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Fault Propagation in Industrial Complexes: $(\min, +)$Algebra Framework

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Abstract

The fault propagation and detection in an industrial complex of plant units networks is modeled by $(\min, +)$algebra linear systems. The diagnosability of the network can be transformed to the matrix inversion in $(\min, +)$. In the paper the inversion of matrices is studied for particular network structures, and diagnosability is solved.

key words. Fault propagation, diagnosability, $(\min, +), (\max, +)$algebras.

1 Introduction

An industrial complex consists on interconnected plant units. The interconnection among the units is realised both material and information flow. It is supposed that the plant units are equipped with fault detectors doing local detection and diagnosis. However, a fault detected in a unit may be the consequence of a fault occurred in another plant unit, propagated its effect to be detected in several other plants. The detection and diagnosis and the propagation of a fault to the detection location may have delay time. This process, with delay in the detection and the propagation can be modeled by $(\min, +)$ linear system of form

$$x(k+1) = A \hat{\otimes} x(k)$$

where the $x(0) = (t_1, t_2, \cdots, t_n)^t$, are the instants when faults occur at the plant units. The diagonal elements of A, a_{ii} represent the delay of the detection and diagnosis at the i-th plant units, and a_{ij} is the sum of the time delay of the propagation between the i-th and j-th plant unit and the delay of the detection and diagnosis at the j-th plant.

The first detection at the i-th plant occurs at time T_i. In terms of $(\min, +)$algebra an algebraic relation can be established between the vectors $x(0)$, and $T = (T_1, T_2, \cdots, T_n)^t$, see equation (3).

It is supposed that $x(0)$ is unknown and T is known.

Our industrial network is diagnosable if from knowledge of the vector T, $x(0)$ is computable (ver [11]).

In the present paper the diagnosability of a particular network structure is considered.

It is supposed that we have fault propogation from the i-th to j-th plant units if $i < j$. In this case at the relation (3), the matrix.

$$\mathrm{X} = A^t \oplus (A^t) \oplus \cdots \oplus (A^t)^n$$

is triangular. Invertibility of triangular matrices are exhaustively analysed in the paper and the obtained results are applied to the problem of diagnosability.

In [6] the effects of delays in communication network are analysed by using DEDS, however $(\min, +)$ or $(\max, +)$ linear systems are not applied. The application of $(\max, +)$ linear systems to the description of the public transportation network systems with delay is quite similar to our fault propagation model, however the considered problem in [7] is completely different.

The next section is an outline of $(\min, +)$ and $(\max, +)$ algebras. The main results are contained in the section 3, see theorems 2 and 3.

2 Preliminaries

In this section, we recall some basic definitions, concepts and theorem which will be used later. For an extensive discussion about $(\max,+)$algebra, $(\min,+)$algebra and similar structures, see to [2].

Let $\varepsilon = -\infty, \hat{\varepsilon} = +\infty$ and R_ε and $R_{\hat{\varepsilon}}$ denote the sets $\mathbb{R} \cup \{\varepsilon\}$ and $\mathbb{R} \cup \{\hat{\varepsilon}\}$.

For $a, b \in R_\varepsilon$ and $R_{\hat{\varepsilon}}$ the operations $\oplus, \hat{\oplus}, \otimes, \hat{\otimes}$ will be defined by $a \oplus b = \max(a, b)$, $a \hat{\oplus} b = \min(a, b)$ and $a \otimes b = a \hat{\otimes} b = a + b$, where will be adopted the conventions $\max(a, -\infty) = \max(-\infty, a) = a$, $\min(a, +\infty) = \min(+\infty, a) = a$, $a + (-\infty) = -\infty + a = -\infty$ and $+\infty + a = a + \infty = +\infty$ for all $a \in R_\varepsilon$ and $R_{\hat{\varepsilon}}$.

