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Weak Compactness criteria for set valued integrals and Radon Nikodym Theorem for vector valued multimeasures

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Abstract

Some criteria for weak compactness of set valued integrals are given. Also we show some of applications to the study of multimeasures on Banach spaces with the Radon-Nikodym property.

1 Introduction

The theory of measurable multifunctions has shown to be useful in many mathematical fields such as Control Theory [1], Convex Analysis [6], Abstract evolution equations [15], etcetera.

It is the purpose of this paper to provide some results about the weakly compactness of the measurable selections of a measurable multifunction, and use them to show a Radon-Nikodym Theorem for multimeasures.

2 Preliminaries

In this section we fix the notations and definitions that we use in this paper. For a Banach space X, its dual space will be denote by X^*. We will also use the following notations:

$$\begin{aligned} P_{f(c)}(X) &= \{A \subseteq X : A \neq \emptyset, \ \ A \quad \text{closed (convex)}\} \\ P_{\omega kc}(X) &= \{A \subseteq X : A \neq \emptyset, \ \ A \quad \text{weakly compact (convex)}\}. \end{aligned}$$

For a subset A of X we set $|A| = \sup_{a\in A}\|a\|$.

Following [3], given a complete finite measure space (Ω, Σ, μ) and a Banach space X, we say that a multifunction $F : \Omega \to P_f(X)$ is μ-**measurable**, if there is a μ-null set $N \in \Sigma$ and a sequence $\{f_n\}$ of μ-measurable functions such that

$$F(\omega) = \mathrm{cl}\{f_n(\omega)\} \quad \text{for all} \quad \omega \in \Omega\backslash N.$$

This definition does not need the hypothesis "X separable"; and according with Pettis measurability theorem [9], it contains the classical definition for separable Banach spaces, due to Castaing Representation [6]. This will allow us to deal with a considerable grade of generality.

Given a measurable multifunction $F : \Omega \to P_f(X)$, we denote by S_F^p the set

$$S_F^p = \{f : \Omega \to X : f \in L_X^p(\mu); \quad f(\omega) \in F(\omega)\mu \cdot a.e\};$$

and for $E \in \Sigma$, by $\int_E F d\mu$, we denote the set

$$\int_E F d\mu = \left\{ \int_E f d\mu : f \in S_F^1 \right\}.$$

We say that a measurable multifunction F is *integrably bounded* if $|F(\cdot)| \in L^1(\mu)$. Following [19, 20], for $\{A_n, A\} \subset P_f(X)$, $A_n's$ *weakly converges to* A ($A_n \overset{\omega}{\to} A$) if for each $x^* \in X^*$, $\sigma(x^*, A_n) \to \sigma(x^*, A)$; where $\sigma(x^*; B) = \sup\{\langle x^*, x\rangle : x \in B\}$; for any non-empty subset B of X. A sequence of measurable multifunctions $\{F_n\}_{n=1}^{\infty}$ is said to be *weakly convergent* to F in $L_X^1(\mu)$ ($F_n \overset{\omega}{\to} F$), if

$$\int_\Omega \sigma(x^*(\omega), F_n(\omega)) d\mu(\omega) \to \int_\Omega \sigma(x^*(\omega), F(\omega) d\mu(\omega)$$

for each $x^* \in (L_X^1(\mu))^*$.

A *multimeasure* is a function $M : \Sigma \to P(X)$ satisfying

(i) $M(\emptyset) = \{0\}$

(ii) If $E_1, E_2 \in \Sigma$ with $E_1 \cap E_2 = \emptyset$, then $M(E_1 \cup E_2) = M(E_1) + M(E_2)$.

(iii) If $\{E_n\}_{n=1}^{\infty}$ is a sequence in Σ with $E_i \cap E_j = \emptyset \ \forall i \neq j$, then

$$\begin{aligned} M\left(\bigcup_{n=1}^{\infty} E_n\right) &= \sum_{n=1}^{\infty} M(E_n) \\ &= \{x \in X;\ \text{for each } n \in \mathbf{N};\ \text{there is } x_n \in M(E_n) \text{ such that} \\ &\quad \sum_{n=1}^{\infty} x_n,\ \text{unconditionally converges to } x\}. \end{aligned}$$

The multimeasure M is called *bounded variation* if

$$\|M\| = \sup \sum_{i=1}^{n} \|M(A_i)\|$$

is finite where the sup is taken over all finite partition of Ω.

