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A SURVEY ON THE CLASSIFICATION PROBLEM OF FACTORS OF VON NEUMANN ALGEBRA.

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CONTENTS

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The present survey article is a thoroughly revised version of the earlier one published in NOTAS DE MATEMATICA, Nº 116, 1991. Unlike the earlier version, here we give sufficient motivations of the various concepts and developments in the classification theory and devote a section to describe the matrix representation of operators, which plays a key role in the whole work. The fantastic achievements of many mathematicians in the classification theory are described here in an easily accessible form, as far as possible, to a general functional analyst.

A SURVEY ON THE CLASSIFICATION PROBLEM OF FACTORS OF VON NEUMANN ALGEBRAS

BY

T.V. PANCHAPAGESAN*

In the famous work 'On Rings of Operators' [19] published by Murray and von Neumann in 1936 is given the type classification theory of factors along with a general measure theoritic construction of those of type I and II, leaving aside the problem of determining the existence of type III-factors. Later, in [25] von Neumann modified the construction given in [19] and gave the construction of type III-factors with some examples in the same. Introducing an isomorphism invariant, known as the property , Murray and von Neumann succeeded in constructing a pair of non-isomorphic type II_1-factors in [21], but couldn't obtain such results for type III-factors. Only in 1956, Pukánsky [28] could produce two non-isomorphic type III-factors, one satisfying the property (L) introduced by him and the other failing this property. Since the publication of the work of Pukánsky [28], many mathematicians got interested in the construction of new non-isomorphic type II_1 or type III -factors on a separable Hilbert space H, which finally culminated in the remarkable discoveries of Powers [27] and Dusa McDuff [18], who showed respectively the existence of a continuum of non-isomorphic type III- and type II_1-factors on H. Since the family of all von Neumann algebras on H has the cardinality of continuum, their results are optimum in this direction.

The results of Powers motivated the work of Araki and Woods [2] on infinite tensor product of type I-factors, which in turn played a crucial role in motivating

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the study of Connes [6]. Making use of the Tomita-Takesaki's theory of modular Hilbert algebras and the unitary co-cycle Radon-Nikodým theorem obtained earlier in [7], Connes classified all type III-factors in terms of type III_λ-factors for $\lambda \in [o,1]$ and derived the structure theorem of type III_λ-factors for $\lambda \in [o,1)$ in his famous memoir [6], which fetched him the Fields medal for that decade. (In this connection, we should not fail to mention that Takesaki [36] too independently obtained the structure theorem for the more general type III von Neumann algebras in the same time, using some of the earlier results of Connes).

The aim of the present survey article is to narrate some of the most important discoveries in the classification theory since the publication of [19]. Though many of the results cited above are treated in the monographs and texts on von Neumann algebras, because of their very advanced nature they are practically inaccessible to a general functional analyst. Therefore, in the present survey we try to give a description of the fantastic achievements of these mathematicians in an easily accesible form, as far as possible, by restricting our study just to that of factors on separable Hilbert spaces only.

By a Hilbert space we mean a complex infinite dimensional one. A separable Hilbert space is thus infinite dimensional and separable. An operator on a Hilbert space H is bounded and linear. An inner product preserving linear transformation from one Hilbert space onto another is called an isomorphism of these Hilbert spaces.

1.-Definition of a factor. Throughout this article H denotes a separable Hilbert space, unless otherwise mentioned. L(H) denotes the Banach algebra of all operators on H with respect to the operator norm and is a C*-algebra. The inner product of H is denoted by $<.,.>$.

For $T \in L(H)$ and $x,y \in H$, let $p_{x,y}(T) = |<Tx,y>|$. Then the locally convex topology τ_w defined by the semi-norms $\{p_{x,y} : x,y \in H\}$ is called the *weak operator* *topology* and is weaker than the norm topology of L(H). It is well known that these two topologies coincide if and only if H is finite dimensional.

A τ_w-closed *-subalgebra R of L(H) is called a *von Neumann algebra* if R contains the identity operator.

Historically speaking, von Neumann introduced this class of operators in [23] and called it a ring of operators. But, later it was justly suggested by Dieudonne to call these classes as von Neumann algebras (vide Introduction of [9]).

For the general theory of von Neumann algebras, the classical reference is [9]. However, an easily readable account is found in [26], which gives an introductory treatment of these algebras. The reader may also refer to Chapter VII of [22].

Given a *-subalgebra R of L(H), the set $\{T \in L(H) : TR=RT,\ R \in R\}$ is called the *commutant* of R and is denoted by R'. The commutant $(R')'$ of R' is called the *double commutant* of R and is denoted by R''. For a *-subalgebra R of L(H), it is easy to observe that R' is a τ_w-closed *-subalgebra of L(H), containing the identity operator and hence is a von Neumann algebra.

Thanks to the double commutant theorem of von Neumann [23], we can give the definition of a von Neumann algebra just algebraically, without using any topological ingredient. In fact, this is the approach adopted by Dixmier in [9]. Now,let us state the double commutant theorem.

THEOREM 1.1. A *-subalgebra R of L(H) is a von Neumann algebra if and only if $R = R''$.

Motivated by certain problems in quantum mechanics and the theory of infinite dimensional representations of groups, Murray and von Neumann made an extensive study of operator algebras in [19]. In this context, they defined the notion of a factor of a von Neumann algebra and their study led to the classification of factors as type I_n, $n \in \mathbb{N}$, type I_∞, type II_1, type II_∞ and type III. In the first paper [19] of 1936, they could give a general method to construct type I and II-factors and thus obtained some examples of these factors. But, as they point out explicity in [19], they were not aware of the existence of any type III-factors at that time. All these details we shall elaborate in the sequel.

To give the definition of a factor we proceed as follows. Suppose C is a non-void subset of L(H). Let R(C) be the smallest von Neumann algebra in L(H), which contains C. Since L(H) itself is a von Neumann algebra and the intersection of a non-void family of von Neumann algebras is a von Neumann algebra, obviously R(C) is well defined. R(C) is called the *von Neumann algebra generated* by C. Let Σ be the class of all von Neumann algebras on H. If we partially order Σ by the inclusion, then L(H) and CI are respectively the greatest and the smallest elements in Σ, where I is the identity operator on H. Given R_1, R_2 in Σ, the supremum $R_1 \vee R_2$ and the infimum $R_1 \wedge R_2$, of R_1 and R_2 with respect to this partial ordering exist in Σ and are given by

$$R_1 \vee R_2 = \mathrm{R}(R_1, R_2)$$

and

$$R_1 \wedge R_2 = R_1 \cap R_2.$$

Clearly, we have

$$(R_1 \vee R_2)' = R_1' \wedge R_2' \qquad (1)$$

Now, by the double commutant theorem we also have

$$(R_1 \wedge R_2)' = R_1' \vee R_2' \quad . \qquad (2)$$

We say that R_1 and R_2 form a *factorisation* if R_1 and R_2 commute elementwise and $R_1 \vee R_2$ = L(H). The notion of factors arises then as a particular case of factorisation and is given by the following

DEFINITION 1.1. For $R \in \Sigma$, suppose $R \vee R'$ = L(H) so that R and R' form a factorisation. Then R is called a *factor*.

If R is a factor, then by the double commutant theorem R' is also a factor. Besides, as $(L(H))' = \mathbb{C}I$, by (1) and Theorem 1.1 a von Neumann algebra R on H is a factor if and only if its centre is $\mathbb{C}I$.

Before ending this section, we remark that all the definitions and results given above for *-algebras of operators on H also hold when H is of arbitrary dimension.

2.-**Relative dimension function of a factor.** Given a factor $\mathbf{M}$ on H, we construct a relative dimension function $D_{\mathbf{M}}$ of $\mathbf{M}$ and use the range of $D_{\mathbf{M}}$ to classify $\mathbf{M}$ as of type I, II or III. We prefer to use the relative dimension functions of a factor to describe the classification in stead of the normal trace, since this approach is more direct and elementary. The definitions and results mentioned in this section are found in [19,22].

Throughout this section $\mathbf{M}$ denotes a factor on H and $P(\mathbf{M})$ is the set of all projections belonging to $\mathbf{M}$. Besides, H can be a unitary space or a separable Hilbert space.

For two projections E and F on H, it is natural to consider E to be smaller than F in size if dim EH $\leq$ dim FH, where dim denotes the dimension of the subspace. Clearly, this is equivalent to say that there exists a linear isometry U from EH onto a closed subspace of FH. On extending U linearly to the whole of H by defining $U(H \setminus EH) = o$, we observe that dim EH $\leq$ dim FH if and only if there exists a partial isometry $U \in L(H)$ with its initial domain EH and final domain a closed subspace of FH. This observation motivaties the following

DEFINITION 2.1. For $E, F \in P(\mathbf{M})$, we write $E \precsim F$ if there exists a partial isometry $U \in \mathbf{M}$ with its initial domain EH and final domain F_1H, where $F_1 \in P(\mathbf{M})$ and $F_1 \leq F$. If $F_1 = F$, then we write $E \sim F$. If $E \precsim F$ and $E \not\sim F$, then we write $E \precnsim F$ or simply, $E \prec F$.

In other words, for projections E and F in $P(\mathbf{M})$, we say $E \precsim F$ if and only if there exists a partial isometry $\mathbf{U} \in \mathbf{M}$ such that $U^*U = E$ and $UU^* = F_1 \leqq F$. Besides, ' ' is an equivalence relation on $P(\mathbf{M})$.

Note that dim EH= dim FH, if $E \sim F$. But, the converse is not true in general, since dim EH= dim FH doesn't guarantee the existence of a partial ismoetry $U \in \mathbf{M}$ for which $U^*U = E$ and $UU^* = F$ hold.

Now we can state the following result on $\precsim$.

THEOREM 2.1. For $E,F \in P(\mathbf{M})$, $E \precsim F$ and $F \precsim E$ imply $E \sim F$. Besides, given $E,F \in P(\mathbf{M})$ one and only one of the relations $E \prec F$, $E \sim F$ or $F \prec E$ holds.

Motivated by the concepts of finite and infinite sets in set theory, we say that $E \in P(\mathbf{M})$ is *finite* (relative to $\mathbf{M}$), if $E \nsim F$ for any subprojection F of E belonging to $\mathbf{M}$; i.e. if $E \sim F \leqq E$ and $F \in P(\mathbf{M})$, then $F = E$. We say that E is *infinite* (relative to $\mathbf{M}$), if it is not finite. In this case, there exists a $F \in P(M)$ such that $E \sim F < E$.

The following lemma of [19] is a key result on which are based the definitions of a fundamental sequence and a relative dimension function of $\mathbf{M}$.

LEMMA 2.1. Let $E,F \in P(\mathbf{M})$, $E \neq o$ and F finite. Then there exists a finite sequence $\{G_i\}_1^p$ of mutually orthogonal projections in $\mathbf{M}$ such that

(i) $E \sim G_1 \sim G_2 \sim \ldots \sim G_p$,

(ii) $\sum_1^p G_i \leqq F$ and

(iii) $F - \sum_{i=1}^{p} G_i \prec E$.

Besides, this number p is uniquely determined by E and F, and is denoted by $[\frac{F}{E}]$.

Note that $[\frac{F}{E}] \in \mathbb{N} \cup \{o\}$ and $[\frac{F}{E}] = o$ if $F \prec E$.

A projection $E \in \mathbf{M}$ is said to be *minimal* if for any projection $F \in \mathbf{M}$ with $F \leqq E$ we have $F = o$ or $F = E$. Since these projections play an exceptional role, this fact is taken care of in the following

DEFINITION 2.2. Let $=\{E_1,E_2,\ldots\}$ be an infinite sequence in P(**M**) with each $E_i \neq$ o and finite. If $[\frac{E_i}{E_{i+1}}] \geq 2$ for all i, then we say that is a *fundamental sequence* in **M**. If E is a minimal projection in **M**, then also $= \{E\}$ is called a *fundamental sequence* in **M**.

We note that the minimal projections are finite ones in **M**. In [19] Murray and von Neumann establish that there exists al least one fundamental sequence in **M**, if there exists a non-zero finite projection in **M**. Given a fundamental sequence in **M**, for two finite non-zero projections E and F in **M** is defined a positive real number $(\frac{F}{E})$ by the following

THEOREM 2.2. If $= \{E_i\}_1^\infty$ is a fundamental sequence in **M** and E,F ε P(M),E,F non-zero and finite, then

$$\lim_i \frac{[\frac{F}{E_i}]}{[\frac{E}{E_i}]} = (\frac{F}{E})$$

exists as a positive real number, where by lim we mean the value at i = 1 when consits of a minimal projection.