Moreover, ε is the neutral element for the $\oplus$, $\hat{\varepsilon}$ is the neutral element for the operation $\oplus$ and absorbing for $\otimes$ and $\hat{\otimes}$, that is, for all $a \in R_\varepsilon$ and $R_{\hat{\varepsilon}}$, $a \otimes \varepsilon = \varepsilon$ and $a \hat{\otimes} \hat{\varepsilon} = \hat{\varepsilon}$. The neutral element for $\otimes$ and $\hat{\otimes}$ is 0 (somethings it is denoted by e). The sets R_ε and $R_{\hat{\varepsilon}}$ together with the operations $\oplus, \hat{\oplus}, \otimes$ and $\hat{\otimes}$, denoted by $\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are called $(\max,+)$algebra and $(\min,+)$algebra. More precisely, $\mathbb{R}_{\max} = (R_\varepsilon, \oplus, \otimes, \varepsilon, e)$ and $\mathbb{R}_{\min} = (R_{\hat{\varepsilon}}, \hat{\oplus}, \hat{\otimes}, \hat{\varepsilon}, e)$ are semiring with respect to the operations $\oplus$ and $\hat{\oplus}$. The $(\max,+)$algebra and $(\min,+)$algebra are moreover conmutative semiring. Furthermore, there are idempotent semiring, because $\oplus$ and $\hat{\oplus}$ are idempotent, that is, for all $a \in R_\varepsilon$ and $R_{\hat{\varepsilon}}$, we have $a \oplus a = a$ and $a \hat{\oplus} a = a$ and therefore does not allow for inverse elements. Indeed, if $a \neq \varepsilon$ or $a \neq \hat{\varepsilon}$ had an inverse element, say b, then $a \oplus b = \varepsilon$ or $a \hat{\oplus} b = \hat{\varepsilon}$ would imply $a \oplus a \oplus b = a \oplus \varepsilon$ or $a \hat{\oplus} a \hat{\oplus} b = a \hat{\oplus} \hat{\varepsilon}$. By idempotency, the left-hand side equals $a \oplus b$ or $a \hat{\oplus} b$, whereas the right-hand side is equal to a. Hence, we have $a \oplus b = a$ or $a \hat{\oplus} b = a$, which contradicts $a \oplus b = \varepsilon$ or $a \hat{\oplus} b = \hat{\varepsilon}$.

$\mathbb{R}_{\max}$ and $\mathbb{R}_{\min}$ are isomorphic algebraic structures.

We now extend $(\max, +)$ and $(\min, +)$algebra operations to matrices in the following way. If $A, B \in \mathbb{R}_{\varepsilon}^{n\times n}$ and $\mathbb{R}_{\hat{\varepsilon}}^{n\times n}$ then

$$\begin{aligned}(A \oplus B)_{ij} &= A_{ij} \oplus B_{ij} = \max(A_{ij}, B_{ij}) \\ (A \hat{\oplus} B)_{ij} &= A_{ij} \hat{\oplus} B_{ij} = \min(A_{ij}, B_{ij}),\end{aligned}$$

for $i = 1, \cdots, m$; $j = 1, \cdots, n$.

If $A \in \mathbb{R}_{\varepsilon}^{m\times p}$ and $\mathbb{R}_{\hat{\varepsilon}}^{m\times p}$, $B \in \mathbb{R}_{\varepsilon}^{p\times n}$ and $\mathbb{R}_{\hat{\varepsilon}}^{p\times n}$ then

$$\begin{aligned}(A \otimes B)_{ij} &= \bigoplus_{k=1}^{p} A_{ik} \otimes B_{kj} = \max_{1\le k\le p}(A_{ik} + B_{kj}) \\ (A \hat{\otimes} B)_{ij} &= \hat{\bigoplus}_{k=1}^{p} A_{ik} \hat{\otimes} B_{kj} = \min_{k}(A_{ik} + B_{kj}),\end{aligned}$$

for $i = 1, \cdots, m$ and $j = 1, \cdots, n$. From the definitions it follows that $\mathbb{R}_{\max}^{n\times n}$ and $\mathbb{R}_{\min}^{n\times n}$ are idempotent semiring.