For a fixed measurable space (Ω, Σ), $c_a(X)$ will denote the Banach space of all X valued countably additive, bounded variation vector measures endowed with the norm of total variation.

3 Weak compactness criteria for S_F^p in $L_X^p(\mu)$

The following result can be found in [3].

Theorem 3.1 *Let $F : \Omega \to P_{fc}(X)$ be an integrably bounded multifunction. Then S_F^1 is weakly compact in $L_X^1(\mu)$ if and only if for almost every $\omega \in \Omega$, $F(\omega)$ is weakly compact.*

A small refinement of above theorem is the following one.

Theorem 3.2 *If $1 \leq p < \infty$ and $F : \Omega \to P_f(X)$ is a measurable multifunction, then the following statement are equivalent:*

(a) *S_F^p is relatively weakly compact in $L_X^p(\mu)$.*

(b) *S_F^p is bounded in $L_X^p(\mu)$ and the multifunction $G : \Omega \to P_{fc}(X)$ defined by $G(\omega) = \overline{c_0}F(\omega)$ takes weakly compact values μ.a.e.*

Proof. Suppose $p = 1$.

$(a \Rightarrow b)$. If S_F^1 is relatively weakly compact in $L_X^1(\mu)$; then it is bounded and by [13] (Theorem 3.2); F is integrably bounded.

Furtheremore, given a sequence $\{f_n\} \subseteq S_F^1$, there is a sequence $g_n \in \overline{c_0}\{f_k \; k \geq n\}$ ([8] Theorem 2.1) such that $g_n(\omega)$ is norm convergent in X μ.a.e. This implies $\overline{c_0}F(\omega)$ weakly compact μ.a.e.

Conversely, if S_F^1 is bounded and $\overline{c_0}F(\omega)$; weakly compact μ.a.e, being F measurable; by definition there is a null set $N_0 \in \Sigma$ and a sequence $f_n : \Omega \to X$ of measurable functions such that $\mu(N_0) = 0$ and $F(\omega) = \text{cl}(f_n(\omega))$ $\forall \omega \in \Omega \backslash N_0$. Applying the measurability Pettis theorem, for each $n \in \mathbf{N}$, there is $N_n \in \Sigma$ with $\mu(N_n) = 0$ and $\text{cl}(f_n(\Omega \backslash N_n)$ is separable.

If we put $N = \bigcup_{n=0}^{\infty} N_n$; we see that $\mu(N) = 0$ and $F(\Omega \backslash N)$ is separable. Let Y be the separable Banach space generated by $F(\Omega \backslash N)$. Then if we define, as in [3],

$$H : \Omega \to P_f(Y)$$

$$H(\omega) = \begin{cases} F(\omega) & \text{if} \quad \omega \in \Omega \backslash N \\ \{0\} & \text{if} \quad \omega \in N; \end{cases}$$

G is a measurable multifunction taking values in a separable Banach space. Applying Theorem 1.5 of [13], we see that $\overline{c_0}H$ is a measurable multifunction. Since $G(\omega) = \overline{c_0}F(\omega) = \overline{c_0}H(\omega)$ μ.a.e; we conclude that G is a measurable multifunction taking values in a separable Banach space. It is not hard to see that G is integrably bounded and $G(\omega) \in P_{\omega Kc}(X)$ μ.a.e. So by Theorem 3.1, $S^1_{\overline{c_0}F}$ is weakly compact in $L_X^1(\mu)$ and consequently S_F^1 is relatively weakly compact.

Let $1 < p < \infty$. Since S_F^p is relatively weakly compact in $L_X^p(\mu)$; and the injection $i : L_X^p(\mu) \to L_X^1(\mu)$ is continuous, the set S_F^p is relatively weakly compact in $L_X^1(\mu)$.

If we put $M = \overline{S_F^p}$; then M is decomposable $i.e$; if $f, g \in M$ and $A \in \Sigma$, then $fX_A + gX_{\Omega \backslash A} \in \Sigma$. Then, according with [13] Theorem 3.1, there is a measurable multifunction $G : \Omega \to P_f(X)$ such that $M = S_G^1$. Since $\overline{S_G^1}$ is weakly compact in $L_X^1(\mu)$, we see that $\overline{c_0}G(\omega)$ is weakly compact μ.a.e. Since $S_G^1 = S_G^p \supset S_F^p$, Corollary 1.2 from [13] implies the conclusion.

