $w \in QH$ we have $Q \geq C_E(\rho_E(w)) \neq 0$. Consequently, by Theorem 65.2 of [5] we have $\rho_E(w_j) \geq \rho_E(w)$ for all $j \in \mathbb{N}$. This contradiction proves that $Q = 0$. Conversely, let $k = \infty$ and $\bigwedge_1^\infty C_E(\rho_E(w_j)) = 0$. If $\nu \in \Sigma(S)$ is such that $\rho_E(w_j) \gg \nu$ for all $j \in \mathbb{N}$, then $C_E(\rho_E(w_j)) \geq C_E(\nu)$ for all j, and therefore,

$$C_E(\nu) \leq \bigwedge_1^\infty C_E(\rho_E(w_j)) = 0.$$

Hence the OSD (7) is simply infinite.

Let the OSD (7) be simply infinite. Then, as observed above, $\bigwedge_1^\infty C_E(\rho_E(w_j)) = 0$ and hence by (i) and (13) we have

$$0 = \bigwedge_1^\infty C_E(\rho_E(w_j)) = \bigwedge_{j=1}^\infty (I - \sum_{n \in M_E, n<j} Q_n) = I - \sum_{i=1}^\infty Q_{n_i}.$$

Thus $\sum_{i=1}^\infty Q_{n_i} = I$ and hence $M_E = \{n_i\}_{i=1}^\infty$. Therefore, $E(\cdot)$ is simply infinite.

(iii) Let the OSD (7) be doubly infinite. Then $k = \infty$ and there exists $\nu \in \Sigma(\mathcal{S})$ such that $C_E(\nu) \neq 0$ and such that $\rho_E(w_j) \gg \nu$ for all $j \in I\!N$. Then

$$C_E(\rho_E(w_j)) \geq C_E(\nu) \text{ for all } j \in I\!N$$

and hence

$$\bigwedge_1^\infty C_E(\rho_E(w_j)) \geq C_E(\nu) \neq 0.$$

Conversely, suppose $k = \infty$ and $\bigwedge_1^\infty C_E(\rho_E(w_j)) = Q \neq 0$. Then as shown in the proof of (ii) there exists $w \in QH$ such that $Q = C_E(\rho_E(w)) \neq 0$ so that $\rho_E(w_j) \gg \rho_E(w) \neq 0$ for all $j \in I\!N$. Thus the OSD (7) is doubly infinite.

Now let the OSD (7) be doubly infinite. Then $k = \infty$ and $\bigwedge_1^\infty C_E(\rho_E(w_j)) = Q(\text{say}) \neq 0$. Moreover, as $I = C_E(\rho_E(w_1) \underset{\neq}{>} C_E(\rho_E(w_{n_2})$ by the second part of Theorem 1, it follows that $Q \neq I$. Therefore, by (13) and (i) we have

$$0 \neq I - Q = \bigvee_1^\infty (I - C_E(\rho_E(w_j))) = \bigvee_{j=1}^\infty (\sum_{n_i<j} Q_{n_i}) = \sum_{i=1}^\infty Q_{n_i}.$$

Thus $\aleph_0 \in M_E$ and $Q_{\aleph_0} = Q$. Consequently, as $M_E = M_E \cap I\!N \cup \{\aleph_0\}$, by (i) we have $M_E = \{n_i\}_1^\infty \cup \{\aleph_0\}$. Hence $E(\cdot)$ is doubly infinite.