In [19] is developed a functional calculus for ( ) , which suggests the following concept.

DEFINITION 2.3. A function D:P(**M**)→ [o,∞] is called a *relative dimension function* of **M** if

(i) D(o)= o;

(ii) E ~ F => D(E)= D(F) and

(iii) EF= o => D(E + F)= D(E) + D(F)

for projections E,F in **M**.

If **M** has a non-zero finite projection E, we can construct a fundamental sequence in **M** by Lemma 8.13 of [19] and define a relative dimension function D_M using $(\frac{F}{E}$ for F ε P(M). More precisely, we have the following

THEOREM 2.3. Let **M** be a factor on H. Then:

(i) If no non-zero finite projection belongs to **M**, let

$$D_{\mathbf{M}}(F) = \begin{cases} o \text{ if } F = o \\ \infty \text{ if } F \in P(M), F \neq o. \end{cases}$$

If **M** has a non-zero finite projection E, let

$$D_{\mathbf{M}}(F) = \begin{cases} o \text{ if } F = o \\ (\frac{F}{E}) \quad \text{if } F \in P(M),\ F \neq o,\ F \text{ finite} \\ \infty \text{ if } F \in P(\mathbf{M}) \text{ and } F \text{ is infinite} \end{cases}$$

where is a fundamental sequence in **M**. In this case, $D_{\mathbf{M}}$ is independent of the fundamental sequence used in the definition.

In both the cases, $D_{\mathbf{M}}$ thus defined is a relative dimension function of **M**.

(ii) If D' is another relative dimension function of **M**, then $D' = cD_{\mathbf{M}}$ for some constant $c \in (o,\infty)$.

(iii) $E \sim F \Longleftrightarrow D_{\mathbf{M}}(E) \lesseqgtr D_{\mathbf{M}}(F)$, where $E \succ F$ if $F \prec E$.

(iv) The range Δ of $D_{\mathbf{M}}$ satisfies the following properties:

(a) $\Delta \quad [o,\infty]$.

(b) $o \in \Delta$; $\sup \Delta = t_o > o$ and $t_o \in \Delta$.

(c) For $t_1, t_2 \in \Delta$, $t_2 > t_1 \Rightarrow t_2 - t_1 \in \Delta$.

(d) If $\{t_i\}_1^\infty \quad \Delta$ with $\sum_1^\infty t_i \leq t_o = \sup \Delta$, then $\sum_1^\infty t_i \in \Delta$.

(v) The only sets Δ which satisfy (a)-(d) of (iv) are the following ones:

(I_n): $\Delta = \{k\delta : k = o,1,\ldots,n\}$ for $n \in \mathbb{N}$, $o < \delta < \infty$

$(I\ )$: $\Delta = \{k\delta : k = o,1,2,\ldots,\infty\}$, $o < \delta < \infty$

(II_1): $\Delta = \{t : o \leq t \leq t_o\}$, $o < t_o < \infty$

$(II\ )$: $\Delta = \{t : o \leq t \leq \infty\}$

(III): $\Delta = \{o, \infty\}$.

If we normalise $D_{\mathbf{M}}$ by a suitable positive multiple (vide (ii)) we can take $\delta =$ 1 in (I_n), and (I_∞) and $t_o = 1$ in (II_1).

Then we have $\Delta=\{o,1,\dots,n\}$ for (I_n); $\Delta=\{o,1,2,\dots,\infty\}$ for (I_∞) and $\Delta=\{t:o \leqq t \leqq 1\}$ for (II_1).

By an *isomorphism* Φ from A onto B, where A and B are *-algebras, we mean a *-isomorphism. If a *-algebra A satisfies a property (P) and if this property(P)holds for an isomorphic image of A, then we say that (P) is an *isomorphism invariant*.It turns out that the range Δ of $D_{\mathbf{M}}$ is an isomorphism invariant and hence is used for the classification of factors. Thus we have the following

DEFINITION 2.4. A factor **M** on H is said to be of *type* I_n, $n \in \mathbb{N}$, *type* I_∞,*type* II_1, *type* II_∞ or *type* III according as the range Δ of the corresponding normalided relative dimension function $D_{\mathbf{M}}$ of **M** is given by $\Delta=\{o,1,\dots,n\}$, $\Delta=\{o,1,2,\dots,\infty\}$, $\Delta=\{t:o \leqq t \leqq 1\}$, $\Delta=\{t:o \leqq t \leqq \infty\}$ or $\Delta=\{o,\infty\}$, respectively. When **M** is of type I_n or I_∞, we say that **M** is of *type* I or *discrete*; when **M** is of type II_1 or II_∞, we say that **M** is of *type* II or *continuous* and finally, when **M** is of type III we also say that **M** is *purely infinite*. When **M** is not of type III, we say **M** is *semi-finite*.

Now we can state the following

THEOREM 2.4. For a factor **M** on H only one of the types I_n, I ,II_1,II_∞ or III can occur. Besides, any two isomorphic factors on separable Hilbert spaces have the same type. If **M** is a factor, then **M** is of type I(respy.,of type II,of type III) if and only if **M'** is so.

NOTE 2.1. The classification theory of Murray and von Neumann [19]has been extended later to arbitrary von Neumann algebras R on Hilbert spaces of arbitrary diemnsion and accordingly, there exist unique mutually orthogonal central projections P_1,P_2 and P_3 of R such that $R\,P_1$ is of type I if $P_1 \neq o$; RP_2 is of type II if $P_2 \neq o$ and RP_3 is of type III if $P_3 \neq o$. Besides, $P_1+P_2+P_3=I$. When $P_3=o$, we say that R is semi-finite; when P_2 is finite and non-zero, we say that RP_2 is of type II_1 and when P_2 is infinite, we say that RP_2 is of type II_∞. For details, the reader may refer. to [9], [26], etc.

3.-Matrix representation of an operator. The results of this section play a crucial role in the sequel. Suppose $H=\sum_1^\infty + H_i$, where all the spaces H_i are isomorphic to the fixed Hilbert space H_1. Here we represent each $T \in L(H)$ as a matrix (T_{ij}) of operators in $L(H_1)$.

Let $U_i:H_1 \to H_i$ be an isomorphism. Considering H_i as a closed subspace of H, it is easy to observe that the adjoint U_i^* is a linear mapping from H onto H_1 such that $U_i^*(H - H_i)= o$ and U_i^* maps H_i isometrically onto H_1. Consequently, $U_i^*U_i$ is the identity operator on H_1 and $U_iU_i^*$ is the projection P_i of H onto H_i. For $T \in L(H)$, let $T_{ij}= U_i^*TU_j$. Then $T_{ij}:H_1 \to H_1$, linear and $\|T_{ij}\|=\|U_i^* TU_j\| \leqq \|T\|$. Thus $(T_{ij})_{ij}$ is a matrix of operators in $L(H_1)$ such that $\| T_{ij}\| \leqq \| T \|$ for all i,j.

Conversely, suppose (T_{ij}) is a matrix of operators $T_{ij} \in L(H_1)$ such that $T_{ij}= U_i^* TU_j$ for some linear mapping $T:H \to H$. Then, for $x \in H$, we have $\sum_{i=1}^\infty \| \sum_{j=1}^\infty T_{ij}U_j^*x\|^2 = \sum_{i=1}^\infty \| \sum_{j=1}^\infty U_i^*TP_jx\|^2= \sum_{i=1}^\infty \|U_i^*Tx\|^2= \sum_{i=1}^\infty \|U_i^*P_iTx\|^2=\sum \| P_iTx\|^2= \|Tx\|^2$, since U_i^* is an isometry on H_i.

Thus $T \in L(H)$ if and only if there exists a constant $C > o$, such that

$$\sum_{i=1}^\infty \|\sum_{j=1}^\infty T_{ij}U_j^*x\|^2 \leqq C^2\|x\|^2 \qquad (1)$$

for $x \in H$. When (T_{ij}) with $T_{ij}= U_i^*TU_j$ satisfies (1), we say that (T_{ij}) is *bounded*.

Thus, the matrix (T_{ij}) of operators $T_{ij} \in L(H_1)$ with $T_{ij}= U_i^* TU_j$ for a linear mapping $T:H \to H$ is bounded if and only if $T \in L(H)$. In this case, we describe T as the matrix (T_{ij}) and observe that

$$Tx= \sum_{j=1}^\infty TP_jx= \sum_{j=1}^\infty \sum_{i=1}^\infty P_iTP_jx= \sum_{j=1}^\infty \sum_{i=1}^\infty U_iU_i^*T\,U_jU_j^*x$$

$$= \sum_{j=1}^\infty \sum_{i=1}^\infty U_iT_{ij}U_j^*x$$

for $x \in H$. This shows that the correspondence $T \sim (T_{ij})$ is a bijective correspondence from $L(H)$ onto all bounded matrices (T_{ij}) of operators $T_{ij} \in L(H_1)$, with $T_{ij}=U_i^*TU_j$.

4.-Factors of type I_n, n $\in \mathbb{N}$ and I_∞. For a unitary space H of dimension n, L(H) is a type I_n-factor. If H is separable, then L(H) is a type I_∞-factor.

If **M** is a type I_n-factor on H, n $\in \mathbb{N} \cup \{\infty\}$, then there exists an orthogonal family $\{E_i\}_1^n$ of minimal equivalent projections in **M** such that $\sum_1^n E_i = I$. Let $H_i = E_i H$. For $T \in$ **M'**, let $T \sim (T_{ij})$, with $T_{ij} = U_{1i}^* T U_{1j}$, where the U_{1i} are partial isometries in **M** with the initial domain H_1 and final domain H_i. Since $T \in$ **M'**, $TE_i = E_i T$ and hence T is reduced by $E_i H$. Let $T_o = T|E_1 H$. Then it is easy to observe that $T_{ij} = \delta_{ij} T_o$ for i,j= 1,2,...,n. Thus $T \sim (\delta_{ij} T_o)$ (vide Section 3). Since **M= M''**, the factor M on the space H must consist of all matrices $A \sim (A_{ij})$ (bounded in the sense of Section 3, if n= ∞) of operators $A_{ij} = U_{1i}^* A U_{1j} \in L(H_1)$, which commute with all matrices of the form $(\delta_{ij} T_o)$, being $T_o = T|H_1$ and $T \in$ **M'**. An operational calcules of these matrices readily shows that $T_o A_{ij} = A_{ij} T_o$ for all i,j and hence $A_{ij} \in$ **(M'**$E_1)'$. But, **(M'**$E_1)'$ can be shown to coincide with E_1**M**E_1 and hence **M** consists of all the matrices of the form (A_{ij}), with $A_{ij} \in E_1$**M**E_1 (and bounded, if n=∞). On the other hand, by the spectral theorem the von Neumann algebra E_1**M**E_1 on H_1 is the norm closure of the linear span of all the projections in E_1**M**E_1. Since E_1 is a minimal projection, this shows that E_1**M**$E_1 = \mathbb{C}E_1$. Consequently, **M**$=\{(\lambda_{ij}) : \lambda_{ij} \in \mathbb{C}$ and bounded if n= ∞}. Thus there exists an isomorphism $U : H \to \sum_1^n + H_1$ such that $UAU^{-1} = (A_{ij}) = (\lambda_{ij})$ for A ε **M**. In particular, **M** is isomorphic to L(K), where $K = \{(\lambda_i)_i^n : \lambda_i \in \mathbb{C}$. $\sum_i^n |\lambda_i|^2 < \infty\}$.

Thus we have proved the following

THEOREM 4.1. Suppose **M** is a type I_n factor on a unitary space H or on a separable Hilbert space H with n $\in \mathbb{N} \cup \{\infty\}$. Then there exists a closed subspace H_1 of H and an isomorphism V from H onto $\sum_1^n + H_1$ such that $VAV^{-1} = (A_{ij}) = (\lambda_{ij})$ for A ε **M**. In particular, **M** is isomorphic to L(K), with dimK= n.

From the above theorem we observe that a factor of type I_n is isomorphic to L(K) with dimK= n, n $\in \mathbb{N} \cup \{\infty\}$. Thus all factors of type I_n (respy., of type I_∞ on a separable Hilbert space) are isomorphic to each other.