For the converse, suppose $\overline{c_0}F(\omega)$ weakly compact for almost every $\omega \in \Omega$ and S_F^p bounded in $L_X^p(\mu)$ then S_F^p is bounded in $L_X^1(\mu)$.

It is not hard to see that

$$S_F^p \subset S^p_{\overline{c_0}F} = S^1_{\overline{c_0}F}.$$

By theorem 3.1, $S^1_{\overline{c_0}F}$ is weakly compact in $L_X^1(\mu)$; which implies S_F^p relatively weakly compact in $L_X^1(\mu)$. Applying corollary 3.4 of [8], we conclude that S_F^p is relatively weakly compact in $L_X^p(\mu)$. ■

Corollary 3.3 *If $F(\omega)$ is convex and weakly compact μ.a.e with F a measurable integrably bounded multifunction, then for $1 \le p < \infty$, S_F^p is weakly compact in $L_X^p(\mu)$ if and only if it is bounded.*

Proof. The condition is necessary to S_F^p be relatively weakly compact.

If $\{f_n\}$ is a sequence in S_F^p converging to f in the weak topology of $L_X^p(\mu)$; then $\{f_n\}$ converges to f in the weak topology of $L_X^1(\mu)$ because the inclusion of $L_X^p(\mu)$ into $L_X^1(\mu)$ is continuous. As it is shown in [3], there is a sub-sequence $\{f_{n_k}\}$ of $\{f_n\}$ such, that $f_{n_k}(\omega) \to f(\omega)$ for almost every $\omega \in \Omega$. This implies $f(\omega) \in F(\omega)$, μ.a.e, and f measurable. Therefore, $f \in S_F^p$. We are done. ■

Corollary 3.4 *Let X be a Banach space and $1 \le p < \infty$. For every measurable and integrably bounded multifunction $F : \Omega \to P_{fc}(X)$, S_F^p is weakly compact if and only if X is reflexive.*

Remark 3.5 According with Theorem 3.2, Theorems 5.2 and 5.5 of [16] hold for any Banach space and any $p \in [1, +\infty)$. While the hypothesis "X is sequentially weakly complete" should be added in Theorem 5.4 of same reference; since according with Rosenthal l_1 dichotomy [22], a Banach space is reflexive if and only if it is sequentially weakly complete and contains no copy of l_1.

Remark 3.6 The weak compactness of S_F plays a key role in the existence of mild solution of evolution inclusions ([17]); with hypothesis $F : \Omega \to P_{\omega kc}(X)$. In [15], in an attempt of giving a different approach in the context of reflexive Banach spaces, the weak compactness is replaced by closedness and boundness. According with 3.4, this is a particular case of [17].

4 Weak limits of sequence of measurable multifunctions

In this section we generalice a result due to Castaing [4] and Papageorgiou [19].

Theorem 4.1 *Let X be a Banach space with X^* having the Radon-Nikodym Property. Let $\{F_n\}$ be a uniformly integrable sequence of measurable multifunctions $F_n : \Omega \to P_{\omega kc}(X)$ satisfying the following conditions:*

(i) *For every $A \in \Sigma$, the set*

$$H_A = \bigcup_{n=1}^{\infty} \int_A F_n d\mu$$

is relatively weakly compact.

(ii) *Any bounded variation vector measure $m : \Sigma \to X$ verifying $m(A) \in \overline{co}(H_A)$ for all $A \in \Sigma$ admits a density in $L_X^1(\mu)$. Then there exists $F : \Omega \to P_{\omega kc}(X)$ integrably bounded and a sequence $\{F_{n_k}\}$ of $\{F_n\}$ such that $F_{n_k} \to F$ in $L_X^1(\mu)$.*

Proof. Since for each $n \in \mathbf{N}$; $F_n : \Omega \to P_{\omega kc}(X)$ is a measurable multifunction, we have that for each $n \in \mathbf{N}$, there is a set $N_n \in \Sigma$ such that $\mu(N_n) = 0$ and $F_n(\Omega \backslash N_n)$ is separable. If $N = \bigcup_{n=1}^{\infty} N_n$ then $\mu(N) = 0$ and the closed subspace Y generated by $\bigcup_{n=1}^{\infty} F_n(\Omega \backslash N)$ is separable. Now we define

$$G_n : \Omega \to P_{\omega kc}(Y)$$

by

$$G_n(\omega) = \begin{cases} F_n(\omega); & \omega \in \Omega \backslash N \\ \{0\}; & \omega \in N. \end{cases}$$