(iv) Let the OSD (7) be finitely infinite. Then by (5) there exists n_0 such that $\rho_E(w_j) \equiv \rho_E(w_{n_0})$ for all $j \geq n_0$. Consequently, $n_0 = n_k + 1$ and hence k is finite whenever such a sequence $\{n_i\}_1^k$ exists. Thus, in this case, k is finite and N is infinite. If such a sequence does not exist, then

$$\rho_E(w_1) \equiv \rho_E(w_2) \equiv ... \equiv \rho_E(w_j) \equiv ...$$

for all $j \in I\!N$ and hence, as shown in the proof of (i), $Q_{\aleph_0} = I$ and $M_E = \{\aleph_0\}$. Conversely, if $N = \infty$ and k is finite, by hypothesis $\rho_E(w_j) \equiv \rho_E(w_{n_k+1})$ for all $j \geq n_k + 1$, and hence the OSD is finitely infinite. If $M_E = \{\aleph_0\}$, then by Theorem 2(iii) $Q_{\aleph_0} = I$ and $\rho_E(x_j) \equiv \mu_{Q_{\aleph_0}}$ for all $j \in I\!N$. Then by Theorem 3.11 of [9] we conclude that $\rho_E(w_j) \equiv \rho_E(x_j) \equiv \mu_{Q_{\aleph_0}}$ for all $j \in I\!N$. Thus, in this case too, the OSD is finitely infinite.

If the OSD (7) is finitely infinite and $M_E \neq \{\aleph_0\}$, then k is finite and by (i), $M_E \cap \mathbb{N} = \{n_i\}_1^k$. Moreover, by (13) we have

$$0 \neq C_E(\rho_E(w_{n_k+1})) = I - \sum_1^k Q_{n_i}.$$

Hence $\aleph_0 \in M_E$ and $Q_{\aleph_0} = C_E(\rho_E(w_{n_k+1}))$. Thus $M_E = \{n_i\}_1^k \cup \{\aleph_0\}$. The other case is that $M_E = \{\aleph_0\}$ in which case $Q_{\aleph_0} = I = C_E(\rho_E(w_1))$ by the second part of Theorem 1 and by Theorem 1(ii). Moreover, $E(\cdot)$ is finitely infinite in both cases.

(v) If N is finite and $\{n_i\}_1^k$ exists, then clearly $n_k = N$ and by (i), $M_E = M_E \cap \mathbb{N} = \{n_i\}_1^k$. If such a sequence does not exist, then by the second part of (i) we have $M_E = \{N\}$ and $Q_N = I$.

(vi)(a) If the OSD (7) is simply infinite, by (ii) we have $M_E = \{n_i\}_1^\infty$. Moreover, by (13) we obtain

$$C_E(\rho_E(w_{n_i})) - C_E(\rho_E(w_{n_{i+1}})) = (I - \sum_1^{i-1} Q_{n_j}) - (I - \sum_1^{i} Q_{n_j}) = Q_{n_i}$$

for $i \in \mathbb{N}$, where $Q_{n_0} = 0$.

Similar arguments combined with (iii), (iv) and (v) prove the validity of (b),(c) and (d). The details are left to the reader.

Other assertions follow from the previous parts.

This completes the proof of of the theorem.

5. OSDs INDUCED BY COBOTSRs AND COBOTSRs INDUCED BY OSDs.

Let the spectral measure $E(\cdot)$ have the CGS-property in H. Then H admits not only OSDs relative to $E(\cdot)$, but also by Theorems 3.6 and 5.6 of [10] it admits COBOTSRs relative to $E(\cdot)$ (see Definitions 3.1 and 5.2 of [10]). Starting with a COBOTSR U (resp. with an OSD) of H relative to $E(\cdot)$, we now construct in a canonical way an OSD (resp. a COBOTSR U) of H relative to $E(\cdot)$.

THEOREM 4. *Suppose U is a COBOTSR of H relative to $E(\cdot)$ with the measure family F. Then there exists an OSD*

$$H = \bigoplus_{1}^{N} Z_E(x_i) \quad (14)$$

of H relative to E with $N \in \mathbb{N} \cup \{\infty\}$ such that

$$C_E(\rho_E(x_i)) = \sum_{\mu \in F: u_E(\mu) \geq i} C_E(\mu)$$

for all those $i \in \mathbb{N}$ for which there exists some $\mu \in F$ with $u_E(\mu) \geq i$. Moreover, $N = \sup\{n : n \in M_E\}$, and the OSD (14) is determined upto equivalence of OSDs. The OSD (14) is called the OSD induced by the COBOTSR U.

