UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS

NOTAS DE MATEMATICA

UNBOUNDED PERTURBATION OF THE CONTROLLABILITY FOR EVOLUTION EQUATIONS

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Notas de Matemática
Serie: Pre-Print
No. 188

Mérida - Venezuela
1999

Notas de Matemática, No. 188
Mérida, January 1999.

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Abstract

In this paper we prove that the controllability for evolution equations in Banach spaces is not destroyed, if we perturb the equation by a "small" unbounded linear operator. This is done by employing a perturbation principle from linear operator Theory. Finally, we apply these to a control system governed by a partial parabolic equation.

Key words. evolution equations, controllability, perturbation principle.

AMS(MOS) subject classifications. primary: 93B05; secondary: 93C25.

1 Introduction

In this work we study an abstract infinite dimensional control system of the form

$$z' = Az + B(t)u(t), \quad z(t) \in Z, \quad u(t) \in U, \quad t > 0, \tag{1.1}$$

where Z, U are Banach spaces and $t \to B(t) : I\!R \to L(U, Z)$ is bounded continuous in the strong operator topology of $L(U, Z)$. A is the infinitesimal generator of a C_0 -semigroup $\{T(t; A)\}_{t \geq 0}$ and the control function u belongs to the space of functions $L^2(0, t_1; U)$.

Necesary and sufficient conditions for the controllability of the system (1.1) can be found in [1], [2].

Here we are interested in answer the following question: If the control system (1.1) is controllable, then for which class of unbounded linear operators P on Z the perturbed system

$$z' = (A + P)z + B(t)u(t), \quad t > 0. \tag{1.1$_P$}$$

is also controllable?. It is easy to prove that if P is a bounded linear operator, which is small enough in the uniform topology of $L(Z)$, then the equation $(1.1)_p$ is also controllable. But, if P is unbounded this result is not true in general.

In this paper we shall answer this question for a very general class of unbounded linear operators $\mathcal{P}(A)$, such that $L(Z) \subset \mathcal{P}(A)$(see section 3).

One of the goals in this work is to prove the following statement: If for some $P_0 \in \mathcal{P}(A)$ the equation $(1.1)_{P_0}$ is controllable according to Definition 2.1, then there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for all $P \in \mathcal{N}(P_0)$ the equation $(1.1)_p$ is also controllable.

* This research was partially supported by CDCHT-ULA

2 Preliminaries

Now, we are ready to give the concept of controllability for the non-autonomous system (1.1). For all $z_0 \in Z$ and admissible control $u \in L^2(0,t_1;U)$, $t_1 > 0$ the equation (1.1) has a unique mild solution given by

$$z(t,z_0,u) = T(t;A)z_0 + \int_0^t T(t-s;A)B(s)u(s)ds, \quad 0 \le t \le t_1. \tag{2.1}$$

Definition 2.1 We shall say that the system (1.1) is exactly controllable on $[0,t_1]$, $t_1 > 0$, if for all $z_0, z_1 \in Z$ there exists a control $u \in L^2(0,t_1;U)$ such that the solution $z(t)$ of (2.1) corresponding to u, verifies: $z(t_1) = z_1$.

One can say that z_1 is reachable from z_0 on time $t_1 > 0$.

Consider the following bounded linear operator

$$G : L^2(0,t_1;U) \to Z, \quad Gu = \int_0^{t_1} T(t-s;A)B(s)u(s)ds. \tag{2.2}$$

Proposition 2.1 *The system (1.1) is controllable on* $[0,t_1]$ *if and only if, the operator* G *is onto, that is to say*

$$GL^2(0,t_1;U) = GL^2 = RanGL^2 = Z.$$

2.1 Perturbation Principle

The results presented in this section follow from a combination of Theorem 19 in [9] pg. 31 and chapter XIII of [5]. It is well known that, if A is the infinitesimal generator of a C_0- semigroup $\{T(t;A)\}_{t\ge 0}$ in the Banach space Z and P is a bounded linear operator in Z ($P \in L(Z)$), then $A+P$ is the infinitesimal generator of a C_0-semigroup $\{T(t;A+P)\}_{t\ge 0}$ which is given by the following formula

$$T(t;A+P)z = T(t;A)z + \int_0^t T(t-s;A)PT(s;A+P)zds, \quad z \in Z. \tag{2.3}$$