5.-Structure theorem for type II_∞-factors. Every type II_∞-factor can be obtained as the tensor product of a type II_1-factor and $L(H_2)$ for a suitable separable Hilbert space H_2. In fact, suppose **M** is a type II_∞-factor on H. Then, by definition, there exists $E \in P(\mathbf{M})$, $E \neq o$, and finite such that $E\mathbf{M}E$ is a type II_1-factor on EH. Consequently, by a classical result on type II_∞-factors there exists an orthogonal sequence of projections $\{E_i\}_1^\infty$ in **M** such that $I = \sum_1^\infty E_n$ and $E \sim E_1 \sim E_2 \sim \ldots$ Then H is isomorphic to $\sum_1^\infty + EH$. Consequently, as discussed in Section 4, it can be shown that $M = \{(A_{ij}) : A_{ij} \in E\mathbf{M}E$, and the matrix is bounded in the sense of Section 4$\}$. This matrix representation is written in the form $M = (E\mathbf{M}E) \times L(H_2)$ where $H_2 = \{(\lambda_i)_1^\infty : \lambda_i \in C, \sum_1^\infty |\lambda_i|^2 < \infty\}$. (Vide § 2 of Chapter I of [9]). Thus we obtain the following structure theorem of type II_∞-factors.

THEOREM 5.1. Every type II_∞-factor **M** on a separable Hilbert space H is of the form $M_1 \times L(H_2)$ for a suitable type II_1-factor $\mathbf{M}_1$, where H_2 is a separable Hilbert space.

Thus the study of type II_∞-factors is reduced to that of type II_1-factors.

6.-Measure theoretic construction of type I and type II-factors. In [25] von Neumann modified the construction given earlier in [19] and constructed factors of type I, II and III on a separable Hilbert space. Till the appearance of [25] the existence of a type III-factor was unknown. In this section we follow [25] and restrict our attention to the construction of type I and type II-factors only, while in the next section we shall take up the study of type III-factors.

Let (X, S, μ) be a σ-finite measure space with $\mu(X) > o$ and let C be an utmost countable subfamily of S such that S is the σ-algebra generated by C, $\bigcup_{C \in C} C = X$, and $\mu(C) < \infty$ for $C \in C$. Further, we assume that for $x, y \in X$ such that $x \in E \Leftrightarrow y \in E$ for all $E \in C$, then $x = y$. In the sequel, all the measure spaces considered are supposed to satisfy the above assumptions.

DEFINITION 6.1. Let G be any at most countable group. We say that G is an (X, S, μ)

-group if the following conditions hold:

(i) For each $g \in G$ there exists a bijective map $T_g: X \to X$ given by $T_g x = xg$ such that $T_{g_2} T_{g_1} = T_{g_2 g_1}$ for $g_1, g_2 \in G$. (This implies $T_e x = x$ and $(T_g)^{-1} x = T_{g^{-1}} x$ for $x \in X$, where e is the identity of G).

(ii) For A X and for $g \in G$, let $Ag = \{xg : x \in A\} = T_g(A)$. Then $A \in \mathcal{S}$ implies $Ag \in \mathcal{S}$.

(iii) The measures μ_g on $\mathcal{S}$ defined by $\mu_g(A) = \mu(Ag)$ for $A \in \mathcal{S}$ and $g \in G$ are absolutely continuous with respect to μ (i.e. $\mu_g \ll \mu$ for all $g \in G$).

The following definition is essential for the construction of factors.

DEFINITION 6.2. Let G be an $(X, \mathcal{S}, \mu)$-group. We say that

(i) G is *free* if $g \neq e$ and $A = \{x \in X : xg = x\}$, then $\mu^*(A) = o$, where μ^* is the outer measure induced by μ on $P(X)$;

(ii) G is *ergodic* if $A \in \mathcal{S}$ such that $\mu(Ag \,\Delta\, A) = o$ for all $g \in G$ implies that either $\mu(A) = o$ or $\mu(X \quad A) = o$;

(iii) G is *measurable* if there exists a σ-finite measure ν on $\mathcal{S}$ such that $\nu \equiv \mu$ (i.e. $\nu \ll \mu$ and $\mu \ll \nu$) and $\nu(A) = \nu(Ag)$ for all $A \in \mathcal{S}$ and $g \in G$ (i.e. ν is G-invariant); and

(iv) G is non-measurable if G is not measurable.

In the sequel we assume that G is an utmost countable $(X, \mathcal{S}, \mu)$-group, which is free and ergodic.

Let

$H_\mu^G = \{F(x,g) : X \times G \to C$ such that $F(.,g)$ is $\mathcal{S}$-measurable for each $g \in G$ and $\sum_{g \in G} \int_X |F(x,g)|^2 d\mu(x) < \infty\}$, with the inner product given by

$$<F_1, F_2> = \sum_{g \in G} \int_X F_1(x,g) F_2(x,g) d\mu(x).$$

Clearly, $H_\mu^G = \sum_{g \in G} + L^2(\mu)$. The hypotheses on $(X, \mathcal{S}, \mu)$ imply that $L^2(\mu)$ is non-trivial and separable. As G is at most countable, H_μ^G is either a unitary space or a separable Hilbert space.

With the aim of constructing a factor we define certain linear transformations on H_μ^G as below.

DEFINITION 6.3. Let $F(.,.) \in H_\mu^G$, $g_o \in G$ and ψ a bounded S-measurable complex function on X. Let $\frac{d\mu_g}{d\mu}$ be the Radon-Nikodým derivative of μ_g with respect to μ for $g \in G$. Then we define

(a) $(\bar{U}_{g_o} F)(x,g) = \left(\frac{d\mu_{g_o}(x)}{d\mu}\right)^{\frac{1}{2}} F(xg_o, gg_o)$;

(b) $(\bar{V}_{g_o} F)(x,g) = F(x, g_o^{-1} g)$;

(c) $(\bar{W}F)(x,g) = \left(\frac{d\mu_{g^{-1}}(x)}{d\mu}\right)^{\frac{1}{2}} F(xg^{-1}, g^{-1})$;

(d) $(\bar{L}_\psi F)(x,g) = \psi(x)F(x,g)$;

and

(e) $(\bar{M}_\psi F)(x,g) = \psi(xg^{-1})F(x,g)$.

The folowing theorem is established in [25].

THEOREM 6.1.

(i) $\bar{U}_g$, $\bar{V}_g$, $\bar{W}$, $\bar{L}_\psi$ and $\bar{M}_\psi$ as in Definition 6.3 are bounded operators on H_μ^G and $\bar{U}_g, \bar{V}_g$ and $\bar{W}$ are even unitary.

(ii) Let $\Omega = \{ \bar{U}_g, \bar{L}_\psi : g \in G, \psi$ as in Definition 6.3 but arbitrary$\}$ and $\hat{\Omega} = \{ \bar{V}_g, \bar{M}_\psi :$ $g \in G$, ψ as in Definition 6.3 but arbitrary$\}$. Then $R(\Omega) = (\hat{\Omega})'$ and $R(\hat{\Omega}) = \Omega'$, where $\Omega' = \{T \in L(H_\mu^G): TA = AT$ for $A \in \Omega\}$, etc.

(iii) $R(\Omega)$ and $R(\hat{\Omega})$ are *spatially isomorphic* and the spatial isomorphism is implemented by $\bar{W}$ in the sense that the isomorphicm $\Phi: R(\Omega) \to R(\hat{\Omega})$ is given by $\Phi(A) = \bar{W} A \bar{W}^{-1}$, $A \in R(\Omega)$. Each is the commutant of the other.

(iv) Since G is free and ergodic, $R(\Omega)$ and $R(\hat{\Omega})$ are factors.

NOTATION 6.1. In the sequel we shall denote $R(\Omega)$ and $R(\hat{\Omega})$ by $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$, rspectively.

In order to define relative dimension functions D_M and $D_{M'}$ of $\mathbf{M}(X,G,\mu)$ and $M'(X,G,\mu)$, we make use of the results of Section 3.

Since $H_\mu^G = \sum_{g \varepsilon G} + L^2(\mu)$, by Section 3 every $T \varepsilon L(H_\mu^G)$ has a matrix representation of the form $(T_{g,h})_{g,h \varepsilon G}$ where each $T_{g,h}$ is a bounded linear operator on $L^2(\mu)$ and $\|T_{g,h}\| \leq \|T\|$ for g,h ε G. However, when T belongs to $\mathbf{M}(X,G,\mu)$ or $\mathbf{M}'(X, G, \mu)$ we can describe T more specifically. To this end, we define the following mappings on $L^2(\mu)$.

DEFINITION 6.4. For $f \varepsilon L^2(\mu)$ let

(A) $(U_g f)(x) \doteq (\frac{d\mu_g(x)}{d\mu})^{\frac{1}{2}} f(xg)$ for $g \varepsilon G$.

and

(B) $(L_\psi f)(x) = \psi(x)f(x)$ for any bounded S-measurable complex function ψ on X.

Then it is known that U_g, L_ψ are bounded operators on $L^2(\mu)$ and U_g is even unitary. Recall that $L^2(\mu)$ is a unitary space or a separable Hilbert space.

Now we can describe $(T_{g,h})$ as below.

THEOREM 6.2. Let T be a bounded operator on H_μ^G with its marix representation $(T_{g,h})_{g,h \varepsilon G}$. Then:

(i) $T \varepsilon \mathbf{M}(X,G,\mu)$ if and only if $T_{g,h} = L_{\psi_{g^{-1}}(x)} U_{h^{-1}g}$ and

(ii) $T \varepsilon \mathbf{M}'(X,G,\mu)$ if and only if $T_{g,h} = L_{\psi_{gh^{-1}}}(xh^{-1})$ where ψ_g is a bounded S-measurable complex function on X.

NOTATION 6.2. In the terminology of Theorem 6.2 we shall write $T\ [[\psi_g(x)]]_{g \varepsilon G}$ for $T \varepsilon \mathbf{M}(X,G,\mu)$ (respy., for $T \varepsilon \mathbf{M}'(X,G,\mu)$). (Note that the totality of the functions $\{\psi_g : g \varepsilon G\}$ is the same for both $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$.)

Making use of the results in [19] and [25] we can determine the types of $M(X,G,\mu)$ and $M'(X,G,\mu)$, when G is measurable and the following theorem describes their type classifcation.

THEOREM 6.3. Suppose G is an utmost countable, free, ergodic and measurable $(X,\mathcal{S},\mu)$-group, with $\nu \equiv \mu$, where ν is a σ-finite G-invariant measure on $\mathcal{S}$. Let ν^* be the outer measure induced by ν. Let $\frac{d\nu(x)}{d\mu} = k(x)$ and let $T \approx [[\psi_g(x)]]_{g \in G}$ for $T \in \mathbf{M}(X,G,\mu)$ or for $T \in \mathbf{M}'(X,G,\mu)$. Let $D_{\mathbf{M}}(E) = {}_X\,\psi_e(x)k(x)d\mu(x)$ and $D_{\mathbf{M}'}(E') = {}_X\,\tilde{\psi}_e(x)k(x)d\mu(x)$ where $E \approx [[\psi_g(x)]]_{g \in G}$, $E' \approx [[\tilde{\psi}_g(x)]]_{g \in G}$, $E \in \mathbf{M}(X,G,\mu)$ and $E' \in \mathbf{M}'(X,G,\mu)$. (Since G is ergodic, the function $k(x)$ is uniquely determined but for a positive constant multiple) Then the following hold:

(i) $D_{\mathbf{M}}$ is a relative dimension function of $\mathbf{M}(X,G,\mu)$ and there exists a projection $E \in \mathbf{M}(X,G,\mu)$ with $o < D(E) < \infty$. Thus $\mathbf{M}(X,G,\mu)$ is a non-type III-factor. A similar result holds for $D_{\mathbf{M}'}$ and $\mathbf{M}'(X,G,\mu)$.

(ii) If $\nu(X) < \infty$ and if there exists $x \in X$ with $\nu^*(\{x\}) > o$, then there exists N X with $\nu^*(N) = o$ such that the one-point sets $\{y\} \in \tilde{\mathcal{S}}$ and $\nu^*(\{y\}) = \nu^*(\{x\})$ for all $y \in X$ N, where $\tilde{\mathcal{S}}$ is the Lebesgue completion of $\mathcal{S}$ with respect to ν. Thus we can take $X = \{x_1, x_2, \ldots, x_n\}$ (say) with $\nu^*(x_i) = \nu^*(x_j) = \varepsilon$ for $i \neq j$, with $o < \varepsilon < \infty$. Then $\mathbf{M}(X,G,\mu)$ nad $\mathbf{M}'(X,G,\mu)$ are of type I_n. Besides, $\frac{1}{\varepsilon}D_{\mathbf{M}}$ and $\frac{1}{\varepsilon}D_{\mathbf{M}'}$ are the normalised relative dimension functions of $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ respectively.

(iii) If $\nu(X) = \infty$ and $\nu^*(\{x\}) > o$ for some $x \in X$, then a result similar to (ii) holds with $X = \{x_i\}_1^\infty$ and $\nu^*(\{x_i\}) = \nu^*(\{x_j\})$ for $i \neq j$. (Note that $\nu^*(\{x\}) < \infty$). Consequently, $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are of type I_∞.