A square max-plus or min-plus matrix corresponds to a graph. Let A be $n \times n$ max-plus matrix or min-plus matrix. The precedence graph $\mathcal{G}(A)$ is a weighted digraph (V, E), where V is the set of nodes (vertices) and E is the set of ordered pairs of vertices, called arcs and of $\mathcal{G}$. If $a_{ij} \neq \varepsilon$ or $\hat{\varepsilon}$, then the ordered pair (i, j) is an element of E and a_{ij} is called its weight. Of the pair (i, j), i is the initial vertex and j the terminal vertex.

The notation A^k in $(\max, +)$algebra or $(\min, +)$algebra denote $A \otimes A \otimes \cdots \otimes A$ or $A \hat{\otimes} \cdots \hat{\otimes} A$ k times, and the value of $(A^k)_{ij}$ equals the maximum or minimum of the weights of all paths of lenght k from vertex i to vertex j.

The role of determinant in the standard matrix algebra is replaced by dominant (to be defined).

Let $A \in \mathbb{R}_{\max}^{n \times n}$, then the dominant of A is defined as:

$$Dom(A) = \begin{cases} \text{hightest exponent in } \; det(z^A) & \text{if} \quad det(z^A) \neq 0 \\ \\ \varepsilon & \text{otherwise} \end{cases}$$

where $z \in (1, +\infty)$ and $z^A \in \mathbb{R}^{n \times n}$, $(z^A)_{ij} = z^{A_{ij}}$.

We define

$$M(+) = \left\{ \begin{array}{ccc} A \in \mathbb{R}_{\max}^{n \times n} & : & \text{the sign of the hightest exponent} \\ & & \text{in} \quad det(z^A) \quad \text{is positive} \end{array} \right.$$

$$M(-) = \left\{ \begin{array}{ccc} A \in \mathbb{R}_{\max}^{n \times n} & : & \text{the sign of the hightest exponent} \\ & & \text{in} \quad det(z^A) \quad \text{is negative} \end{array} \right. .$$

Then, if $A \in M(+)$ we shall say that A has positive sign: $Sign(A) = +$; similarly, $A \in M(-)$ will be equivalent with $sign(A) = -$.

The Cramer's rule in $(\max, +)$algebra, in order to obtain the solution of the linear equation $A \otimes x = b$, is given by

$$x_j \otimes Dom(A) = Dom(A_i), \quad 1 \leq i \leq n, \tag{1}$$

where A_i is obtainded from A by replacement of the i-th column by b.

In general, a solution to $A \otimes x = b$ in $(\max, +)$algebra may not exist, not even if $dom(A) > \varepsilon$; and if it exists, it not necessarily unique.

Theorem 1 *If the pair* (A, b) *satisfies* $Sign(A) = Sign(A_i), 1 \leq i \leq n$ *and* $dom(A) > \varepsilon$*, then* (1) *yields a solution of the equation* $A \otimes x = b$*.*

Proof. (See [9]). ∎

3 Statement of the Problem

An interconnected system of n subsystems is considered. Suppose that the system i has a fault at $-t_i$ time $1 \leq i \leq n$ and the fault is propagated forward, that is, from i-th plant unit the fault is propagated to j-th only for $j \geq i$.