Proof. Let $\nu_i \equiv \bigvee\{\mu \in F : u_E(\mu) \geq i\}$. As F is countable by Theorem 5.6 of [10], ν_i is well defined and is unique upto equivalence. Obviously, $\nu_1 \gg \nu_2 \gg ... \gg ....$ By Theorem 5.6(iii) of [10] we have

$$\mu_{Q_n} \equiv \bigvee\{\mu \in F : u_E(\mu) = n\}, \text{ for } n \in M_E.$$

Then

$$\nu_i \equiv \bigvee\{\mu \in F : u_E(\mu) \geq i\} \equiv \bigvee\{\mu_{Q_n} : n \in M_E, n \geq i\}.$$

Suppose $M_E = \{n_i\}_1^\infty \cup \{\aleph_0\}$. Similarly, other cases can be dealt with. For $1 \leq i \leq n_k$, we have

$$\begin{aligned} \oplus_1^{n_k} L_2(\nu_i) &= ((\oplus_{n_1}(\oplus_{i=1}^{\infty}(L_2(\mu_{Q_{n_i}})) \oplus (\oplus_{n_1}(L_2(\mu_{Q_{\aleph_0}}))) \bigoplus \\ &\oplus ((\oplus_{n_2-n_1}(\oplus_{i=2}^{\infty}(L_2(\mu_{Q_{n_i}})) \oplus (\oplus_{n_2-n_1}(L_2(\mu_{Q_{\aleph_0}}))) \\ &\oplus ... \bigoplus((\oplus_{n_k-n_{k-1}}(\oplus_{i=k}^{\infty}(L_2(\mu_{Q_{n_i}})) \oplus (\oplus_{n_k-n_{k-1}}(L_2(\mu_{Q_{\aleph_0}}))). \end{aligned}$$

Therefore, by rearranging we have

$$\begin{aligned} \bigoplus_1^{n_k} L_2(\nu_i) &= (\bigoplus_{n_1}(L_2(\mu_{Q_{n_1}})) \bigoplus(\bigoplus_{n_2}(L_2(\mu_{Q_{n_2}})) \bigoplus \cdots \bigoplus(\bigoplus_{n_k}(L_2(\mu_{Q_{n_k}})) \bigoplus \\ &\bigoplus(\bigoplus_{i=k+1}^{\infty} \bigoplus_{n_k} L_2(\mu_{Q_{n_i}})) \bigoplus(\bigoplus_{n_k}(L_2(\mu_{Q_{\aleph_0}})). \end{aligned}$$

Since this holds for all n_k, $k \in \mathbb{N}$, we conclude that

$$K = \bigoplus_1^{\infty} L_2(\nu_i) = (\bigoplus_{i=1}^{\infty} \bigoplus_{n_i} L_2(\mu_{Q_{n_i}})) \bigoplus(\bigoplus_1^{\infty} L_2(\mu_{Q_{\aleph_0}})).$$

Since by Lemma 5.3 of [10], $\{\mu_{Q_{n_i}}\}_1^\infty \cup \{\mu_{Q_{\aleph_0}}\}$ is the measure family of a COBOTSR U_0 of H relative to $E(\cdot)$ and since U and U_0 are equivalent by Theorem 4.2 of [10], then by Definition 3.1 of [10] we conclude that there exists an isomorphism V_U from H onto K satisfying

$$V_U E(\cdot) V_U^{-1}(f_i) = (\chi_{(\cdot)} f_i), \text{ for } (f_i) \in K.$$

Consequently, by Definition 4.1 of [9] V_U is an ordered spectral representation of H relative to $E(\cdot)$ as $\nu_1 \gg \nu_2 \gg \ldots$. Hence by the discussion on pp.228-229 of [9] there exists an OSD $H = \oplus_1^\infty Z_E(x_i)$ of H relative to $E(\cdot)$ such that $\rho_E(x_i) \equiv \nu_i$. Then by Theorems 66.2 and 66.5 of [5] we conclude that $C_E(\rho_E(x_i)) = C_E(\nu_i) = \sum_{\mu \in F, u_E(\mu) \geq i} C_E(\mu)$. Then (14) is unique upto equivalence by Theorem 3.11 of [9].

