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Abstract

In this work we give a necessary and suficient condition for the local null contro llability of the following linear time varying system $x'(t) = A(t)x(t) - B(t)u(t)$. We extend the results from [2] and [8] to this case.

Key words. null controllability, time varying systems

AMS(MOS) subject classifications. primary: 93B05; secondary: 93C25.

1 Introduction and Definitions

In this paper we study the local null controllability of the linear time varying system

$$\dot{x}(t) = A(t)x(t) - B(t)u(t), \quad t \in \mathbb{R}. \tag{1.1}$$

where $x(t) \in \mathbb{R}^n$, $A(t)$ and $B(t)$ continuous matrices of dimension $n \times n$ and $n \times l$ respectively; the control function $u(\cdot)$ belongs to the admisible control set:

$$C_\Omega = \{u(\cdot) \in L^2_{loc}(\mathbb{R}; \mathbb{R}^l) : \; u(t) \in \Omega, \; a.\ e \ on \ \mathbb{R}\}, \tag{1.2}$$

$$C_\Omega(\tau) = \{u(\cdot) \in L^2(0, \tau; \mathbb{R}^l) : \; u(t) \in \Omega, \; a.\ e \ on \ [0, \tau]\}, \tag{1.3}$$

where Ω a non-empty set of $\mathbb{R}^l$.
For any admisible control $u \in C_\Omega$ and $x_0 \in \mathbb{R}^n$ the solution $x = x_u$ of (1.1) that satisfies the initial condition $x_u(0) = x_0$ is given by the variation constant formula

$$x_u(t) = \Phi(t)x_0 - \int_0^t \Phi(t)\Phi^{-1}(s)B(s)u(s)ds, \tag{1.4}$$

where $\Phi(t)$ is the principle fundamental metrix of the linear system $x' = A(t)x$. We shall denote by $V(t, \Omega)$ the set of points in $\mathbb{R}^n$ transferible to the origin in time t by an admissible trayectory $x_u(\cdot)$ of (1.1) i.e.,

$$V(t, \Omega) = \{\int_0^t \Phi^{-1}(s)B(s)u(s)ds : \; u \in C_\Omega\} \tag{1.5}$$

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Definition 1.1 The system (1.1) is called locally null controllable in time $t > 0$, if

$$0 \in \text{int}V(t, \Omega).$$

If we put

$$V_\infty(\Omega) = \bigcup_{t>0} V(t, \Omega),$$

then we shall say that the system (1.1) is locally null controllable in free time if

$$0 \in \text{int}V_\infty(\Omega),$$

where int$V$ denotes the interior of the set V.

When $A(t) = A$ and $B(t) = B$ are constants, the system (1.1) can be written as follows

$$x' = Ax + Bu, \quad t \geq 0. \tag{1.6}$$

A necessary and sufficient condition for the local null controllability of system (1.6) is given by the following theorems

Theorem 1.1 *(see [5]) If $0 \in int\Omega$, then the following statements are equivalent:*

a) $0 \in intV(t, \Omega), \quad t > 0$

b) $Sp\{B\mathbb{R}^l, AB\mathbb{R}^l, \ldots, A^{n-1}B\mathbb{R}^l\} = \mathbb{R}^n$.

In [2] and [8] we can find the following generalization of the above theorem.

Theorem 1.2 *If $0 \in \Omega$, the system (1.6) is locally null controllable in free time, if and only if, the following conditions holds.*

a) $Sp\{e^{-At}B\Omega : t \geq 0\} = \mathbb{R}^n$

b) there is not a real eigenvector of A^* outer normal of the set $B\Omega$ at zero.

The goal of this work is to give a necessary and sufficient condition for the local null controllability of the time-varying system (1.1).

2 Main Theorems

Now, we are ready to formulate and prove the main theorems of this work.

Theorem 2.1 *If $0 \in int\ \Omega$, then system (1.1) is null controllable in time $t_1 > 0$ if and only if the following condition holds.*

$$B^*(t)\Phi^{-1^*}(t)\xi = 0, \quad 0 \leq t \leq t_1 \Longrightarrow \xi = 0 \tag{2.7}$$

Proof . Suppose that system (1.1) is locally null controllable in time $t_1 > 0$. i. e.,

$$0 \in \text{int}V(t_1, \Omega).$$

Assume that there exists $\xi \neq 0,\ \xi \in I\!R^n$ such that

$$B^*(t)\Phi^{-1^*}(t)\xi = 0, \quad t \in [0, t_1].$$

Then, for all $\eta \in I\!R^l$ and all $t \in [0, t_1]$ we have

$$\langle B^*(t)\Phi^{-1*}(t)\xi, \eta\rangle \ = \ \langle \xi,\ \Phi^{-1}(t)B(t)\eta\rangle = 0.$$

Therefore,

$$\langle \xi,\ \int_0^{t_1} \Phi^{-1}(t)B(t)u(t)\rangle = 0 \ \ \forall u \in C_\Omega(t_1).$$

Hence, $\text{int}V(t_1, \Omega) = \emptyset$, which is a contradiction.