Figure 1:

If we denote by $-a_{ij}$ the time needed to pass the fault from the system i to the system j and for $-T_j$ the time where we detect for first time some fault at the jth system. Then $-T_j$ is given by

i n

j

Figure 2:

$$\begin{aligned} -T_j &= \min\big(-t_1 \hat{\otimes} (-a_{1j}), -t_2 \hat{\otimes} (-a_{2j}), \cdots, -t_i \hat{\otimes} (-a_{ij}), \cdots \ -t_1 \hat{\otimes} (-a_{1\ell}) \hat{\otimes} (-a_{\ell j}), \\ &\quad \cdots, -t_i \hat{\otimes} (-a_{i\ell}) \hat{\otimes} (-a_{\ell j}), \cdots\big) \\ &= -\max\left(t_1 \otimes a_{1j}, t_2 \otimes a_{2j}, \cdots, t_i \otimes a_{ij}, \cdots, t_1 \otimes a_{1\ell} \otimes a_{\ell j}, \cdots, t_i \otimes a_{i\ell} \otimes a_{\ell j}, \cdots\right) \end{aligned}$$

$$T_j = \max\left(t_1 \otimes a_{1j}, t_2 \otimes a_{2j}, \cdots, t_i \otimes a_{ij}, \cdots, t_1 \otimes a_{1\ell} \otimes a_{\ell j}, \cdots, t_i \otimes a_{i\ell} \otimes a_{\ell j}, \cdots\right) . \quad (2)$$

We consider the vectors T and t define by $T = (T_1, T_2, \cdots, T_n)^t$; $t = (t_1, t_2, \cdots, t_n)^t$ and the A^t matrix define by.

$$A^t = \begin{pmatrix} a_{11} & \varepsilon & \cdots & \varepsilon \\ a_{12} & a_{22} & \ddots & \\ \vdots & \vdots & \ddots & \varepsilon \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

In the $(\max,+)$algebra the equation (2) can be written as:

$$T = (A^t \oplus (A^t)^2 \oplus \cdots \oplus (A^t)^n) \otimes t. \quad (3)$$

If X denotes the matrix

$$\mathrm{X} = A^t \oplus (A^t)^2 \oplus \cdots \oplus (A^t)^n,$$

then (3) take the form

$$T = \mathrm{X} \otimes t \quad (4)$$

where X is a triangular matrix of the form

$$\mathrm{X} = \begin{pmatrix} x_{11} & \varepsilon & \cdots & \varepsilon \\ x_{21} & x_{22} & & \\ \vdots & \vdots & \ddots & \varepsilon \\ x_{n1} & x_{n2} & & x_{nn} \end{pmatrix}$$

Theorem 2 *If there exists a unique solution of the system* $T = \mathrm{X} \otimes t$*, then the dominant of the matrix* X_i *equals the sum of element of diagonal of* X_i *and the solution is* $t_i = T_i - x_{ii}$ *for all* $i = 1, \cdots, n$*.*

Proof. We have the proof by induction over n.

It is obvious that for $n = 1, 2$ the theorem is true. Suppose that for $1, 2, \cdots, n$ the theorem is true and that for $n + 1$ is false. Then it can be supposed that there exists A, hence X, such that the extended system

$$\begin{pmatrix} x_{11} & \varepsilon & \cdots & \varepsilon & \varepsilon \\ x_{21} & x_{22} & \ddots & & \vdots \\ \vdots & & \ddots & \varepsilon & \vdots \\ x_{n1} & \cdots & \cdots & x_{nn} & \varepsilon \\ x_{n+1,1} & & & x_{n+1,n} & x_{n+1,n+1} \end{pmatrix} \otimes \begin{pmatrix} t_1 \\ \vdots \\ \vdots \\ t_n \\ t_{n+1} \end{pmatrix} = \begin{pmatrix} T_1 \\ \vdots \\ \vdots \\ T_n \\ T_{n+1} \end{pmatrix} \tag{5}$$

has the unique solution $(t_1, \cdots, t_n, t_{n+1})$. However, at least one X_i is not dominant by its diagonal and $t_1, t_2, \cdots, t_n$ is also solution of the first n equations: By induction

$$\begin{aligned} t_1 &= T_i - x_{11}, \\ &\vdots \\ t_n &= T_n - x_{nn}, \end{aligned}$$