Now let $\sup\{n : n \in M_E\} = k$. Note that $k = \aleph_o$ if M_E is infinite or if $\aleph_o \in M_E$. If $k \in \mathbb{N}$, then by Proposition 3.3(v) of [10] there exists some $\mu \in F$ such that $u_E(\mu) = k$ and hence ν_k and therefore, $\rho_E(x_k)$ exist. Obviously, $\rho_E(x_i)$ with $x_i \neq 0$ is not defined for $i > k$ and hence the OSD-multiplicity N of the OSD coincides with k. If $k = \aleph_0$, then $\rho_E(y_i)$ exists and y_i es non zero for each $i \in \mathbb{N}$ and hence the OSD-multiplicity N is $\aleph_0$.

This completes the proof of the theorem.

Remark 2. For self-adjoint operators (resp. for spectral measures) on a separable Hilbert space similar result was proved by Plesner and Rohlin [10, p. 143] (resp. Brown [1, pp.155-156]).

Given an OSD of H relative to $E(\cdot)$, the following theorem gives in a canonical way the construction of a COBOTSR of H relative to $E(\cdot)$.

THEOREM 5. *Let $E(\cdot)$ have the CGS-property in H and let us consider the OSD (7) of Theorem 3. Then the following assertions hold:*

(i) The multiplicity set M_E is determined by $\rho_E(w_i)$, $1 \leq i \leq N, i \in \mathbb{N}$.

(ii) The central projections Q_n, $n \in M_E$, are determined by $\{C_E(\rho_E(w_i))_1^N\}$.

(iii) There exists a COBOTSR U of H relative to $E(\cdot)$ with the measure family $F = \{\rho_E(Q_n w_1)\}_{n \in M_E}$ and U is called the COBOTSR indeuced by the OSD (7).

(iv) For $n \in M_E$,

$$Q_n H = \bigoplus_{j=1}^{n} Z_E(Q_n w_j)$$

is an OSD of $Q_n H$ with $\rho_E(Q_n w_1) \equiv \rho_E(Q_n w_2) \equiv \ldots \equiv \rho_E(Q_n w_j) \equiv \ldots$ for $j \in \mathbb{N}, j \leq n$. Moreover, $\mu_{Q_n} \equiv \rho_E(Q_n w_j)$, $1 \leq j \leq n$, $j \in \mathbb{N}$.

Proof.
(i) and (ii) hold by Theorem 3.

(iii) By the second part of Theorem 1, $C_E(\rho_E(w_1)) = I$. Consequently, by Theorem 66.2 of [5],

$$C_E(\rho_E(Q_n w_1)) = C_{[WQ_n w_1]} = Q_n C_{[Ww_1]} = Q_n C_E(\rho_E(w_1)) = Q_n$$

for $n \in M_E$. Thus, by Theorem 65.2 of [5], $\rho_E(Q_n w_1) \equiv \mu_{Q_n}$ for $n \in M_E$. Then (iii) holds by Lemma 5.4 of [9].

(iv) By (13), $C_E(\rho_E(w_i)) \perp Q_n$ for $n \in M_E$ with $n < i$ and $C_E(\rho_E(w_i))Q_n = Q_n$ for $n \in M_E$ with $n \geq i$. Therefore, by Theorems 66.2 and 66.5 of [5] we have

$$C_E(\rho_E(Q_n w_i)) = C_{[WQ_n w_i]} = Q_n C_{[Ww_i]} = Q_n C_E(\rho_E(w_i)) = \begin{cases} 0 & \text{if } n < i \\ Q_n & \text{if } n \geq i \end{cases}$$