Now, suppose condition (2.7) holds and $\text{int}V(t_1, \Omega) = \emptyset$. Consider the following linear bounded operator

$$G(t_1) : L_2(0, t_1; I\!R^l) \longrightarrow I\!R^n$$

$$G(t_1)u = \int_0^{t_1} \Phi^{-1}(s)B(s)u(s)ds. \tag{2.8}$$

Now, we shall use the following theorem from [7]

Theorem.

Let X and Y two Banach spaces and $G : X \longrightarrow Y$ a bounded lineal operator. Then the following statements hold

(α) $\underline{\text{Ran}(G)} = Y \iff \exists \alpha > 0$ such that $\|G^*y^*\|_{X^*} \geq \alpha\|y^*\|_{Y^*}, \quad (y^* \in Y^*)$.

(β) $\overline{\text{Ran}(G)} = Y \iff G^*$ is one to one.

Where $\text{Ran}(G) = G(X)$ denotes the range of G.

In this case $Y = I\!R^n,\ \ X = L^2(0; t_1; I\!R^l)$ and $G = G(t_1)$. Since any linear finite dimensional subspace is closed, then $\overline{\text{Ran}(G(t_1))} = \text{Ran}G(t_1)$. Therefore, ($\alpha$) and ($\beta$) are equivalents.

Claim 1.

Condition (2.7) holds, if and only if $\text{Ran}G(t_1) = I\!R^n$. i.e.,

$$\text{Ran}G(t_1) = G(t_1)L_2(0, t_1; I\!R^l) = I\!R^n. \tag{2.9}$$

In fact, it follows from the adjoint $G^*(t_1)$, of $G(t_1)$ and applying foregoing theorem, once we have computed $G^*(t_1)$.

$$G^*(t_1) : I\!R^n \longrightarrow L^2(0, T; I\!R^l)$$

$$\begin{aligned}
\langle \xi, G(t_1)u\rangle_{I\!R^n, I\!R^n} &= \int_0^{t_1} \langle \xi, \Phi^{-1}(s)B(s)u(s)\rangle_{I\!R^n, I\!R^n} ds \\
&= \int_0^{t_1} \langle B^*(s)\Phi^{-1*}(s)\xi, u(s)\rangle ds \\
&= \langle B^*(\cdot)\Phi^{-1*}(\cdot)\xi, u\rangle_{L^2, L^2}.
\end{aligned}$$

Hence,

$$G^*(t_1)\xi = B^*(\cdot)\Phi^{-1*}(\cdot)\xi \in L^2(0, t_1; \mathbb{R}^m).$$

So,

$$G^*(t_1)\xi = B^*(\cdot)\Phi^{-1*}(\cdot)x = 0 \Longrightarrow \xi = 0.$$

i.e.,

$$B^*(t)\Phi^{-1^*}(t)\xi = 0, \quad 0 \le t \le t_1 \Longrightarrow \xi = 0.$$

Claim 2. .
If $G(t_1)L^2(0, t_1; \mathbb{R}^l) = \mathbb{R}^n$, then $G(t_1)C(0, t_1; \mathbb{R}^l) = \mathbb{R}^n$. In fact, for any $x \in \mathbb{R}^n$ there exists $u \in L^2(0, t_1; \mathbb{R}^l)$ such that $G(t_1)u = x$. Since $C(0, t_1; \mathbb{R}^l)$ is dense in $L^2(0, t_1; \mathbb{R}^l)$, there exists a sequence $(u_n)_{n \ge 1}$ in $C(0, t_1; \mathbb{R}^l)$ with $u_n \to u$ as $n \to \infty$. By the continuity of $G(t_1)$, we get that $G(t_1)u_n \to G(t_1)u$ as $n \to \infty$. Therefore, we have that

$$G(t_1)C(0, t_1; \mathbb{R}^l) = \overline{G(t_1)C(0, t_1; \mathbb{R}^l)} = \mathbb{R}^n.$$

Since $0 \in \text{int}\Omega$, then there exists $r > 0$ such that the open ball $B(0, r)$ is contained in Ω. Therefore, the set

$$A(0, r) = \{u \in C(0, t_1; \mathbb{R}^l) \; : \; \|u(t)\| < r \quad t \in [0, t_1]\}$$

is an open set in $C(0, t_1; \mathbb{R}^l)$ such that $A(0, r) \subset C_\Omega(t_1)$.