(iv) If $\nu^*(\{x\}) = o$ for each $x \in X$, then $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are of type II_1 if $\nu(X) < \infty$ and of type II_∞ if $\nu(X) = \infty$. When $\nu(X) < \infty$, $\frac{1}{\nu(X)} D_{\mathbf{M}}$ and $\frac{1}{\nu(X)} D_{\mathbf{M}'}$ are the normalised relative dimension functions of $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$, respectively.

With the general construction established in the above, following [19] we now give some examples of type II-factors.

EXAMPLES 6.1 **(Type II-factors)**. Let X be the set $X_\infty = \mathbb{R}$ or set $X_1 = [o,1)$, the set

$\mathbb{R}$ mod 1. Let $\mathcal{S} = \mathcal{B}(X)$, the σ-algebra of all Borel sets in X and let μ be the Borel restriction of the Lebesgue measure.

We take G to be any one of the following additive groups.

(α) $G_\Theta = \{m + n\Theta : m, n \in Z\}$, Θ an irrational number.

(β) $G_{rat} = \{$all rational numbers in $\mathbb{R}\}$.

(γ) $G_{rat,p} = \{\frac{m}{p^n} : m \in Z$, $n = o,1,2,\ldots\}$, where p is any given number 2,3,... (not necessarily prime!).

For $g \in G$ we define $xg = x + g$ for $x \in X_\infty$ and $xg = x + g \pmod 1$ for $x \in X_1$. Then it can be shown that G is a free, ergodic $(X,\mathcal{S},\mu)$-group. Since μ is translation invariant in X_∞ as well as in X_1, we have that G is measurable with $\nu = \mu$. Thus by Theorem 6.3, $\mathbf{M}(X_\infty, G, \mu)$ and $\mathbf{M}'(X_\infty, G, \mu)$ are type II -factors, while $\mathbf{M}(X_1, G, \mu)$ and $\mathbf{M}'(X_1, G, \mu)$ are type II_1-factors.

NOTE 6.1. Since the family of the groups G_Θ is uncountable, apparently we have given above a continuum of type II_1 and type II_∞-factors on a separable Hilbert space. But, all the type II_1-factors given in Examples 6.1 are spatially isomorphic to each other. (Vide Section 8).

7.-Construction of type III-factors. When the $(X,\mathcal{S},\mu)$-group G is non-measurable von Neumann showed in [25] that the factors $\mathbf{M}(X,\mathcal{S},\mu)$ and $\mathbf{M}'(X,\mathcal{S},\mu)$ of Notation 6.1 are of type III. The following result gives a sufficient condition for G to be non-measurable.

THEOREM 7.1. Suppose G is a countable $(X,\mathcal{S},\mu)$-group which is free and ergodic. Let $G_o = \{g \in G : \mu(A) = \mu(Ag)$ for all $A \in \mathcal{S}\}$. Then G_o is a free $(X,\mathcal{S},\mu)$-group and is measurable with $\nu = \mu$. If G_o is ergodic and $G_o \neq G$, then G is non-measurable.

THEOREM 7.2. If G is a free, ergodic, non-measurable $(X,\mathcal{S},\mu)$-group, then the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ of Section 6 (vide Notation 6.1) are of type III.

Making use of Theorems 7.1 and 7.2, the following example of a type III-factor on a separable Hilbert space is given in [25].

EXAMPLE 7.1 (A type III-factor). Let $X=\mathbb{R}$ and $\mathcal{S}=\mathcal{B}(\mathbb{R})$, the σ-algebra of all Borel subsets of $\mathbb{R}$. Let μ be the Borel restriction of the Lebesgue measure in $\mathbb{R}$. We take G to be the group of transformations $\{T(\rho,\sigma): \rho > o, \rho, \sigma$ rational$\}$, where

$$T(\rho,\sigma)x = \rho x + \sigma, x \in \mathbb{R}$$

and the group operation of G is given by composition of transformations. Clearly, G is a free $(X,\mathcal{S},\mu)$-group and is countably infinite. The group G_o of Theorem 7.1 is given by

$$\begin{aligned} G_o &= \{T(\rho,\sigma): \mu(T(\rho,\sigma)A) = \mu(A) \text{ for } A \in \mathcal{S}\} \\ &= \{T(\rho,\sigma): \rho\mu(A) = \mu(A) \text{ for } A \in \mathcal{S}\} \\ &= \{T(1,\sigma): \sigma \text{ rational}\} \end{aligned}$$

and hence $G_o \neq G$. Besides, G_o is isomorphic to G_{rat} (vide Examples 6.1) which is ergodic and hence G_o itself is ergodic. Therefore, by Theorem 7.1, G is non-measurable and hence, by Theorem 7.2, $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are type III-factors.

Note that $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are spatially isomorphic by Theorem 6.1(iii).

Before proceeding further, we make some comments on [19] and [25]. In [19], Murray and von Neumann gave the type classification theory of factors on H and constructed the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ assuming that G is an at most countable $(X,\mathcal{S},\mu)$-group, which is free and ergodic such that $\mu(Ag)=\mu(A)$ for all $A \in \mathcal{S}$. In other words, in the terminology of Theorem 6.3, they assumed $\nu=\mu$ and hence were led to the construction of factors of type I and II only. At that time, they wondered whether there exists any type III-factor at all. It was only in 1940, von Neumann modified the construction given in [19] introducing the terminology of measurable and non-measurable $(X,\mathcal{S},\mu)$-group and thus obtained in [25] the construction of factors of type I, II and III. These results have been described above in Section 6 and in the present section.

8.-Hyperfinite type II_1-factors. In [21] Murray and von Neumann answered affirmatively the question whether there exist at least two non-isomorphic type II_1-factors on H. This they achived by studying the class of type II_1-factors known as approxi mately finite type II_1-factors. The main results of [21] will be presented in this section as well as in the next two sections. Here we restrict our study to isomorphism property of these factors and show that the type II_1-factors in Examples 6.1 are spatially isomorphic.

DEFINITION 8.1. A factor $\mathcal{R}$ on H is said to be hyperfinite (=approximately finite or ATI= almost type I) if there exists an increasing sequence $(\mathbf{M}_i)_1^\infty$ of discrete factors M_i of finite type I_{n_i} (so that n_i divides n_{i+1}) such that $\mathcal{R}$ is the von Neumann algebra generated by $\bigcup_1^\infty M_i$.

Murray and von Neumann use the terminology 'approximately finite' and Dixmier [9] calls it hyperfinite, which is also referred to as ATI by Connes.

The sequence (n_i) involved in Definition 8.1 doesn't play any role in determining the algebraic type of $\mathcal{R}$ when $\mathcal{R}$ is a type II_1-factor. In fact, the following result is obtained in [21].

THEOREM 8.1. Hyperfinite type II_1-factors exist on H and any two hyperfinite type II_1-factors on separable Hilbert spaces are isomorphic.

In Theorem 6.3 we can guarantee that $\mathbf{M}(X,\mathcal{S},\mu)$ and $\mathbf{M}'(X,\mathcal{S},\mu)$ are hyperfinite when G satisfies some more conditions. In fact, the following theorem has been given in [21].

THEOREM 8.2. Suppose in Theorem 6.3 the $(X,\mathcal{S},\mu)$-group G further satisfies one of the following conditions:

(*) There exists a sequence $G_1 \quad G_2 \quad \ldots$ of finite subgroups of G such that $G=\bigcup_1^\infty G_i$.

(**) G is abelian.

Then the factors $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are hyperfinite type II_1-factors, whenever they are of type II_1.

A detailed proof of Theorem 8.2 corresponding to the condition (*) is found in [21], but the proof corresponding to (**) is postponed to a future publication, which however didn't take place. Nevertheless, later in 1963 Dye [11] obtained the said result as a particular case of a more general situation.

Returning to the factors $\mathbf{M}(X_1,G_\Theta)$, $\mathbf{M}(X_1,G_{rat})$ and $\mathbf{M}(X_1,G_{rat,p})$ of Examples 6.1, we observe that they are hyperfinite type II_1-factors by Theorem 8.2 as the groups are abelian (while G_{rat} and $G_{rat,p}$ also satisfy (*)) and hence by Theorem 8.1 they are isomorphic. Now by Theorem XI of [19], Theorem XI of [20] and by the isomorphism between these factors we deduce the following

COROLLARY 8.1. The factors $\mathbf{M}(X_1,G)$ of Examples 6.1 are spatially isomorphic hyperfinite type II_1-factors, where G is any one of the groups G_Θ, G_{rat} and $G_{rat,p}$.

9.-**A simple group-theoretic construction of type II_1-factors.** In [21] Murray and von Neumann gave a simplified version of the measure theoretic construction of Section 6, imposing a stringent condition on the group G to obtain type II_1-factors. Before explaining this construction, we make the remark that this construction played a very crucial role in the works of Dusa McDuff [17,18] and Sakai [30,31] to obtain a continuum of non-isomorphic type II_1 and type III-factors. Vide Sections 12 and 13.

Suppose $X=\{x_o\}$, $S=\{\{x_o\},\emptyset\}$ and $\mu(\emptyset)=o$, $\mu(\{x_o\})=1$. Given a countably infinite group G, let $x_o g=x_o$ for all $g\in G$, so that G is an (X,S,μ)-group. In this case, H_μ^G reduces to the separable Hilbert space $\ell^2(G)$, which is given by

$$\ell^2(G)=\{\ f:G\to C \text{ such that } \sum_{g\in G}|f(g)|^2<\infty\}$$

with the inner product

$$<f_1,f_2> = \sum_{g\in G} f_1(g)f_2(g).$$

Besides, in this case the unitary operators U_{g_o}, V_{g_o}, W of Definition 6.3 assume the simple forms $\hat{U}_{g_o}$, $\hat{V}_{g_o}$, $\hat{W}$, respectively, where

$$(\hat{U}_{g_o}f)(g)=f(gg_o)$$

$$(\hat{V}_{g_o}f)(g)=f(g_o^{-1}g)$$

and $$(\hat{W}f)(g) = f(g^{-1})$$

for g_o, $g \in G$ and $f \in \ell^2(G)$.

Then by Theorem 6.1(i), $\hat{U}_{g_o}$, $\hat{V}_{g_o}$ and $\hat{W}$ are unitary operators on $\ell^2(G)$. Since the bounded S-measurable functions on X reduce to constant functions the von Neumann algebras $R(\Omega)$ and $R(\hat{\Omega})$ of Theorem 6.1(ii) reduce to those generated by $\{\hat{U}_g : g \in G\}$ and $\{\hat{V}_g : g \in G\}$, respectively. Let us denote them by $\mathcal{U}(G)$ and $\mathcal{V}(G)$, respectively . Then $\mathcal{U}(G)$ and $\mathcal{V}(G)$ are spatially isomorphic to each other by $\hat{W}$ and one is the commutant of the other. Since these algebras play an important role in the construction of type II_1 -and type III- factors of later sections, we give the following

NOTATION 9.1. $\mathcal{U}(G)$ and $\mathcal{V}(G)$ denote the von Neumann algebras generated by $\{\hat{U}_g : g \in G\}$ and $\{\hat{V}_g : g \in G\}$, respectively.

THEOREM 9.1. $(\mathcal{U}(G))' = \mathcal{V}(G)$ and $(\mathcal{V}(G))' = \mathcal{U}(G)$. Besides, $\mathcal{U}(G)$ and $\mathcal{V}(G)$ are spatially isomorphic and the spatial isomorphism is implemented by $\hat{W}$.

Since $X = \{x_o\}$ with $\mu(\{x_o\}) = 1$ and $x_o g = x_o$ for all $g \in G$, evidently G is neither free nor ergodic. Thus one is led to find some other conditions, now on the group G, to ensure that $\mathcal{U}(G)$ and $\mathcal{V}(G)$ are factors. To this end, Murray and von Neumann introduced the following concept in [21].

DEFINITION 9.1. A group G is called an infinite conjugacy class group (in abbreviation, an ICC-group) is for each $g \neq e$ the conjugacy class $C_g = \{h^{-1}gh : h \in G\}$ is infinite.

Obviously, an ICG-group is a non-commutative infinite group.

Now we can state the following interesting

THEOREM 9.2. For a countable group G, $\mathcal{U}(G)$ and $\mathcal{V}(G)$ are factors on the separable Hilbert space $\ell^2(G)$ if and only if G is an ICC-group. In this case, $\mathcal{U}(G)$ and $\mathcal{V}(G)$ are type II_1-factors. If G satisfies besides the condition (*) of Theorem 8.2, then these are hyperfinite type II_1-factors.