for $n \in M_E$. Moreover, for each $n \in M_E$, by Theorem 60.2 of [5] the projections $E_i' = [WQ_n w_i]$, $1 \leq i \leq n$, $i \in \mathbb{N}$, form an orthogonal family of abelian projections in W'. Since these projections have the same central support Q_n, it follows from Proposition 1 and Theorem 65.2 of [5] that

$$Q_n H = \bigoplus_{i=1}^{n} Z_E(Q_n w_i), \;\; n \in M_E$$

is an OSD of $Q_n H$ with $\rho_E(Q_n w_1) \equiv \rho_E(Q_n w_2) \equiv ....$ As $C_E(\rho_E(Q_n w_i)) = Q_n$ it follows that $\rho_E(Q_n w_i) \equiv \mu_{Q_n}$ for $n \in M_E$ and for $1 \leq i \leq n$, $i \in \mathbb{N}$.

This completes the proof of the theorem.

6. Spatial isomorphism of abelian von Neumann algebras

Using the results of [7,9,12] we give a necessary and sufficient condition for an involution preserving isomorphism between two abelaian von Neumann algebras to be spatial. When the algebras have countably decomposable commutants, we deduce the condition in terms of the multiplicity functions m_p and m_c of the respective canonical spectral measures of the algebras. The last result is an analogue of Theorem 7.8 of Stone [16], though it is not a generalization.

Let $\mathcal{A}$ be an abelian von Neumann algebra on a Hilbert space H and let $\mathcal{M}$ be its maximal ideal space. For each $f \in C(\mathcal{M})$, let T_f be the operator in $\mathcal{A}$ whose image under the Gelfand mapping is f. Let $\mathcal{B}(\mathcal{M})$ be the σ-algebra of the Borel subsets of $\mathcal{M}$. The set function $G : \mathcal{B}(\mathcal{M}) \to \mathcal{A}$ defined by $G(\sigma) = T_{\chi(\Upsilon(\sigma))}$, where $\Upsilon(\sigma)$ is clopen and $\Upsilon(\sigma)\Delta\sigma$ is meagre in $\mathcal{M}$. Then $G(\cdot)$ is projection valued and is σ-additive in the strong operator topology. Then $G(.)$ is called the canonical spectral measure of $\mathcal{A}$.

For the concepts of primitive projections in $\mathcal{A}$, proper $\mathcal{A}$-base and the Kelley multiplicity function ϕ of $\mathcal{A}$, the reader is referred to [7]. It is shown in [12] that a projection $P \in \mathcal{A}$ is primitive if and only if P is cyclic and has UH-multiplicity, where we consider $\mathcal{A}$ as the von Neumann algebra generated by the range of its canonical spectral measure $G(\cdot)$. Moreover, for a primitive projection

$P \in \mathcal{A}$, let e_P be the clopen set in $\mathcal{M}$ such that $P = T_{\chi_{e_P}}$. If P has UH-multiplicity n, then as shown in [12], we have $\phi(t) = n$ for all $t \in e_P$. The following proposition is proved in [12] , which is the same as result 5.1 mentioned on p.605 of [7].

PROPOSITION 2. *There exists a maximal orthogonal family $\mathcal{F}$ of primitive projections in $\mathcal{A}$. For $F \in \mathcal{F}$, let $J_{\mathcal{F}}$ be a proper $\mathcal{A}$-base for F. Let $L_2(\mathcal{M}, \mathcal{B}(\mathcal{M}), \rho_G(x)) = L_2(\rho_G(x))$. Choose $x_F \in J_F$, $F \in \mathcal{F}$. Let card.$J_F = n_F$. Then there exists an isomorphism U from H onto $K = \bigoplus_{F\in\mathcal{F}} \bigoplus_{n_F} L_2(\rho_G(x_F))$ such that*

$$UAU^{-1}(f_{F,j})_{F\in\mathcal{F},j\in J_F} = (gf_{F,j}), \ \ (f_{F,j})_{F\in\mathcal{F},j\in J_F} \in K$$

for $A \in \mathcal{A}$, where $g \in C(\mathcal{M})$ with $T_g = A$.