Now, we shall see that: if P is an unbounded linear operator which is not too irregular relative to A, then $A+P$ is the infinitesimal generator of a C_0-semigroup $\{T(t;A+P)\}_{t\ge 0}$, but, the formula 2.3 is not true in general.

We shall denote by $\mathcal{D}(S)$ the domain of an operator S in a Banach space W, $L(W)$ the space of bouded and linear operator defined on W and $\sigma(S)$ the spectrum of the linear operator S. With these notation in mind, we will consider the following class of unbounded linear operators: If A is the infinitesimal generator of a C_0-semigroup $\{T(t;A)\}_{t\ge 0}$ we denote $\mathcal{P}(A)$ the class of closed linear operators P satisfying the conditions

(I) $\mathcal{D}(A) \subseteq \mathcal{D}(P)$,

(II) for each $t > 0$, there exists a constant $h(t) \ge 0$ such that

$$\|PT(t,A)z\| \le h(t)\|z\|, \quad \forall z \in \mathcal{D}(A),$$

(III) the integral $\int_0^1 h(t)dt$ exists.

Remark 2.1 A is bounded, if and only if $A \in \mathcal{P}(A)$.

The following Theroem can be found in [9]. pg 631.

Theorem 2.1 *Let A be the infinitesimal generator of a C_0semigroup $\{T(t;A)\}_{t\geq 0}$ in Z. If $P \in \mathcal{P}(A)$, then $A+P$ defined on $\mathcal{D}(A+P) = \mathcal{D}(A)$ is the infinitesimal generator of a C_0-semigroup $\{T(t;A+P)\}_{t\geq 0}$. Furthermore,*

$$T(t;A+P)z = \sum_{0}^{\infty} S_n(t), \quad t \geq 0, \tag{2.4}$$

where

$$S_0(t) = T(t;A) \quad \textit{and} \quad S_n(t)z = \int_0^t T(t-s;A)PS_{n-1}(s)zds, \quad n \geq 1, \quad z \in Z,$$

and the serie (2.4) is absolutely convergent in the uniform norm of $L(Z)$, uniformly with respect to t in each finite interval. For each n and z the function $S_n(t)z$ is continuous for $t \geq 0$.

The following facts can be found in [9].

(a) $\bigcup_{t>0} T(t;A)z \subseteq \mathcal{D}(P)$,

(b) the mapping $z \to PT(t;A)z, \;\; z \in \mathcal{D}(A)$, has a unique extension to a bounded operator defined in on Z. In order to simplify the notation, we will call this extension $PT(t)$.

(c) $PT(t)z$ is continuous in $t > 0$ at each $z \in Z$. If $\omega_0 = \lim_{t\to\infty} \log \|T(t)\|/t$, then

$$\limsup_{t\to\infty} \frac{\log \|PT(t)\|}{t} \leq \omega_0.$$

(d) if $R(\lambda) > \omega_0$, then

$$P\mathcal{R}(\lambda;A)z = \int_0^\infty e^{-\lambda t} PT(t)zdt, \quad z \in Z;$$

where $\mathcal{R}(\lambda;A) = (A - \lambda I)^{-1}$.