As an application of the last part of the above theorem, we give below an example of a hyperfinite type II_1-factor as a $\mathcal{U}(G)$.

EXAMPLE 9.1. Suppose G is the subgroup of the permutation group of $\mathbb{N}$ formed by all those permutations which leave all but a finite number of elements fixed. Then G is an ICC-group and $G = \bigcup_{1}^{\infty} G_n$ with $G_n \uparrow$, where G_n is the subgroup of all those permutations which leave all but $\{1,2,\ldots,n\}$ fixed. Consequently, by Theorem 9.2 the factors (G) and (G) are hyperfinite type II_1-factors.

10.-Example of a non-hyperfinite type II_1-factor . All the type II_1-factors constructed in the earlier sections turn to be hyperfinite and thus isomorphic to each other by Theorem 8.1. Then the following question arises naturally: Does there exist any non-hyperfinite type II_1-factor on H? Murrray and von Neumman answered this question affirmatively in [21] by introducing an isomorphism invariant called the property and then constructing a factor on H failing the property .

DEFINITION 10.1. We say that a type II_1-factor **M** on H has the property if for each $\varepsilon > o$ and for each finite set $\{T_1, T_2, \ldots, T_n\}$ of elements in **M** there exists a unitary $U = U(T_1, T_2, \ldots, T_n) \in \mathbf{M}$ with $Tr_{\mathbf{M}}(U) = o$ and

$$[[U^{-1} T_k U - T_k]] < \varepsilon \quad \text{for } k = 1,2,\ldots,n$$

where $[[A]] = (Tr_{\mathbf{M}}(A^*A))^{\frac{1}{2}}$ and $Tr_{\mathbf{M}}$ is the relative trace of **M**.

Here the relative trace $Tr_{\mathbf{M}}$ is an extension of $D_{\mathbf{M}}$ to all hermitian elements in M with $Tr_{\mathbf{M}}(I) = 1$ and satisfying certain properties. (Vide [21]).

THEOREM 10.1. The property is an isomorphism invariant. If **M** is a hyperfinite type II_1-factor on H, then **M** satisfies the property . Thus all hyperfinite type II_1-factors on separable Hilbert spaces satisfy the property .

In [21] Murray and von Neumann introduced a sufficient condition on the ICG-group G to ensure that (G) be not hyperfinite. Let us state this result.

THEOREM 10.2. Let G be a countable ICC-group and suppose there exists a set F G with the following properties:

(i) There exists a $g_1 \in G$ such that

$$F \cup g_1 F g^{-1} = G \quad \{e\}$$

and

(ii) There exists a $g_2 \varepsilon G$ such that the sets F, $g_2 F g_2^{-1}$ and $g_2^{-1} F g_2$ are disjoint.

Then the factors (G) and $\mathcal{V}$(G) do not possess the property .

As an application of Theorems 10.1 and 10.2 we give below the construction of a non-hyperfinite type II_1-factor.

EXAMPLE 10.1 **(A non-hyperfinite type II_1-factor)**. Let G be the free group generated by two elements a and b. Clearly, G is a countable ICC-group. Let F be the set of those $g \varepsilon G$ which when written as a power product of a and b of minimum length end with a^n, $n = \pm 1, \pm 2, \ldots$ It is an easy exercise to verify the properties (i) and (ii) of Theorem 10.2 for the set F. Consequently, by Theorem 10.2 the type II_1-factors (G) and (G) do not satisfy the property Γ and hence are non-hyperfinite by Theorem 10.1.

The above example, and Theorems 8.1 and 10.1 imply the following

THEOREM 10.3. There exist at least two non-isomorphic type II_1-factors on a separable Hilbert space H, one being hyperfinite and the other non-hyperfinite.

Though Murray and von Neumann could provide more examples of non- hyperfinite type II_1-factors in [21], they could establish the existence of just two non-isomorphic type II_1-factors in terms of the property Γ . However, their method and ideas were exploited later by Dixmier and Lance [10] and Dusa McDuff [17,18], the latter being successful in constructing even an uncountable family of non-isomorphic type II_1-factors on a separable Hilbert space. Vide Section 12.

11.-**Pukansky's examples of two non-isomorphic type III-factors**. Though von Neumann constructed some type III-factors on a separable Hilbert space H in [25],he didn't study any isomorphism invariant to obtain some non-isomorphic type III-factors.The first contribution in this direction was due to Pukánsky [28], who introduced an isomorphism invariant called the property (L) and constructed two non-isomorphic type III-factors, one satisfying (L) and the other failing (L). Later, these examples played a fundamental role in the construction of an uncountable family of non-isomorphic type III-factors given by Powers [27], Sakai [30] and Connes [6]. Vide Sections 13,14 and 16.

Following Pukánsky [28] and [32] we present the construction of these factors of Pukánsky. For details, the reader may refer to [32].

DEFINITION 11.1. A von Neumann algebra R on H is said to satisfy the property (L) if there exists a sequence of unitary elements $(U_k)_1^\infty$ in R such that $U_k \to o$ in τ_w and $\|U_k A U_k^* - A\| \to o$ as $k \to \infty$.

EXAMPLE 11.1 **(Type III-factor $M_{(p_n)_1^\infty}$)**. Let $\Omega_n = \{o,1\}$, $n \in \mathbb{N}$ and let $X = \prod_1^\infty \Omega_n$. Let μ_n be the measure on $P(\Omega_n)$ defined by $\mu_n(\{o\}) = \frac{1-p_n}{2}$ and $\mu_n(\{1\}) = \frac{1+p_n}{2}$, where $o < \delta < p_n < 1-\delta$ for some fixed $\delta > o$. Let $\mu = \prod_1^\infty \mu_n$ be the product measure on the corresponding σ-algebra in X. Let $G = \{w=(w_n)_1^\infty : w_n \neq o$ occurs for a finite number of n's only$\}$. Then, with respect to addition given coordinatewise mod 2, G is a countable group. For $g \in G$ and $w \in X$, let $wg = w+g$, where $(w+g)_i = w_i + g_i \pmod 2$. Then it can be shown that G is a free, ergodic, non-measurable (X,S,μ)-group. Consequently, by Theorem 7.2, $\mathbf{M}(X,G,\mu)$ and $\mathbf{M}'(X,G,\mu)$ are type III-factors on the separable space H_μ^G. Besides, these factors satisfy the property (L) if $\frac{1-p_n}{2} = p$ and $\frac{1+p_n}{2} = q$ for all n. (Vide [28]). For later use, let us denote $\mathbf{M}(X,G,\mu)$ corresponding to $(p_n)_1^\infty$ by $M_{(p_n)}$.

EXAMPLE 11.2 (Type III-factor $\mathbb{P}$). Let G be the free group generated by two elements. Then G is countably infinite. For each $g \in G$, let $X_g = \{o,1\}$. Let μ_g be the measure defined by $\mu_g(\{o\}) = p$ and $\mu_g(\{1\}) = q$ with $o < p < q$ and $p+q = 1$. Let $X = \prod_{g \in G} X_g$ and let (X,S,μ) be the associated product measure space. Let

$$G_o = \{x=(x_g)_{g \in G} : x_g \neq o \text{ for a finite number of g's only}\}.$$

Let $\mathcal{G} = \{(x,g) : x \in G_o, g \in G\}$. For each elements $\alpha = (x^o, g_o) \in \mathcal{G}$, let us define the transformation

$$T_\alpha : X \to X \text{ given by}$$

$$T_\alpha x = x\alpha = (x_{g_o g} + x_g^o)_{g \in G}$$

where $x_{g_o g} + x_g^o = x_{g_o g} + x_g^o \pmod 2$. Then these mappings T_α are bijective on X. For the law of composition $\alpha\beta = (x,g_o)(y,h_o) = (x^{h_o} + y,\ g_o h_o) = r$, where $x^{h_o} = (x_{h_o g})_{g \in G}$, G is a semigroup. Since G has the identity element $(\mathbf{o},e)$ and the inverse of (x,g) in G is given by $(x^{g^{-1}}, g^{-1})$, we observe that G is a group. Also it can be shown that G is a free, ergodic and non-measurable (X,S,μ)-group. Consequently, the corresponding $\mathbf{M}(X,S,\mu)$ of Theorem 7.2 is a type III-factor on the separable space H_μ^G . Pukánski [28] showed that this factor fails the property (L). For later use, we shall denote this factor by $\mathbb{P}$. (Note that in the study of Pukánski [28] or that of Saks [32], the factor $\mathbb{P}$ is not distinguished for different pairs (p_1,q_1) and (p_2,q_2)).

Since the property (L) is an isomorphism invariant, the above examples imply the following

THEOREM 11.1. There exist al least two non-isomorphic type III-factors on a separable Hilbert space H, with one satisfying the property (L) and the other failing it.

12.- A continuum of non-isomorphic type II_1-factors. After the publication of "On Rings of Operators IV" in 1943, for many years were known only two non-isomorphic type II_1-factors. In 1963 J.Schwartz introduced an isomorphism invariant called the property (P) and using (P) distinguished two non-isomorphic non-hyperfinite type II_1-factors. After the publication of [34], many mathematicians got interested in the construction of new non-isomorphic type II_1-factors. Using the notions of central and hyper-central sequences in a type II_1-factor, Dixmier and Lance constructed two new examples of non-isomorphic type II_1-factors in [10]. Also were given new type II_1-factors by Wai-mee-Ching [4], Sakai [29] and Zeller-Meir [39]. Thus were known nine non-isomorphic type II_1-factors before the publications of [17] and [18] by Dusa McDuff.

In this section we briefly sketch some of the ideas used by Dusa McDuff [17, 18] and describe the construction of a continuum of non-isomorphic type II_1-factors following Sakai [32]. For details of the proof, the reader is recommended to refer to Sakai [32,pp.183-192].

Motivated by the hypothesis in Lemma 6.2.1 of [21] (vide Theorem 10.2 above), Dixmier and Lance introduced in [10] the notion of a residual subgroup H of G, according to which the hypothesis in the said lemma of [21] implies that $\{e\}$ is a residual subgroup of the ICC-group G. Since it is not known whether the finite product of residual subgroups is residual, Dusa McDuff defined in [17] a much stronger notion of strongly residual subgroups for which the said property holds and considered strongly residual sequences of subgroups in G. Using these notions, and proving many technically deep lemmas, she constructed an uncountable family of type II_1-factors in [18].

Let $G_1, G_2, \ldots; H_1, H_2, \ldots$ be two sequences of groups. We denote by $(G_1, G_2, \ldots; H_1, H_2, \ldots)$ the group generated by the G_i's and the H_i's with additional relations that H_i, H_j commute elementwise for $i \neq j$ and G_i, H_j commute elementwise for $i \leq j$. Let $L_1 = (Z, Z, \ldots; Z, Z, \ldots)$. Let L_k be defined inductively by $L_k = (Z, Z, \ldots; L_{k-1}, L_{k-1} \ldots)$ for $k > 1$.

Let π be a sequence of positive integers. Let $M_n(\pi) = \sum_{i=1}^{n} + L_{p_i}$ if $\pi = (p_1, p_2, \ldots)$; and $M_n(\pi) = \sum_{i=1}^{n} + L_{p_i}$ for $n \leq n_o$ and $M_n(\pi) = \sum_{i=1}^{n_o} + L_{p_i}$ for $n > n_o$, if $\pi = (p_1, \ldots, p_{n_o})$. Let $G(\pi) = (Z, Z, \ldots; M_1(\pi), M_2(\pi) \ldots)$. Then one has the following

THEOREM 12.1. If $\pi_1 = (p_i)$ and $\pi_2 = (q_i)$ are two sequences of positive integers such that $\pi_1 \neq \pi_2$ as sets, then $(G(\pi_1))$ and $(G(\pi_2))$ (vide Notation 9.1) are non-isomorphic type II_1-factors. None of these factors is hyperfinite.

13. Sakai's construction of uncountably many non-hyperfinite type III and type II-

factors. In the set up of W*-algebras, Sakai [30,32] extended the notion of central sequences and using the type III-factor $\mathbb{P}$ of Section 11 and the ICC-groups $G(\pi)$ of Section 12 above constructed a continuum of non-isomorphic type III-factors and deduced the existence of a continuum of non-isomorphic type II -factors. Let us briefly sketch the construction of Sakai [32].

A B*-algebra W is called a W*-algebra if there exists a Banach space W_* such that W is the Banach space dual of W_*. Let W denote a W*algebra in the sequel. The weak*-topology $\sigma(W,W_*)$ is called the σ-topology of W. A *-homomorphism $\Phi: W_1 \to W_2$ between two W*-algebras W_1 and W_2 is called a W*-homomorphism if it is continuous for the σ-topologies of W_1 and W_2.