If $\mathcal{A}'$ is countably decomposable, then $G(\cdot)$ has the CGS-property in H and using Theorem 3 above, we showed in [12] that the Kelley multiplicity function ϕ is given by $\phi(t) = \max(m_p(t), m_c(t))$ for $t \in \mathcal{M}\backslash(\bar{p}_G\backslash p_G)$, where $p_G = \{t \in \mathcal{M} : G(\{t\}) \neq 0\}$ and m_p and m_c are the multiplicity functions corresponding to $G(\cdot)$ as in Definitions 6.4 and 6.8 of [9].

Let $\mathcal{F}$, x_F, J_F and n_F be as in Proposition 2. Let $\mathcal{S}_o$ be the σ-ring of all Borel sets of $\mathcal{M}$ which are contained in some clopen set corresponding to a cyclic projection in $\mathcal{A}$. Let $\nu_{\mathcal{F}}(\sigma) = \sum_{F\in\mathcal{F}} \rho_G(x_F)(\sigma)$ for $\sigma \in \mathcal{S}_o$. It is clear that $\nu_{\mathcal{F}}$ is a σ-finite measure on $\mathcal{S}_o$. Adopting the von Neumann definition of ordinal and cardinal numbers, so that each ordinal is identical with the set of all smaller ordinals, and a cardinal is an ordinal which cannot be put in 1-1 correspondence with a smaller ordinal, let C be the supremum of the values of ϕ. Let γ be the counting measure on the σ-ring of all countable subsets of C. Now we give below a proof of the result 5.2 mentioned in [7] in which we also include the special case when the commutant of $\mathcal{A}$ is countably decomposable.

THEOREM 6. *Let Γ be the subset of $\mathcal{M} \times C$, consisting of ordered pairs (t, c) with $\phi(t) \geq c$ and let $\eta_{\mathcal{F}}$ be $\nu_{\mathcal{F}} \times \gamma$ restricted to subsets of Γ. Then there exists an isomorphism V from H onto $L_2(\eta_{\mathcal{F}})$ such that $VAV^{-1}k(t,c) = g(t)k(t,c)$ for $k \in L_2(\eta_{\mathcal{F}})$, where $g \in C(\mathcal{M})$ with $T_g = A$. If $\mathcal{A}'$ is countably decomposable and if m_p and m_c are the multiplicity functions of the canonical spectral measure $G(\cdot)$ of $\mathcal{A}$, then the above result holds with Γ replaced by $\Gamma_o = \{(t,c) : max(m_p(t), m_c(t)) \geq c\}$ in $\mathcal{M} \times C$.*

Proof. Let $K, \mathcal{F}, F, x_F, J_F$ and n_F be as in Proposition 2. Let C_F be the set all of all cardinals $\{c : c \leq n_F\}$. Let $\Phi_F : C_F \to J_F$ be a bijective map. For $k \in L_2(\eta_{\mathcal{F}})$, let $k_{F,\Phi_F(c)}(t) = k(t,c)$ if $t \in e_F$. Then clearly $k_{F,\Phi_F(c)}$ is Borel measurable on e_F and moreover,

$$\begin{aligned}
\int_\Gamma |k(t,c)|^2 d(\eta_{\mathcal{F}}) &= \int_C (\int_{\mathcal{M}} |k(t,c)|^2 d\nu_{\mathcal{F}}) d\gamma = \int_C (\int_{\cup_{F\in\mathcal{F}} e_F} |k(t,c)|^2 d\nu_{\mathcal{F}}) d\gamma \\
&= \int_C (\sum_{F\in\mathcal{F}} \int_{e_F} |k(t,c)|^2 d\rho_G(x_F)) d\gamma = \sum_{F\in\mathcal{F}} \int_{C_F} (\int_{e_F} |k(t,c)|^2 d\rho_G(x_F)) d\gamma \\
&= \sum_{F\in\mathcal{F}} \sum_{c\in C_F} \int_{e_F} |k_{F,\Phi_F(c)}(t)|^2 d\rho_G(x_F) \quad (15).
\end{aligned}$$