(e) If $\omega > \omega_0$, then there exists $M_\omega < \infty$ such that

$$\|T(t)\| \leq M_\omega e^{\omega t}, \quad \text{and} \quad \|PT(t)\| \leq M_\omega e^{\omega t}, \quad t \geq 0.$$

(f) for all $\beta > o$

$$\int_0^\beta \|PT(t)\|dt < \infty.$$

(h) If $\gamma = \int_0^\infty e^{-\omega t}\|PT(t)\|dt < 1$, then

$$\|S_n(t)\| \leq M_\omega e^{\omega t}\gamma^n, \quad n \geq 0.$$

Proposition 2.2 *Let A be the infinitesimal generator of a C_0-semigroup $\{T(t;A)\}_{t\geq 0}$ of type ω_0. Define the function*

$$d_A(P_1,P_2)=\int_0^1 \|(P_1-P_2)T(t;A)\|dt,\quad P_1,P_2\in\mathcal{P}(A), \tag{2.5}$$

and for a fixed $\omega>\omega_0$ the function

$$\delta_A(P_1,P_2)=\int_0^\infty e^{-\omega t}\|(P_1-P_2)T(t;A)\|dt\quad P_1,P_2\in\mathcal{P}(A). \tag{2.6}$$

Then $\delta_A(P_1,P_2)$ and $d_A(P_1,P_2)$ are equivalent metrics on $\mathcal{P}(A)$. i.e., there exist constants M_A and m_A such that

$$m_A\delta_A(P_1,P_2)\leq d_A(P_1,P_2)\leq M_A\delta_A(P_1,P_2),\quad P_1,P_2\in\mathcal{P}(A).$$

Remark 2.2 *If P_1-P_2 is bounded, then*

$$d_A(P_1,P_2)\leq (\int_0^1\|T(t;A)\|dt)\|(P_1-P_2)\|.$$

Theorem 2.2 *The function $P\in\mathcal{P}(A)\to T(t;A+P)\in L(Z)$ is continuous. i.e.,*

$$\lim_{d_A(P,P_0)\to 0}\|T(t;A+P)-T(t;A+P_0)\|=0,$$

uniformly with respect to t in each interval of the form $[0,\beta]$, $\beta>0$.

Furthermore. If $\delta_A(P,P_0)<1$, then there exists a constant $M=M(P_0)$ such that

$$\|T(t;A+P)-T(t;A+P_0)\|\leq\frac{\delta_A(P,P_0)}{1-\delta_A(P,P_0)}Me^{\omega t},\quad t\geq 0.$$

3 Main Results

From the foregoing section we have that $(\mathcal{P}(A),d_A)$ is a metric space endowed with the metric d_A. Now, we are ready to study the following family of control systems.

$$z'=(A+P)z+B(t)u(t),\quad t>0,\quad P\in(\mathcal{P}(A),d_A). \tag{3.1$_P$}$$

Theorem 3.1 *If for some $P_0\in(\mathcal{P}(A),d_A)$ the linear control system (3.1)$_{P_0}$ is controllable on $[0,t_1]$, then there exists a neighborhood $\mathcal{N}(P_0)$ of P_0 such that for each $P\in\mathcal{N}(P_0)$ the linear control system (3.1)$_P$ is also controllable on $[0,t_1]$.*

Proof Without lose of generality, we shall suppose that $p_0=0$. Next, consider the following linear and bounded operator

$$G_P:L^2(0,t_1;U)\to Z,\quad G_Pu=\int_0^{t_1}T(t-s;A+P)B(s)u(s)ds. \tag{3.1}$$

From proposition 2.1, it is enough to prove that G_P is onto for all P in a neighborhood $\mathcal{N}(0)$ of zero. Since (1.1) is controllable on $[0, t_1]$ the operator G given by (2.2) is onto. i.e., $\text{Ran}G = Z$.

It is well known from linear operators theory that; if $\text{Ran}G = Z$, then there exists a number $\alpha > 0$ such that for all $W \in L(L^2(0, t_1, U), Z)$ with $\|W - G\| < \alpha$, we have that $\text{Ran}W = Z$.