Given a W*-algebra W, there exists a faithful W*-representation Φ of W into L(K) of some Hilbert space K(K can be finite dimensional or of arbitrary dimension) such that $\Phi(W)$ is a *-subalgebra closed in the weak operator topology of L(K)(vide Section 1.16 of Sakai [32]). Then we say that W has a faithful W*-representation (Φ,K). Besides, when W contains the identity, then $\Phi(W)$ is a von Neumann algebra on K.

Let $\mathbf{T}=\{\psi: \psi$ a σ-continuous positive linear form on $W\}$. For each $\psi \in \mathbf{T}$, let $\alpha_\psi(x)= (\psi(x^*x))^{\frac{1}{2}}$ for $x \in W$. The locally convex topology defined on W by the family $\{\alpha_\psi: \psi \in \mathbf{T}\}$ of semi-norms is called the s-topology of W. If $(X_n)_1^\infty$ is a uniformly bounded sequence in W, we say that (X_n) is a *central sequence* if $X_nX - XX_n \to o$ in s-topology for all $X \in W$.

From the theory of tensor products of von Neumann algebras (vide [32])we have that $\mathbb{P} \times \mathbf{M}$ is a factor for any factor $\mathbf{M}$ and is of type III, where $\mathbb{P}$ is as in Example 11.2.

Considering $A_i=\mathbb{P} \times (G(\pi_i))$, i= 1,2 as W*-algebras with identity and assuming them to be isomorphic for two different sequences of positive integers π_1 and π_2 (where $(G(\pi_i))$ are as in Section 12), Sakai [32] arrives at a contradiction after proving many intermediate lemmas, in which the above generalized notion of central sequences plays a key role.

THEOREM 13.1 (Sakai [32]). Let π_1 and π_2 be two sequences of positive integers which are different as sets. Let $G(\pi_1)$ and $G(\pi_2)$ be the ICC-groups constructed in Section 12 above. Then $\mathbb{P}$ x $(G(\pi_1))$ and $\mathbb{P}$ x $(G(\pi_2))$ are non-isomorphic type III-factors. Besides, these factors are non-hyperfinite (vide Definition 8.1). Thus there exists a continuum of non-isomorphic non-hyperfinite type III-factors on a separable Hilbert space.

NOTE 13.1. In the next section, following Powers [27] we also give the construction of a continuum of non-isomorphic hyperfinite type III-factors.

Since $\mathbb{P}$ is of type III, $\mathbb{P}$ is isomorphic to $\mathbb{P}$ x L(H) for a separable space H. Consequently, we deduce from Theorem 13.1 the following

THEOREM 13.2 (Sakai [32]). If H is separable and if $\pi_1, \pi_2, G(\pi_1)$ and $G(\pi_2)$ are as in Theorem 13.1, then L(H) $\bar{x}$ $(G(\pi_1))$ and L(H) $\bar{x}$ $(G(\pi_2))$ are non-isomorphic type II_∞-factors. Consequently, there exists a continuum of non-isomorphic type II -factors on a separable Hilbert space.

For the details of this section the reader may refer to Sakai [32,pp.193-202].

14.-Powers'construction of a continuum of non-isomorphic hyperfinite type III-factors. The construction of Powers [27] is based on the infinite product of a sequence of type I_2-factors, each one being considered as a C*-algebra with identity.The reader may refer to [12] for details of the construction of infinite tensor products of C*-algebras.

Suppose that $\mathcal{B}_n = \mathcal{B}$ is a type I_2-factor on a separable Hilbert space H for each $n \in \mathbb{N}$. Let (p_n) be a sequence of positive numbers $o < p_n < \frac{1}{2}$. For $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $(\alpha, \beta, r, \delta$ complex numbers), let

$$\psi_{p_n}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \alpha p_n + \delta(1-p_n).$$

Then ψ_{p_n} is a state (= positive linear from with $\|\psi_{p_n}\| = 1$) on B_n. Then let $\Psi_{(p_n)} = \overset{\infty}{\underset{1}{X}} \psi_{p_n}$ be the infinite product state of $(\psi_{p_n})_1^\infty$ on $\overset{\infty}{\underset{1}{X}} B_n$ (vide Section 1.23, Chapter 1 of Sakai [32]).

It is known that the state $\Psi_{(p_n)}$ induces a *-representation of $A = \overset{\infty}{\underset{1}{X}} B_n$ on a Hilbert space $H_{\Psi(p_n)}$ (vide p.40 of [32]).The **von Neumann algebra** $R = (\pi_{\Psi(p_n)}(A))''$ is called the W*-infinite tensor product of $(B_n)_1^\infty$ by the infinite product state $\Psi_{(p_n)}$. In this particular case, R is a factor.

If there exists a positive number δ with $\delta < p_n < \frac{1}{2} - \delta$ for each n, then it can be shown that $(\pi_{\Psi(p_n)}(A))'' = R$ is a type III-factor and that R is spatially isomorphic to the factor $M_{(p_n)}$ of Example 11.1.(Vide p.206 of [32]).

When we take $p_n = \lambda$ for all n with $o < \lambda < \frac{1}{2}$, the associated type III-factor $M_{(p_n)}$ is denoted by M_λ and is called the *Powers factor* of λ.

Introducing an isomorphism invariant called the property L_λ, Powers [27] obtained the following

THEOREM 14.1. For $\lambda_1, \lambda_2 \in [o, \frac{1}{2})$ with $\lambda_1 \neq \lambda_2$, their Powers factors M_{λ_1} and M_{λ_2} are non-isomorphic hyperfinite type III-factors. Consequently, there exists an uncountable family of non-isomorphic hyperfinite type III-factors on a separable Hilbert space.

The reader may note the difference between Theorems 13.1 and 14.1.

15.-ITPFI-factors. In [24] von Neumann observed that certain type III-factors could be obtained as an infinite tensor product of finite type I-factors. But no proof of his statement was given in any of his publications. Only in 1963, Bures [3] gave the proof of the above assertion along with a partial type classification of these infinite products. Such infinite tensor products of finite type I-factors are themselves factors and are called ITPFI-factors.

In the earlier section we saw that the Powers factors M_λ are ITPFI-factors of special type, with the constituent factors being of type I_2. Analysing the work of Powers [27], Araki and Woods studied in [2] the complete type classification of general ITPFI-factors by introducing the isomorphism invariants r_∞ and ρ. Without going into details of a rigorous definition of an ITPFI-factor **M**, let us simply mention some of the principal results of Araki and Woods [2], reformulated in a form comparable with the later results of Connes (vide the next section).

Let us denote the Powers factor M_λ by R_x, where $\lambda = \frac{x}{1-x}$ so that $x \in (o,1)$ as λ varies in $(o,\frac{1}{2})$. We define R_o as the type I_∞-factor and R_1 as the hyperfinite type II_1-factor on a separable Hilbert space H. (Note that these are unique upto isomorphism). The *asymptotic ratio set* $r_\infty(\mathbf{M})$ for an ITPFI-factor **M** defined in terms of the eigen values sets corresponding to the tracial states of the constituent factors is shown in [2] to be the same as the set $\{o \leqq x < \infty : \mathbf{M} \sim \mathbf{M} \times R_{f(x)}\}$, where '$\sim$' denotes 'isomorphic' and $f(x) = x$ for $o \leqq x \leqq 1$ and $f(x) = x^{-1}$ for $1 < x < \infty$. This result suggested the definition of $r_\infty(\mathbf{M}) = \{o \leqq x < \infty : \mathbf{M} \sim \mathbf{M} \times R_{f(x)}\}$ for an arbitrary factor **M**.

For two factors R_1 and R_2, it is known that $R_1 \times R_2$ is also a factor, which is of type III(respy., of type II) if R_1 or R_2 is of type III(respy., if one of them is of type II and the other is semi-finite).

Araki and Woods [2] proved that $r_\infty(\mathbf{M})$ is an isomorphism invariant and Araki [1] showed that $r_\infty(\mathbf{M})$ must be one of the sets $\{o\}, \{1\}$, $S_o = \{o,1\}$, $S_x = \{o,1,x^n : n \in Z\}$, $o < x < 1$ and $S_1 = [o,\infty)$. (Here the original notation is changed in terms of the invariant S of [6]).

THEOREM 15.1 ([2]). Except for the case S_o, $r_\infty(\mathbf{M}) = r_\infty(N)$ for two ITPFI-factors **M** and **N** implies that **M** and N are isomorphic.

The other isomorphism invariant $\rho(\mathbf{M})$ for an arbitrary factor **M** is given in [2] as below:

$$\rho(\mathbf{M}) = \{o \leq x < \infty : R_{f(x)} \sim R_{f(x)} \times \mathbf{M}\} .$$

Using the invariant ρ, is obtained in [2] the following interesting

THEOREM 15.2([2]). There exists a continuum of non-isomorphic ITPFI-factors in the class S_o.

It is interesting to observe that all the Powers factors $M_\lambda (o < \lambda < \frac{1}{2})$ belong to the class S_o, which are already known to be non-isomorphic ITPFI-factors. (Vide Theorem 14.1).

Thus for the first time, after the publication of [21], one had identified factors given by different constructions. The classification by r_∞ and ρ was generalized later by Krieger [14,15,16] to factors constructed from ergodic transformations For more information on ITPFI-factors the reader may refer to Woods [38].

16.Results of Connes [6] **and Takesaki** [36,37]. Using Tomita-Takesaki's theory of modular Hilbert algebras and the non-commutative integration theory, Connes [6] gave an isomorphism invariant T(**M**) for an arbitrary von Neumann algebra **M** and deduced from the following result Theorem 14.1 above and the non-isomorphism of the non-hyperfinite family $\mathcal{J}(G) \times M_\lambda, o < \lambda < \frac{1}{2}$, with G as in Example 10.1.

THEOREM 16.1([6]). If **M** is an ITPFI-factor, then $T_o \in T(\mathbf{M})$ if and only if $\exp(\frac{-2\pi}{T_o}) \in \rho(\mathbf{M})$, where $\rho(\mathbf{M})$ is the invariant given by Araki and Woods in [2].(Vide Section 15.)

Another interesting result about T(M) given in [6] is the following.

THEOREM 16.2([6]). Every subgroup G of $\mathbb{R}$ is the set T(**M**) of a countably decomposable factor **M**. When G is countably infinite, **M** is a factor on a separable Hilbert space. Besides, there exists a countably decomposable type III-factor **M** such that $T(\mathbf{M}) = \mathbb{R}$.

In [6] Connes gave another isomorphism invariant S(**M**) for a factor **M** and showed that **M** is semi-finite if and only if S(**M**)={1}. He also proved that the invariant T(**M**) doesn't determine S(**M**), in the sense that two factors $\mathbf{M}_1$ and $\mathbf{M}_2$ with

$T(M_1) \neq T(M_2)$ can have $S(M_1) = S(M_2)$.

THEOREM 16.3([6]). For an ITPFI-factor **M** of type III, $S(M)=r_\infty(M)$, where $r_\infty(M)$ is the asymptotic ratio set of **M**(vide Section 15).

Connes [6] also gave an examples of a non-ITPFI-factor **M** for which $S(M) \neq r_\infty(M)$ Also is given in [6] a non-hyperfinite ITPFI-factor, contrary to the factors M_λ of Powers.

The most important results of Connes [6] are those which characterize type III-factors. In this direction , he introduced the following

DEFINITION 16.1. Let **M** be a factor and $\lambda \in [0,1]$. We say that **M** is of type III_λ if

$$S(M) = \{0,1\} \text{ for } \lambda = 0; \; S(M) = \{0,1,\lambda^n : n \in Z\} \text{ for } 0 < \lambda < 1;$$

and $$S(M) = [0,\infty) \text{ for } \lambda = 1.$$

Since $0 \in S(M)$ for $\lambda \in [0,1]$, it follows that every type III_λ-factor is necesarily of type III. Connes [6] proved the following result in the reverse direction.

THEOREM 16.4([6]). For every countably decomposable factor **M** of type III there corresponds a unique $\lambda \in [0,1]$ such that **M** is of type III_λ so that every type III-factor **M** on a separable Hilbert space is of type III_λ for some unique $\lambda \in [0,1]$.

He also gave the following theorem of characterization of type III_λ-factors for $\lambda \in [0,1)$.

THEOREM 16.5([6]).

(i) All factors **M** of type III_λ for $\lambda \in (0,1)$ can be realized as the crossed product of a type II_∞-factor by a suitable automorphism Θ of .

(ii) A factor **M** of type III_0 is the crossed product of a von Neumann algebra of type II_∞ with nonatomic centre by a trace diminishing automorphism Θ of which is ergodic on the centre of .