and that X_i^n is dominated by its diagonal, where X_i^n is given by

$$\mathrm{X}_i^n = \begin{pmatrix} x_{11} & \varepsilon & & T_1 & \cdots & \cdots & \varepsilon \\ & x_{22} & & & & & \vdots \\ & & & & & & \vdots \\ & & x_{i-1,i-1} & T_{i-1} & & \varepsilon & \varepsilon \\ & & x_{i,i-1} & T_i & & \varepsilon & \\ & & & T_{i+1} & & \varepsilon & \\ & & & & & & \\ x_{n1} & . \quad . & . \quad . & T_n & & x_{n,n} & \end{pmatrix}.$$

Now, we consider the determinant of $z^{\mathrm{X}_i^n}$ for $i \leq n$:

Because the dominant is diagonal by hypothesis ($i \leq n$), then

$$x_{11} + \cdots + x_{i-1,i-1} + T_i + x_{i+1,i+1} + \cdots + x_{nn} \text{ is bigger than any of the term of } det(z^{\mathrm{X}_i^n}). \quad (6)$$

Now, by summation the term $x_{n+1,n+1}$ in (6) we obtain that $x_{11} + \cdots + x_{i-1,i-1} + T_i + x_{i+1,i+1} + \cdots + x_{nn} + x_{n+1,n+1}$ is bigger than any terms of $det(z^{\mathrm{X}_i^n}) + x_{n+1,n+1}$.

On the other hand, if the determinant of $z^{\mathrm{X}_i^{n+1}}$ is considered, that is,

$$det \begin{pmatrix} z^{x_{11}} & 0 & & \cdots & z^{T_1} & & \cdots & 0 \\ z^{x_{21}} & z^{x_{22}} & & & & & & \vdots \\ & & & & & & & \vdots \\ & & & z^{x_{i-1,i-1}} & z^{T_{i-1}} & & & \vdots \\ \cdot\ \cdot\ \cdot\ \cdot & & & z^{x_{i,i-1}} & z^{T_i} & & \vdots & \\ & & & & z^{T_{i+1}} & z^{x_{i+1,i+1}} & & \\ & & & & & & & \\ z^{x_{n1}} & \cdot & & & z^{T_n} & & z^{x_{n,n}} & \\ z^{x_{n+1,1}} & & & & z^{T_{n+1}} & & z^{x_{n+1,n}} & z^{x_{n+1,n+1}} \end{pmatrix}$$

$$= z^{x_{i+1,i+1}+\cdots+x_{n+1,n+1}} \; Det \begin{pmatrix} z^{x_{11}} & 0 & \cdots & \cdots & z^{T_1} \\ z^{x_{21}} & z^{x_{22}} & & & \vdots \\ & & & 0 & \\ & & & z^{x_{i-1,i-1}} & z^{T_{i-1}} \\ & & & z^{x_{i,i-1}} & z^{T_i} \end{pmatrix}$$

then $x_{11} + \cdots + x_{i-1,i-1} + T_i + x_{i+1,i+1} + \cdots + x_{nn} + x_{n+1,n+1} \geq x_{i+1,i+1} + \cdots + x_{nn} + x_{n+1,n+1} +$ any exponent of $det(z^{\mathrm{X}_i^{n+1}})$. Hence, the dominant is diagonal.

Now, we consider the case $i = n + 1$, that is, we consider the determinant of the matrix $\mathbb{P}$, where $\mathbb{P}$ is given by

$$\mathbb{P} = \begin{pmatrix} z^{x_{11}} & 0 & \cdots & 0 & z^{T_1} \\ z^{x_{21}} & z^{x_{22}} & & & \vdots \\ \vdots & & \ddots & 0 & \vdots \\ z^{x_{n1}} & \dots & \dots & z^{x_{nn}} & z^{T_n} \\ z^{x_{n+1,1}} & & & z^{x_{n+1,n}} & z^{T_{n+1}}. \end{pmatrix}$$