Now, using Theorem 2.2 we get that

$$\begin{aligned} \|G_P u - Gu\| &= \| \int_0^{t_1} (T(t-s; A+P) - T(t-s; A))B(s)u(s)ds \| \\ &\leq \frac{\delta_A(P,0)}{1-\delta_A(P,0)} M e^{\omega t_1} \int_0^{t_1} \|B(s)\| \|u(s)\| ds \\ &\leq \frac{\delta_A(P,0)}{1-\delta_A(P,0)} M e^{\omega t_1} \|B\|_{L^2} \|u\|_{L^2}. \end{aligned}$$

Hence

$$\|G_P - G\| \leq \frac{\delta_A(P,0)}{1-\delta_A(P,0)} M e^{\omega t_1} \|B\|_{L^2}.$$

Then we can take $\delta_A(P, 0)$ small enough such that

$$\frac{\delta_A(P,0)}{1-\delta_A(P,0)} M e^{\omega t_1} \|B\|_{L^2} < \alpha. \tag{3.2}$$

i.e.,

$$\delta_A(P,0) < \frac{\alpha}{\|B\|_{L^2} M \exp(\omega t_1) + \alpha} = r. \tag{3.3}$$

Therefore, the neighborhood $\mathcal{N}(0)$ is given by

$$\mathcal{N}(0) = \{P \in \mathcal{P}(A) : \delta_A(P,0) < r\}. \tag{3.4}$$

□

4 Applications

In this section we shall present an application of Theorem 3.1 to a control system governed by a partial differential equation of parabolic type. More precisely, we shall consider the following control system

$$z_t = z_{xxxx} + a(x)z_x + b(x,t)u(x,t), \quad t > 0, \quad 0 < x < 1, \tag{4.1}$$

$$z(0,t) = z(1,t) = z_{xx}(0,t) = z_{xx}(1,t) = 0. \tag{4.2}$$

Where a belongs to the space $C[0, 1]$ of continuous functions with the sup-norm and $b : [0, 1] \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous and bounded function.

Let $Z = L^2(0, 1)$ and consider the operators $Az = z_{xxxx}$ with domian $D(A)$ which consists of

$$D(A) = \{z \in Z : Az \in Z; \quad z(0) = z(1) = z_{xx}(0) = z_{xx}(1) = 0\},$$

$P_a z = a(\cdot)z_x, \quad z \in D(B)$ which consists of

$$D(A) = \{z \in Z : P_a z \in Z\},$$

and $B(t)u(x) = b(x,t)u(x), \quad u \in Z.$

The operator A generates a C_0-semigroup $\{T(t;A)\}_{t\geq 0}$ on the space Z, which is given by the following formula:

$$T(t;A)z = \sum_{n=1}^{\infty} 2e^{-n^4\pi^4 t} \sin \pi x \int_0^1 \sin n\pi y . z(y)dy, \tag{4.3}$$

and it is easy to show that

$$\|P_a T(t;A)z\| \leq \frac{M\|a\|}{t^{1/4}}\|z\|, \quad z \in D(A), \quad t > 0. \tag{4.4}$$

Therefore, the system (4.1) -(4.2) can be written as follow

$$z' = (A + P_a)z + B(t)u(t), \quad t > 0, \quad a \in C[0,1]. \qquad (4.4)_a$$

We suppose that the unperturbed system

$$z' = Az + B(t)u(t), \quad t > 0, \tag{4.5}$$

is controllable on $[0, t_1], \quad t_1 > 0$. From (4.4) we get that $P_a \in \mathcal{P}(A)$ and

$$\begin{aligned} d_A(P_a, 0) &= \int_0^1 \|P_a T(t;A)\|dt \\ &\leq M\|a\| \int_0^1 \frac{dt}{t^{1/4}} \\ &= \frac{4M}{3}\|a\|. \end{aligned}$$

Hence, if the system(4.5) is controllable on $[0, t_1]$, then there exists a neighborhood $\mathcal{N}(0)$ of zero in the space $C[0,1]$ such that for each $a \in \mathcal{N}(0)$ the system $(4.4)_a$ is also controllable on $[0, t_1]$.

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