It is known from [13] that a result similar to (i) and (ii)above doesn't hold for type III_1-factors.

The work of Connes [6] has many interesting other results, which we omit here for lack of space. Besides, Theorem 16.5 is a remarkable achievement in the classification theory of type III-factors and the work of Connes [6] is so important and original that it fetched him the Fields medal of that decade.

Finally, we include the structure theorem of arbitrary type III-von Neumann algebras obtained by Takesaki [36] independent of Connes [6].

THEOREM 16.6([36]). A von Neumann algebra R of type III is uniquely expressable as the crossed product of a von Neumann algebra R_0 of type II_∞ by a one-parameter automorphism group which leaves a trace of R_0 relatively invariant, but not invariant.

For details of this section refer to [6], [36] and [37].

Finally we observe that so far no structure theory of type II_1 factors is known, even though distinct uncountable families of non-isomorphic type II_1-factors have been constructed by different authors. Vide [5,18,31].

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GENERALIZATION OF A THEOREM OF ALEXANDROFF

BY

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The classical theorem of Alexandroff [1,p.590] states that a bounded regular complex valued additive set function μ defined on an algebra A of subsets of a compact topological space is σ-additive. Also it is known that such a set function μ admits a unique regular σ-additive extension to the σ-algebra generated by A. See Theorems III.5.13 and III.5.14 of Dunford and Schwartz [7]. The first result was generalized by Dinculeanu and Kluvanek [2,Theorem 3] for a regular locally convex space-valued additive set function μ defined on a ring of subsets of a locally compact space. In the present note we give an abstract set theoretic generalization of the above classical theorems for an additive G-valued set function μ, when G is a Hausdorff abelian topological group and μ satisfies, among other things, certain regularity property defined in terms of two fixed classes of sets $\mathcal{G}$ and $\mathcal{L}$. From this abstract study, we also deduce new results for G-valued additive set functions defined on rings of sets in arbitrary topological spaces.

1.-$(\mathcal{G},\mathcal{L})$-Alexandroff regularity and σ-additivity. In the sequel, G denotes an abelian Hausdorff topological group, with

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its operation denoted by +. $\mathcal{B}$ is a base of closed symmetric neighbourhoods of 0 in G and $\mathcal{R}$ is a ring of subsets of a set Ω ($\neq\emptyset$), unless otherwise stated. μ is a G-valued additive set function on $\mathcal{R}$. $\mathcal{G}$ and $\mathcal{L}$ are two fixed non-void families of subsets of Ω.

NOTATION 1.1. If $A \subset \Omega$ and W is a neighbourhood of 0 in G, we write $A \in \mathcal{R}_W(\mu)$ to mean that $\mu(E) \in W$ for all $E \in \mathcal{R}$ with $E \subset A$.

DEFINITION 1.2. The G-valued additive set function μ is said to be $\mathcal{G}$-Alexandroff (resp. $\mathcal{L}$-Alexandroff) regular on $\mathcal{R}$ if given $E \in \mathcal{R}$ and a neighbourhood W of 0 in G, there exists $U \in \mathcal{G}$ and $A \in \mathcal{R}$ (resp. $K \in \mathcal{L}$ and $B \in \mathcal{R}$) such that $E \subset U \subset A$ with $A \setminus E \in \mathcal{R}_W(\mu)$ (resp. $B \subset K \subset E$ with $E \setminus B \in \mathcal{R}_W(\mu)$). We say that μ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular on $\mathcal{R}$ if it is both $\mathcal{G}$-Alexandroff and $\mathcal{L}$-Alexandroff regular.

We note that $\emptyset \in \mathcal{L}$ if μ is $\mathcal{L}$-Alexandroff regular on $\mathcal{R}$ and that $\Omega \in \mathcal{G}$ if $\mathcal{R}$ is an algebra in Ω and μ is $\mathcal{G}$-Alexandroff regular on $\mathcal{R}$. At this stage we don't impose any condition on $\mathcal{L}$ or on $\mathcal{G}$.

By the additivity of μ the following result holds.

PROPOSITION 1.3. The G-valued additive set function μ on $\mathcal{R}$ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular on $\mathcal{R}$ if and only if for each $E \in \mathcal{R}$ and $W \in \mathcal{B}$ there exists $K \in \mathcal{L}$, $U \in \mathcal{G}$, $A,B \in \mathcal{R}$ such that $B \subset K \subset E \subset U \subset A$ with $A \setminus B \in \mathcal{R}_W(\mu)$.

DEFINITION 1.4. We say that $\mathcal{L}$ has the $\mathcal{G}$-c.c.p. (i.e. $\mathcal{L}$ has the countable compactness property relative to $\mathcal{G}$) if for each $K \in \mathcal{L}$, every countable covering of K by members of $\mathcal{G}$ has a finite subcovering.

THEOREM 1.5. Let μ be a G-valued $(\mathcal{G},\mathcal{L})$-Alexandroff regular additive set function on $\mathcal{R}$. If $\mathcal{G}$ is closed for intersection and $\mathcal{L}$ has the $\mathcal{G}$-c.c.p., then μ is σ-additive.

PROOF. Let $\{E_i\}_1^\infty$ be a disjoint sequence in $\mathcal{R}$ with $E=\bigcup_1^\infty E_i \in \mathcal{R}$. Let $W \in \mathcal{B}$. Then there exists a finite family $(q_i)_1^k$ of continuous quasinorms on G and $\varepsilon > o$ such that $W_\varepsilon = \bigcap_1^k B_{q_i}(o,\varepsilon) \subset W$, where $B_{q_i}(o,\varepsilon) = \{x \in G : q_i(x) < \varepsilon\}$. By the hypothesis of regularity there exists $K \in \mathcal{L}$, $U \in \mathcal{G}$, $A, B \in \mathcal{R}$ such that $B \subset K \subset E \subset U \subset A$ with $A \setminus B \in \mathcal{R}_{W_{\varepsilon/4}}(\mu)$. Besides, for each i there exists $U_i \in \mathcal{G}$, $A_i \in \mathcal{R}$ such that $E_i \subset U_i \subset A_i$ with $A_i \setminus E_i \in \mathcal{R}_{W_{\varepsilon/2^{i+3}}}(\mu)$, $= 1,2,..$Since $\mathcal{G}$ and $\mathcal{R}$ are closed for intersection, by replacing U_i by $U \cap U_i$ and A_i by $A \cap A_i$, we shall assume further that $E_i \subset U_i \subset A_i \subset A$. As $\mathcal{L}$ has the $\mathcal{G}$-c.c.p. there exists m such that $K \subset \bigcup_1^m U_i$. Then for $n \geqq m$, we have $B \subset \bigcup_1^n A_i$. Now, $\bigcup_1^n A_i \setminus B \in \mathcal{R}$ with $\bigcup_1^n A_i \setminus B \subset A \setminus B$ so that $\bigcup_1^n A_i \setminus B \in \mathcal{R}_{W_{\varepsilon/4}}(\mu)$. If $D = (\bigcup_1^n A_i) \setminus (\bigcup_1^n E_i)$, let $H_i = A_i \setminus E_i$ and $F_i = H_i \setminus \bigcup_{j<i} H_j$ with $H_o = \emptyset$. Then

$$\mu(D) = \mu(D \cap \bigcup_1^n (A_i \quad E_i)) = \sum_1^n \mu(D \cap F_i)$$

$$= \sum_1^n \{\mu(D \cap H_i) - \mu(D \cap H_i \cap (\bigcup_{j<i} H_j)\}$$

$$\in \sum_{1}^{n} W_{\varepsilon/2^{i+2}} \subset W_{\varepsilon/4}.$$

Consequently,

$$\mu(E) - \sum_{1}^{n}\mu(E_i) = \mu(E) - \mu(\bigcup_{1}^{n} E_i)$$

$$= \mu(E) - \mu((\bigcup_{1}^{n} A_i) \setminus D)$$

$$= \mu(E) - \mu(B) - \mu(\bigcup_{1}^{n} A_i \setminus B) + \mu(D)$$

$$\in W_\varepsilon \subset W.$$

Since W is closed and G is Hausdorff, $\mu(E) = \sum_{1}^{\infty}\mu(E_i)$.

The following result is now immediate from the above theorem and from Theorem XI.3.6(2) of Dugundji [6].

THEOREM 1.6. Let μ be a G-valued additive set function on a ring $\mathcal{R}$ of subsets of a topological space X and let $\mathcal{G}$ be the family of all open subsets of X. If $\mathcal{L}$ is the family of all countably compact (or compact) subsets of X or if $\mathcal{L}$ is the family of all closed subsets of X when X is countably compact and if μ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular on $\mathcal{R}$, then μ is σ-additive.

REMARKS 1.7. Theorem 1.6 gives an improved version of the theorem of Alexandroff (Theorem III.5.13 of [7]) for G= $\mathbb{C}$,since μ is not required to be bounded and X is not assumed to be compact. The result of von Neumann [8, Theorem 10.1.20] on a ring of sets in a topological space is also a particular case of the above theorem. Finally, Theorem 3 of Dinculeanu and

Kluvanek [2] is also generalized to G-valued additive set functions in the above theorem.

2.- $(\mathcal{G},\mathcal{L})$-Alexandroff regularity of the σ-additive extension. Suppose μ is a σ-additive exhausting G-valued set function on a ring R of subsets of Ω with $\mu(R)$ contained in a sequentially complete set in G. Then it is known from Drewnowski [5] that μ admits a unique σ-additive extension $\tilde{\mu}$ to $S(R)$, the σ-ring generated by R. The object of this section is to give a sufficient condition to ensure that the $(\mathcal{G},\mathcal{L})$-Alexandroff regularity of μ imply that of $\tilde{\mu}$ on $S(R)$.

To start with, we recall some definitions and results from Drewnowski [3,4,5], which play a key role in the sequel.

A topology τ on R is called a ring topology, if the operations $(A,B) \to A \Delta B$ and $(A,B) \to A \cap B$ from $(R,\tau) \times (R,\tau) \to (R,\tau)$ are continuous. A ring topology τ on R is called an FN-topology if for each τ-neighbourhood U of $\emptyset$ there exists a τ-neighbourhood V of $\emptyset$ such that for each $A \in V$, $\{B: B \subset A,\ B \in R\} \subset U$

If $\mu: R \to G$ is additive and B is a base of closed symmetric neighbourhoods of 0 in G, then $\{R_W(\mu): W \in B\}$ is a base of neighbourhoods of $\emptyset$ for an FN-topology $\Gamma(\mu)$ on R, which is the coarsest FN-topology on R with respect to which μ is continuous (See 1.9 of [3]). $\Gamma(\mu)$ is called the FN-topology induced by μ on R.

A set function $\mu : \mathcal{R} \to G$ is said to be exhausting if $\mu(E_n) \to o$, whenever $\{E_n\}_1^\infty$ is a disjoint sequence in $\mathcal{R}$. An FN-topology τ on $\mathcal{R}$ is said to be order continuous if $\{E_n\}_1^\infty \subset \mathcal{R}$, $E_n \searrow \emptyset$ imply $E_n \to \emptyset$ in the topology τ.

The following result is immediate from Theorems 8.3, 8.4 and 8.5 of [5].

THEOREM 2.1. (Drewnowski). Suppose λ is a σ-additive exhausting G-valued set function on the ring $\mathcal{R}$. Then there exists a unique order continuous FN-topology $\Gamma(\lambda)^\sim$ on $S(\mathcal{R})$ such that $\Gamma(\lambda)^\sim|\mathcal{R} = \Gamma(\lambda)$, where $\Gamma(\lambda)$ is the FN-topology induced by λ on $\mathcal{R}$. Besides, $\mathcal{R}$ is $\Gamma(\lambda)^\sim$-dense in $S(\mathcal{R})$.

Let λ be as in Theorem 2.1. Then, for $E \in S(\mathcal{R})$, there exists a net $\{E_\alpha\}$ in $\mathcal{R}$ such that $E_\alpha \to E$ in the topology $\Gamma(\lambda)^\sim$. If the range of λ is contained in a complete set $H \subset G$, then $\tilde{\lambda}(E) = \lim_\alpha \lambda_\alpha(E_\alpha)$ exists and belongs to H. Besides, the set function $\tilde{\lambda}$ on $S(\mathcal{R})$ is well defined, extends λ and is σ-additive on $S(\mathcal{R})$. Further, such a σ-additive extension $\tilde{\lambda}$ of λ to $S(\mathcal{R})$ is unique. (See Theorem 9.2 of [5]). Finally, by Remarks 1 on p.411 of [5], it suffices to assume that H is sequentially complete for the above results to hold. Thus we can state the following

THEOREM 2.2 (Drewnowski). Let $\lambda : \mathcal{R} \to G$ be σ-additive and exhausting with $\lambda(\mathcal{R})$ contained in a sequentially complete set H in G. Then there exists a unique σ-additive extension $\tilde{\lambda}$ of λ to $S(\mathcal{R})$, with $\tilde{\lambda}(S(\mathcal{R})) \subset H$. Besides, for $E \in S(\mathcal{R})$,

$$\tilde{\lambda}(E) = \lim_{\alpha} \lambda(E_\alpha)$$

whenever the net $\{E_\alpha\} \subset \mathcal{R}$ converges a E in the topology $(\lambda)^{\sim}$.