Now, expanding this determinant by the last row, we obtain that

$$det(\mathbb{P}) = z^{T_{n+1}} det(\mathbb{P}_0) + \sum_{i=1}^{n} (-1)^i z^{x_{n+1,n-i+1}} det(\mathbb{P}_i) \tag{7}$$

where $\mathbb{P}_0$ and $\mathbb{P}_i$ matrices will be defined by

$$\mathbb{P}_0 = \begin{pmatrix} z^{x_{11}} & 0 & \cdots & 0 \\ & z^{x_{22}} & & \\ & & \ddots & 0 \\ z^{x_{n1}} & \dots & \dots & z^{x_{nn}} \end{pmatrix}, \tag{8}$$

$$\mathbb{P}_i = \left(\begin{array}{cccc|ccccc} z^{x_{11}} & 0 & \cdots\cdots & 0 & 0 & \cdots & \cdots & 0 & z^{T_1} \\ \ddots & & & \vdots & & & & \vdots & \\ \vdots & \ddots & & \vdots & & & & \vdots & \\ \vdots & & \ddots & 0 & & & & \vdots & \\ z^{x_{n-i,1}} & & & z^{x_{n-i,n-i}} & 0 & \cdots & \cdots & 0 & z^{T_{n-i}} \\ \hline z^{x_{n-i+1,1}} & & & z^{x_{n-i+1,n-i}} & 0 & \cdots & \cdots & 0 & z^{T_{n-i+1}} \\ \vdots & & & & z^{x_{n-i+2,n-i+2}} & \cdots & & & \\ z^{x_{n-1,1}} & & & z^{x_{n-1,n-i}} & \vdots & \cdots & z^{x_{n-1,n-1}} & 0 & z^{T_{n-1}} \\ z^{x_{n,1}} & \cdots & \cdots & Z^{x_{n,n-i}} & z^{x_{n,n-i+2}} & \cdots & z^{x_{n,n-1}} & z^{x_{nn}} & z^{T_n} \end{array} \right). \tag{9}$$

Developing the determinant of $\mathbb{P}_i$ by the penult column, successively, in order to obtain a triangular matrix whose principal diagonal elements are $z^{x_{11}}, \cdots, z^{x_{n-i,n-i}}, z^{T_{n-i+1}}$, we have that $det(\mathbb{P}_i) = (-1)^{i+1} z^{x_{nn}+x_{n-1,n-1}+\cdots+x_{n-i+2,n-i+2}} \times z^{x_{11}+x_{22}+\cdots+x_{n-i,n-i}+T_{n-i+1}}$.

Then, by substitution of this value of determinant in the equation (7), we have

$$\begin{aligned} det(\mathbb{P}) &= z^{T_{n+1}+x_{11}+\cdots+x_{nn}} - \sum_{i=1}^{n} z^{x_{11}+\cdots+x_{n-i,n-i}+T_{n-i-1}} \\ &\quad z^{x_{n+1,n-i+1}+x_{nn}+x_{n-1,n-1}+\cdots+x_{n-i+2,n-i+2}} \end{aligned} \tag{10}$$

If $T_{n+1}+x_{11}+\cdots+x_{nn}$ is not equal to dominant of $\mathbb{P}$, then this sum must be less that one of the terms in (10).

In particular, $T_{n+1}+x_{11}+\cdots+x_{nn} < x_{11}+\cdots+x_{n-i,n-i}+T_{n-i+1}+x_{n+1,n-i+1}+x_{n-1,n-1}+\cdots+x_{n-i+2,n-i+2}+\cdots+x_{nn}$. But this imply that

$$T_{n+1}+x_{n-i+1,n-i+1} < T_{n-i+1}+x_{n+1,n-i+1}. \tag{11}$$

For (5) and inductive hypothesis

$$\max(T_1-x_{11}+x_{n+1,1}, T_2-x_{22}+x_{n+1,2}, \cdots, T_{n-i+1}-x_{n-i+1,n-i+1}+x_{n+1,n-i+1}+\cdots)$$

must be equal to T_{n+1}, but this is contradictory with (11). Hence, there is not exist solution if the dominant of is different from the diagonal.