Applying the Open Maping Theorem [7] to the operator $G(t_1) : C(0, t_1; \mathbb{R}^l) \to \mathbb{R}^n$, we get that there exist $\delta > 0$ and a ball $B(0, \delta)$ in $\mathbb{R}^n$ such that

$$B(0, \delta) \subset G(t_1)A(0, r) \subset G(t_1)C_\Omega(t_1) \subset V(t_1, \Omega).$$

Hence $\text{int}V(t_1, \Omega) \ne \emptyset$.

□

Theorem 2.2 *Suppose that $0 \in \Omega$. If the system (1.1) is locally null controllable in free time, then the following condition holds.*

(A) $B^*(t)\Phi^{-1^*}(t)\xi = 0, \quad t \in [0, t^*] \Rightarrow \xi = 0$ *for some* $t^* > 0$.

Proof . Assume that

$$0 \in \text{int}V_\infty = \text{int} \bigcup_{t>0} V(t, \Omega)$$

From [5], we know that $V(t, \Omega)$ is a convex set and since $0 \in \Omega$ we get that

$$V(t_1, \Omega) \subseteq V(t_2, \Omega) \;\; \text{if} \;\; t_1 \le t_2.$$

Then

$$\text{int}V_\infty = \bigcup_{t>0} \text{int}V(t, \Omega).$$

Therefore, there exists $t^* > 0$ such that $0 \in \text{int}V(t^*, \Omega)$.

Then, $\text{intRan}(G(t^*)) \ne \emptyset$; where $G(t^*)$ is the operator defined by (2.8). Hence, by the linearity of $G(t^*)$ we get $Ran(G(t^*)) = \mathbb{R}^n$, which is equivalent to condition (A). □

2.1 Algebraic Condition

In order to get an algebraic condition in terms of $A(t)$, $B(t)$ and Ω for the locall null controllability of the system (1.1), we shall write the matrix $A(t)$ as follow

$$A(t) = \sum_{i=1}^{\infty} a_i(t) A_i, \qquad (2.10)$$

where A_i, $i = 1, 2, \dots, m$ are real constants $n \times n$ matrices and the Lie algebra generated by these matrices has dimension m. Under this condition Wei and Norman in [10] proved that the fundametal matrix $\Phi(t)$ can be written as the product of exponential matrices,

$$\Phi(t) = e^{g_1(t)A_1} e^{g_2(t)A_2} \dots e^{g_m(t)A_m}, \quad t \in [0, T], \qquad (2.11)$$

for some $T > 0$. Where $g_i : [0, T] \to \mathbb{R}$ are continuous functions.

It shoud be pointed out that the representation (2.11) is valid only on $[0, T]$. However, if $A(t)$ is a 2×2 matrix, then it is valid for all t in $\mathbb{R}$.

Theorem 2.3 *Suppose that* $0 \in \Omega$. *If the system (1.1) is locally null controllable on* $[0, T]$, *then the following condition holds:*
(B) There is not a common real eigenvector of the matrices $A_1^*, A_2^*, \dots, A_m^*$ *outer normal of* $B(t)\Omega$ *at zero for all* $t \in [0, T]$.

Proof . Suppose that $0 \in \text{int}V(T, \Omega)$ and there exists a common vector $\xi \in \mathbb{R}^n$, $\xi \neq 0$ such that

$$A_i^* \xi = \lambda_i \xi, \quad i = 1, 2, \dots, m$$

and

$$< \xi, B(t)v > \leq 0, \quad v \in \Omega, \quad t \in [0, T].$$

Now, from the representation (2.11) of $\Phi(t)$ we get that

$$\Phi(t)^{-1*} \xi = e^{-g_m(t)A_m^*} e^{-g_{m-1}(t)A_{m-1}^*} \dots e^{-g_1(t)A_1^*} \xi = e^{h(t)} \xi, \quad t \in [0, T],$$

where $h(t) = -\sum_{i=1}^{m} \lambda_i g_i(t)$. Since $e^{h(t)} > 0$ we get the following

$$\langle \xi, \Phi^{-1} B(t)v \rangle = \langle e^{h(t)} \xi, B(t)v \rangle = e^{h(t)} \langle \xi, B(t)v \rangle \leq 0.$$

Then

$$\langle \xi, \int_0^T \Phi^{-1}(s) B(s) u(s) ds \rangle \leq 0, \quad u \in C_\Omega(T),$$

which is a contradiction to the fact that $0 \in \text{int}V(T, \Omega)$. □

Conjecture.
Suppose that $0 \in \Omega$. Then conditions (A) and (B) of Theorems 2.2, 2.3, together imply the system (1.1) is locally null controllable.

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