In the sequel, unless otherwise stated, μ is a G-valued σ-additive exhausting set function on the ring $\mathcal{R}$ with $\mu(\mathcal{R})$ contained in a sequentially complete subset H of G. $\tilde{\mu}$ denotes the unique σ-additive extension of μ to $\mathcal{S}(\mathcal{R})$. $\mathcal{R}_\sigma$ denotes the class $\{E = \bigcup_1^\infty E_n : E_n \in \mathcal{R}$ for each $n\}$.

LEMMA 2.3. Suppose $\Omega \in \mathcal{S}(\mathcal{R})$. Given $E \in \mathcal{S}(\mathcal{R})$ and $W \in \mathcal{B}$, there exists $\{E_n\}_1^\infty \subset \mathcal{R}$ such that $E \subset \bigcup_1^\infty E_n$ with $(\bigcup_1^\infty E_n) \setminus E \in \mathcal{S}(\mathcal{R})_W(\tilde{\mu})$.

PROOF. Choose $W_o \in \mathcal{B}$ such that $W_o + W_o \subset W$. By the result 4.3 of [4], the condition (*) on p.92 of Sion [9] is equivalent to the exhausting property of μ, since by hypothesis the range of μ is contained in a sequentially complete set. Since $\tilde{\mu}$ is unique on $\mathcal{S}(\mathcal{R})$, by Theorem 3.3 of Sion [9] we have

$$\tilde{\mu}(E) = \lim\{\tilde{\mu}(\alpha) : \alpha \in \mathcal{R}_\sigma^+(E)\}$$

where $\mathcal{R}_\sigma^+(E) = \{\alpha \in \mathcal{R}_\sigma : E \subset \alpha\}$ is directed by $\alpha \leq \beta$ if and only if $\alpha \supset \beta$. Thus there exists $\alpha_o \in \mathcal{R}_\sigma^+(E)$ such that

$$\tilde{\mu}(\alpha) - \tilde{\mu}(E) \in W_o \qquad (1)$$

for $\alpha \geq \alpha_o, \alpha \in \mathcal{R}_\sigma^+(E)$. Let $F \in \mathcal{S}(\mathcal{R})$ with $E \subset F \subset \alpha_o$. By a similar argument applied to F, there exists $\beta_o \in \mathcal{R}_\sigma^+(F)$ such that

$$\tilde{\mu}(\beta) - \tilde{\mu}(F) \in W_o \qquad (2)$$

for $\beta \geqq \beta_o, \beta \in R^+_\sigma(F)$. Clearly, $\alpha_o \cap \beta_o \in R^+_\sigma(E) \cap R^+_\sigma(F)$. Consequently, by (1) and (2) we have

$$\tilde{\mu}(\alpha_o \cap \beta_o) - \tilde{\mu}(E) \in W_o$$

and

$$\tilde{\mu}(\alpha_o \cap \beta_o) - \tilde{\mu}(F) \in W_o$$

so that $\tilde{\mu}(F) - \tilde{\mu}(E) \in W$. This shows that there exists $\{E_n\}_1^\infty \subset \mathcal{R}$ such that $\alpha_o = \bigcup_1^\infty E_n$, $E \subset \bigcup_1^\infty E_n$ and $(\bigcup_1^\infty E_n) \setminus E \in \mathcal{S}(\mathcal{R})_W(\tilde{\mu})$.

REMARKS 2.4. The hypothesis that $\Omega \in \mathcal{S}(\mathcal{R})$ is redundant in Lemma 2.3, since Theorem 3.3 of [9] can be shown to be valid with suitable modifications by considering the hereditary σ-ring generated by $\mathcal{R}$, when $\mathcal{S}(\mathcal{R})$ is not a σ-algebra.

LEMMA 2.5. Given $W \in \mathcal{B}$ and E, F in $\mathcal{R}$ with $E \subset F$ and $F \setminus E \in \mathcal{R}_W(\mu)$, then $F \setminus E \in \mathcal{S}(\mathcal{R})_W(\tilde{\mu})$.

PROOF. Let $A \in \mathcal{S}(\mathcal{R})$ with $A \subset F \setminus E$. Let $A_\alpha \to A$ in the topology $\Gamma(\mu)^\sim$, where $\{A_\alpha\}$ is a net in $\mathcal{R}$. Then $A_\alpha \cap (F \setminus E) \to A \cap (F \setminus E) = A$ in $\Gamma(\mu)^\sim$, since $\Gamma(\mu)^\sim$ is a ring topology on $\mathcal{S}(\mathcal{R})$. Consequently, by Theorem 2.2

$$\tilde{\mu}(A) = \lim_\alpha \mu(A_\alpha \cap (F \setminus E)) \in \bar{W} = W.$$

Thus $F \setminus E \in \mathcal{S}(\mathcal{R})_W(\tilde{\mu})$.

THEOREM 2.6. Suppose μ is a G-valued σ-additive and exhausting set function on the ring $\mathcal{R}$ with $\mu(\mathcal{R})$ contained in a sequentially complete set in G. Suppose the family $\mathcal{G}$ of subsets of Ω is closed for countable unions.

(i) If μ is $\mathcal{G}$-Alexandroff regular on $\mathcal{R}$, then the σ-additive extension $\tilde{\mu}$ of μ is $\mathcal{G}$-Alexandroff regular on $S(\mathcal{R})$.

(ii) If $\Omega \in S(\mathcal{R})$, $\{\Omega \setminus U : U \in \mathcal{G}\} \subset \mathcal{L}$ and μ is $\mathcal{G}$-Alexandroff regular on $\mathcal{R}$, then $\tilde{\mu}$ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular on $S(\mathcal{R})$.

PROOF.

(i) Let $E \in S(\mathcal{R})$ and let $W \in \mathcal{B}$. Choose $\hat{W} \in \mathcal{B}$ such that $\hat{W} + \hat{W} \subset W$. Then there exists a finite family $(q_i)_1^k$ of continuous quasinorms on G and an $\varepsilon > o$ such that $\hat{W}_\varepsilon = \bigcap_{i=1}^{k} B_{q_i}(o,\varepsilon) \subset \hat{W}$. By Lemma 2.3 and Remarks 2.4, there exists $(E_n)_1^\infty \subset \mathcal{R}$ such that $E \subset \bigcup_1^\infty E_n$ and

$$(\bigcup_1^\infty E_n) \setminus E \in S(\mathcal{R})_{\hat{W}}(\tilde{\mu}) \qquad (1)$$

By hypothesis, for each n there exists $U_n \in \mathcal{G}$, $A_n \in \mathcal{R}$ such that $E_n \subset U_n \subset A_n$ and $A_n \setminus E_n \in \mathcal{R}_{\hat{W}_{\varepsilon/2^{n+1}}}(\mu)$. If $U = \bigcup_1^\infty U_n$, then by hypothesis on $\mathcal{G}$, $U \in \mathcal{G}$. If $A = \bigcup_1^\infty A_n$, then $A \in S(\mathcal{R})$ and $E \subset \bigcup_1^\infty E_n \subset U \subset A$. If $F \in S(\mathcal{R})$ with $F \subset A \setminus \bigcup_1^\infty E_n$, then $F = \bigcup_1^\infty F \cap (A_n \setminus E_n)$. Now, by Lemma 2.5

$$\tilde{\mu}(F \cap (A_n \setminus E_n)) \in \hat{W}_{\varepsilon/2^{n+1}}$$

for each n. Let $B_n = F \cap (A_n \setminus E_n)$ and

$$H_n = \bigcup_{k=1}^{n} B_k. \quad \text{Then}$$

$$\tilde{\mu}(H_n) = \sum_1^n \tilde{\mu}(B_k \setminus \bigcup_{j<k} B_j) \quad (\text{where } B_o = \emptyset)$$

$$= \sum_1^n \{\tilde{\mu}(B_k) - \tilde{\mu}(B_k \cap \bigcup_{j<k} B_j)\}$$

$$\varepsilon \sum_1^n \hat{W}_{\varepsilon/2^k} \subset \hat{W}_\varepsilon \subset \hat{W}.$$

Consequently,

$$\tilde{\mu}(F) = \lim \tilde{\mu}(H_n) \;\varepsilon\; \hat{W} \qquad (2)$$

as $\hat{W}$ is closed. Thus for $B \;\varepsilon\; S(R)$ with $B \subset A \setminus E$ we have

$$B = B \cap (A \setminus \bigcup_1^\infty E_n) \cup B \cap ((\bigcup_1^\infty E_n) \setminus E)$$

so that $\tilde{\mu}(B) \;\varepsilon\; \hat{W} + \hat{W} \subset W$ by (1) and (2). Therefore, $\tilde{\mu}$ is $\mathcal{G}$-Alexandroff regular on $S(R)$.

(ii) Let $\Omega \;\varepsilon\; S(R)$ and $\{\Omega \setminus U : U \;\varepsilon\; \mathcal{G}\} \subset \mathcal{L}$. If $E \;\varepsilon\; S(R)$ and $W \;\varepsilon\; \mathcal{B}$, then by (i) there exists $U \;\varepsilon\; \mathcal{G}$ and $A \;\varepsilon\; S(R)$ such that $\Omega \setminus E \subset U \subset A$ with $A \setminus (\Omega \setminus E) \;\varepsilon\; S(R)_W(\tilde{\mu})$. Then $\Omega \setminus A \subset \Omega \setminus U \subset E$, $\Omega \setminus U \;\varepsilon\; \mathcal{L}$ and $E \setminus (\Omega \setminus A) = A \setminus (\Omega \setminus E) \;\varepsilon\; S(R)_W(\tilde{\mu})$. Hence $\tilde{\mu}$ is $(\mathcal{G}, \mathcal{L})$-Alexandroff regular on $S(R)$.

Combining Theorems 1.5 and 2.6 we obtain the following

THEOREM 2.7. Let μ be a G-valued exhausting additive set function on the ring R with $\mu(R)$ contained in a sequentially complete set in G. Suppose the family $\mathcal{G}$ is a lattice of sets closed for countable unions and the family $\mathcal{L}$ has the $\mathcal{G}$-c.c.p. If μ is $(\mathcal{G}, \mathcal{L})$-Alexandroff regular on R, then μ has a unique σ-additive extension $\tilde{\mu}$ on $S(R)$ and $\tilde{\mu}$ is $\mathcal{G}$-Alexandroff regular on $S(R)$. If $\Omega \;\varepsilon\; S(R)$ and $\{\Omega \setminus U : U \;\varepsilon\; \mathcal{G}\} \subset \mathcal{L}$, then $\tilde{\mu}$ is further $(\mathcal{G}, \mathcal{L})$- Alexandroff regular on $S(R)$.

The following result is immediate from the above theorem and Theorem 1.6.

THEOREM 2.8. Let μ be a G-valued exhausting additive set function on a ring $\mathcal{R}$ of subsets of a topological space X, with $\mu(\mathcal{R})$ contained in a sequentially complete set in G. Suppose $\mathcal{G}$ is the family of all open sets in X and $\mathcal{L}$ is a family of subsets of X having the $\mathcal{G}$-c.c.p. If μ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular on $\mathcal{R}$, then μ has a unique σ-additive extension $\tilde{\mu}$ on $\mathcal{S}(\mathcal{R})$ and $\tilde{\mu}$ is $\mathcal{G}$-Alexandroff regular on $\mathcal{S}(\mathcal{R})$. If X is countably compact, we can take $\mathcal{L}$ to be the family of all closed subsets of X. If X is countably compact, $X \in \mathcal{S}(\mathcal{R})$ and $\mathcal{L} = \{C \subset X : C \text{ closed}\}$, then $\tilde{\mu}$ is $(\mathcal{G},\mathcal{L})$-Alexandroff regular.

REMARKS 2.9. Since a bounded complex valued additive set function on a ring of sets is necessarily exhausting, Theorem III.5.14 of [7] mentioned in the outset is indeed a very special case of Theorem 2.8.

R E F E R E N C E S

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