$\blacksquare$

Theorem 3 *If the dominant of* X_i *matrix is equal to the sum of elements of diagonal of* X_i*, then the system* $T=\mathrm{X}\otimes t$ *has a unique solution, given by*

$$t_i = T_i - x_{ii}, \quad i=1,\cdots,n$$

Proof. (Existence)

For the existence is necessary to prove that $Sign(\mathrm{X}) = Sign(\mathrm{X}_i) \quad i=1,\cdots,n$ and then apply the Theorem 1.

We have to calculate $det(z^{\mathrm{X}}),\ \ z \in (1, +\infty)$, in order to obtain the $Sign(\mathrm{X})$. But,

$$det(z^{\mathrm{X}}) = \begin{vmatrix} z^{z_{11}} & 0 & \cdots & 0 \\ \vdots & z^{x_{22}} & \ddots & \\ \vdots & & \ddots & 0 \\ z^{x_{n1}} & \cdots & \cdots & z^{x_{nn}} \end{vmatrix} = z^{x_{11}+x_{22}+\cdots+x_{nn}}$$

therefore, $Dom(\mathrm{X}) = x_{11} + x_{22} + \cdots + x_{nn}$ and $Sign(\mathrm{X}) = +$.

By hypothesis $Dom(\mathrm{X}_i) = x_{11} + x_{22} + \cdots + T_i + \cdots + x_{nn}$ and $Sign(\mathrm{X}_i) = +$. Hence, there exist the solution of the system and given by

$$\begin{aligned} t_i \otimes Dom(\mathrm{X}) &= Dom(\mathrm{X}_i), \quad \text{that is,} \\ t_i &= T_i - x_{ii}, \quad i = 1, \cdots, n. \end{aligned}$$

(Uniqueness)

We apply induction over n. It is clear that for $n = 1$ the solution is unique. Suppose that for $1, 2, \cdots, n$ there exists a unique solution and that for $n + 1$ it is false.

We consider the extended system, given in (5) where by induction $t_1 = T_1 - x_{11}, \cdots, t_n = T_n - x_{nn}$ is the unique solution and t_{n+1} is another solution no neccessarily equals to $T_{n+1} - x_{n+1,n+1}$.

Now, by (10) and hypothesis

$$\begin{aligned} x_{11} + x_{22} + \cdots + x_{nn} + T_{n+1} \quad > \quad & x_{11} + \cdots + x_{n-i,n-i} + T_{n-i+1} + x_{n+1,n-i+1} \\ & + x_{n+1,n-i+1} + x_{n-1,n-1} + \cdots + x_{n-i+2,n-i+2} \\ & + \cdots + x_{nn} \end{aligned}$$

but this imply

$$T_{n+1} + x_{n-i+1,n-i+1} > T_{n-i+1} + x_{n+1,n-i+1}. \tag{12}$$

By (5) and inductive hypothesis $\max(T_1 - x_{11} + x_{n+1,1}, T_2 - x_{22} + x_{n+1,2}, \cdots\cdots,$ $T_{n-i+1} - x_{n-i+1,n-i+1} + x_{n+1,n-i+1}, \cdots, x_{n+1,n+1} + t_{n+1}) = T_{n+1}$.

Therefore , $T_{n+1} = x_{n+1,n+1} + t_{n+1}$, that is,

$$t_{n+1} = T_{n+1} - x_{n+1,n+1},$$

then the solution is unique.

∎

4 Conclusion.

It is easy to see that there is no solution for $(\max,+)$ linear equations, for all diagonal matrix and for all right hand side. Hence the problem of existence and uniqueness may be complicated even in this particular case. Hence our positive results have importance and there are not trivial. The applications is usefull in practice.

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