The sequence G_n is a sequence of measurable multifunctions satisfying $\bigcup_{n=1}^{\infty} \int_A G_n d\mu = H_A$ and since X^* has the Radon-Nikodym Property, by [23], every separable subspace of X has a separable dual. So Y^* is separable. Applying Theorem 5.1 of [4] we find a measurable multifunction

$$F : \Omega \to P_{\omega kc}(Y) \subset P_{\omega kc}(X)$$

and a subsequence G_{n_k} of G_n such that $G_{n_k} \overset{\omega}{\to} F$ in $L^1_X(\mu)$. Since for each $n \in \mathbb{N}$; $G_n = F_n$ μ.a.e, we conclude that $F_{n_k} \to F$ in $L^1_X(\mu)$. We are done. ∎

An operator theoritical applications may be interesting.

Theorem 4.2 *Let X and Y be Banach spaces and $T : X \to Y$ a weakly compact operator. If $F_n : \Omega \to P_{\omega kc}(X)$ is a sequence of μ measurable multifunctions which is uniformly integrable and bounded in $L^1_X(\mu)$, then there is a subsequence $\{F_{nk}\}$ of $\{F_n\}$ and $G : \Omega \to P_{\omega kc}(Y)$ such $TF_{nk} \overset{\omega}{\to} G$ in $L^1_X(\mu)$.*

Proof. Since $T : X \to Y$ is a weakly compact operator, the factorization scheme of [7] provides a reflexive Banach space Z and a pair of bounded linear operator T_1, T_2 such that $T = T_2 \circ T_1$; with $T_1 : X \to Z$ & $T_2 : Z \to Y$. If we concentrate ourselve on $T_1 F_n : \Omega \to P_{\omega kc}(Z)$; we find out that $\{T_1 F_n\}_{n=1}^{\infty}$ is a sequence of bounded on $L^1_X(\mu)$ and uniformly integrable multifunctions. Hence for each $A \in \Sigma$, $\bigcup_{n=1}^{\infty} \{\int_A T_1 F_n d\mu\}$ is bounded in Z and, by reflexivity, relatively weakly compact. Since both Z and Z^* have the Radon-Nikodym Property, Theorem 4.1 implies the existence of a measurable multifunction $F : \Omega \to P_{\omega kc}(Z)$ and a subsequence $\{F_{n_k}\}$ of $\{F_n\}$ such that

$$\int_A \sigma(T_1 F_{n_k}, z^*) d\mu \to \int_A \sigma(F, z^*) d\mu$$

for each $z^* \in Z^*$.

Now, given $y^* \in Y^*$, $y^* T_2 \in Z^*$ and, on the other hand;

$$\sigma(T F_{n_k}, y^*) = \sigma(T_1 F_{n_k}, y^* T_2)$$

and

$$\sigma(T_2 F, y^*) = \sigma(F, y^* T_2).$$

So the conclusion follows with $G = T_2 F$. ∎

It is worth to notice that by applying the above factorization scheme, Papageorgiou [16] has gotten the following result for separable Banach spaces. Since this result easily extend for arbitrary Banach spaces, we state it without separability asumption:

Theorem 4.3 *Let $F_n : \Omega \to P_{fc}(X)$ be a sequence of measurable multifunctions and $W \in P_{\omega kc}(X)$ such that $F_n(\omega) \subseteq W$ μ.a.e. for all $n \in \mathbf{N}$. Then there are $F : \Omega \to P_{\omega kc}(X)$ and a subsequence $\{F_{nk}\}$ of $\{F_n\}$ such that $F_{n_k} \overset{\omega}{\to} F$ in $L^1_X(\mu)$.*

5 Multimeasures and the Radon-Nikodym Property

Definition 1 Let $M : \Sigma \to P_{\omega kc}(X)$ be a multimeasure, and $\mu : \Sigma \to [0, +\infty)$ be a positive measure. M is called μ-representable if there is a μ-measurable multifunction $F : \Omega \to P_{\omega kc}(X)$ integrably bounded such that

$$M(A) = \int_A F d\mu \quad \forall A \in \Sigma.$$

Proposition 5.1 *Let $M : \Sigma \to P_{\omega kc}(X)$ be a multifunction μ-representable by F. Then*

(a) *$M(\Sigma) = \bigcup_{A\in\Sigma} M(A)$ is separable.*

(b) *F is unique.*

Proof.