Hence $\sum_{F\in\mathcal{F}}\sum_{c\in C_F}\int_F |k_{F,\Phi_F(c)}(t)|^2 d\rho_G(x_F) < \infty$ and thus $(k_{F,\Phi_F(c)})_{F\in\mathcal{F},c\in C_F} \in K$. Then clearly the map $\Psi : L_2(\eta) \to K$ given by $\Psi(k) = (k_{F,\Phi_F(c)})_{F\in\mathcal{F},c\in C_F}$ is an isometry. Conversely, let $k_{F,\Phi_F(c)} \in L_2(\rho_G(x_F))$ for $F \in \mathcal{F}$ and for $c \in C_F$ such that $(k_{F,\Phi_F(c)})_{F\in\mathcal{F},c\in C_F} \in K$. Let $k(t,c) = k_{F,\Phi_F(c)}(t)$ if $t \in F$ and $c \in C_F$ and let $k(t,c)$ assume any complex value if $(t,c) \in \Gamma$ with $t \in \mathcal{M}\backslash \cup_{F\in\mathcal{F}} F$. Then clearly k is measurable in Γ since $G(\mathcal{M}\backslash \cup_{F\in\mathcal{F}} F) = 0$. Then by a calculation similar to that in (15) one can show that $k \in L_2(\eta_{\mathcal{F}})$ and that $\Psi(k) = (k_{F,\Phi_F(c)})_{F\in\mathcal{F},c\in C_F}$. Thus Ψ is an onto isomorphism from $L_2(\eta_{\mathcal{F}})$ onto K. For $g \in C(\mathcal{M})$ and $k \in L_2(\eta_{\mathcal{F}})$, let $(gk)(t,c) = g(t)k(t,c)$. If $V = \Psi^{-1}U$, where U is as in Propisition 2, then V is an isomorphism of H onto $L_2(\eta_{\mathcal{F}})$ and by the same proposition we have $VAV^{-1}k(t,c) = g(t)k(t,c)$ where $g \in C(\mathcal{M})$ with $T_g = A$.

To prove the second part we observe that $\mathcal{M}\backslash(\bar{p}_G\backslash p_G)$ is open and dense in $\mathcal{M}$ and $G(\bar{p}_G\backslash p_G) = 0$ since $\bar{p}_G\backslash p_G$ is nowhere dense in $\mathcal{M}$. Since $\phi(t) = \max(m_p(t), m_c(t))$ for $t \in \mathcal{M}\backslash(\bar{P}_G\backslash p_G))$ by Theorem 7 of [12], now the second part is immediate from the first.

COROLLARY 2. *Let $\mathcal{A}_i$ be abelian von Neumann algebras (resp. with $\mathcal{A}_i'$ countably decomposable) for $i = 1, 2$. If Φ is an involution preserving isomorphism from $\mathcal{A}_1$ onto $\mathcal{A}_2$ and if ϕ_1 and ϕ_2 are the Kelley multiplicity functions of $\mathcal{A}_1$ and $\mathcal{A}_2$ (resp. $m_p^{(i)}$ and $m_c^{(i)}$ are the multiplicity functions on the maximal ideal space of the Gelfand space $\mathcal{M}_i$ of the canonical spectral measure $G_i(\cdot)$ for $i = 1, 2$), then Φ is spatial if and only if $\phi_1(t) = \phi_2(ht)$ (resp. if and only if $m_p^{(1)}(t) = m_p^{(2)}(h(t))$ and $m_c^{(1)}(t) = m_c^{(2)}(h(t))$) for $t \in \mathcal{M}_1$ where $h : \mathcal{M}_1 \to \mathcal{M}_2$ is the homeomorphism induced by Φ (see Theorem IV.6.26 of [3]).*

Remark 3.. The above corollary can be considered as an analogue of Theorem 7.8 of Stone [16] for abelian von Neumann algebras with countably decomposable commutants, though it is not its generalization.

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