(a) If there is a μ-measurable multifunction $F : \Omega \to P_{\omega kC}(X)$ so that F is integrably bounded and $\int_A F d\mu = M(A)$, $\forall A \in \Sigma$, then by definition there is $N \in \Sigma$ such that $\mu(N) = 0$ and $\bigcup_{\omega\in\Omega\setminus N} F(\omega)$ is separable. Let Y be the separable subspace generated by $\bigcup_{\omega\in\Omega\setminus N} F(\omega)$. Then for each selector f of F, we have $\int_A f d\mu \in Y$; which implies that $\bigcup_{A\in\Sigma} M(A)$ is separable.

(b) By (a) , we can suppose X separable. Now we apply Theorem III. 35 of [6]. ■

Theorem 5.2 *Let X be a Banach space. The following statements are equivalent:*

(a) *Both X and X^* have the Radon-Nikodym Property.*

(b) *For every complete finite measure space (Ω, Σ, μ) and any μ continuous bounded variation multimeasure $M : \Sigma \to P_{\omega kc}(X)$; with $M(\Sigma)$ separable, there is a μ-measurable integrably bounded multifunction $F : \Omega \to P_{\omega kc}(X)$ such that $M(A) = \int_A F d\mu$.*

(c) *For every probability space (Ω, Σ, μ), and every μ-continuous bounded variation multimeasure $M : \Sigma \to P_{\omega_k}(X)$, with $M(\Sigma)$ separable there is an integrably bounded multifunction $F : \Omega \to P_{\omega_k}(X)$ such that*

$$M(A) = \int_A F d\mu, \quad \forall A \in \Sigma.$$

Proof. $(a \Rightarrow b)$.

Since $M(\Sigma)$ is separable, there is no loss of generality assuming X separable.

Since X^* has the Radon-Nikodym Property, then it is separable and the proof follows as either in [5] or [12].

$(b \Rightarrow a)$. If X does not have the Radon Nikodym Property, there is a separable subspace Y of X which lacks such a property. So there is a vector measure $m : \Sigma \to Y$; bounded variation and $m << \mu$; which is not μ-representable where $\Omega = [0,1]$; Σ the Borel σ-algebra and μ the Lebesgue measure. Therefore the Radon-Nikodym Property on X is a sufficient condition.

Suppose X^* lacks the Radon-Nikodym Property. By the proposition in [11], if $\Omega = \{-1,1\}^{\mathbb{N}}$ is the Cantor group and μ the normalized Haar measure on Ω, there is a subset $H \subseteq L^1_X(\mu)$ such that

(i) H is uniformly bounded.

(ii) $\{\int_A f d\mu\}_{f\in H}$ is relatively weakly compact for each $A \in \Sigma$.

(iii) H is not relatively weakly compact in $L^1_X(\mu)$.

Now we define

$$G = \left\{ f = \sum_{i=1}^{n} g_i \mathcal{X}_{A_i};\ g_i \in H,\ A_i \in \Sigma;\ A_i \cap A_j = \emptyset\ \forall i \neq j\ \&\ \bigcup_{i=1}^{n} A_i = \Omega \right\}.$$

Since G is a bounded decomposable subset of $L^1_X(\mu)$ so is $\overline{G}$. So there is a μ-measurable integrably bounded multifunction $F' : \Omega \to P_f(X)$ so that $S^1_{F'} = \overline{G}$. Take $F = \overline{c_0}F'$: Then F is integrably bounded and by Krein-Smulyan theorem $M(A) = \{\int_A F d\mu\}$ is a weakly convex valued multimeasure. Since $H \subseteq S^1_F$, this set is not relatively weakly compact and by Theorem 3.2; $F(\omega)$ is not weakly compact μ.a.e.

$(a \Rightarrow c)$. Take $M : \Sigma \to P_{\omega_k}(X)$ satisfying hypothesis (c) . Since X has the Radon-Nikodym property, by [24], $clM(A)$ is convex for each $A \in \Sigma$. So $M(A)$ is convex, and weakly compact for each $A \in \Sigma$. Therefore, we have reduced the problem to the implication $a \Rightarrow b$.

$(c \Rightarrow a)$. If $M : \Sigma \to P_{\omega_k}(X)$ is a multimeasure such that $\forall A \in \Sigma$,

$$M(A) = \int_A F d\mu \quad \text{for some} \quad F : \Omega \to P_{\omega_k}(X),$$

integrably bounded, then by [18], $clM(A)$ is convex for each $A \in \Sigma$. So

$$M(A) = \overline{c_0}\left(\int_A F d\mu\right) = \int_A \overline{c_0} F d\mu;$$

and by the implication $b \Rightarrow a$, the proof is over. ∎

Remark 5.3 Above result improve the conclusion in [14] Theorem 5.3, with a rather different proof.

If we put $S_M = \{m : \Sigma \to X;\quad m \in c_a(X),\ \ m(A) \in M(A)\ \ \forall A \in \Sigma\}$, when the multimeasure M is compact valued, the following holds.

Theorem 5.4 *For a Banach space X, the following are equivalent statements*

(a) *X has the Radon-Nikodym property.*

(b) *If $M : \Sigma \to P_k(X)$ is a μ-continuous bounded variation multimeasure for which S_M is compact in $c_a(X)$ then there is an integrably bounded multifunction $F : \Omega \to P_{kc}(X)$ such that*

$$M(A) = \int_A F d\mu.$$

Proof. Suppose X has the Radon-Nikodym property. Then by [24] Theorem 2.7, $M(\Sigma)$ is relatively compact in X. Therefore $M(\Sigma)$ is separable.

For each $m \in S_M$, there is $f_m \in L^1_X(\mu)$ so that

$$m(A) = \int_A f_m d\mu \quad \forall A \in \Sigma$$

and S_M isomorphic to $\{f_m\}_{m \in S_M} \subseteq L^1_X(\mu)$. Furthermore, by [10] we have that for each $A \in \Sigma$,

$$M(A) = \left\{ \int_A f_m d\mu \right\}_{m \in S_M}.$$

Since $\{f_m\}_{m \in S_M}$ is a decomposable compact subset of $L^1_X(\mu)$; we have that $\{f_m\}_{m \in S_M}$ is also separable in $L^1_X(\mu)$; and hence we can suppose X separable. So by [13], there is an integrably bounded multifunction $F : \Omega \to P_{fc}(X)$ such that $S^1_F = \{f_m\}_{m \in S_M}$. Therefore

$$M(A) = \int_A F d\mu \quad \text{for each} \quad A \in \Sigma$$

with S^1_F compact in $L^1_X(\mu)$. This implies $F(\omega)$ weakly compact μ.a.e and by [2] proposition 7, $F(\omega)$ is actually compact μ.a.e.

Conversely, if (b) holds, it happens for single vector measure, the very definition of the Radon Nikodym property. ■

Since the unit ball of $L^\infty([0,1])$ is not compact in $L^1[0,1]$, the multimeasure M can be represented by a compact valued multifunction without being S_M a compact subset of $c_a(X)$, as it is shown in next Theorem.

Theorem 5.5 *Let X be a Banach Space. The following are equivalent:*

(a) *For every $F : [0,1] \to P_{\omega kc}(X)$ μ-measurable respect to the Lebesgue measure, with $|F| \in L^\infty(\mu)$, $M(A) = \int_A F d\mu$ is compact for each $A \in \Sigma$.*

(b) *X is finite dimensional.*

Proof. $(b \Rightarrow a)$. If X is finite dimensional, then for each $A \in \Sigma$, $\int_A F d\mu \subset B(0, M)$, where $M = \sup$ ess $|F|$. This implies $M(A)$ compact.

$(a \Rightarrow b)$. Suppose X is infinite dimensional. Then there is a convex separable subset W in B_X such that W is not compact; which implies the existence of a sequence $\{x_k\} \subset W$ without any convergent subsequence. Put $F : [0,1] \to P_{\omega kc}(X)$ such that $F(\omega) \equiv W$ $(\omega \in [0,1])$. Then for each $k \in \mathbb{N}$, $f_k \equiv x_k$ is a measurable selection of F and, if μ is the Lebesgue measure on $[0,1]$ then for any $t > 0$, $\left\{ \int_0^t f_k d\mu \right\}$ is not compact in X, which implies that $M : [0,1] \to P_{\omega kc}(X)$ is not compact valued. ■

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