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UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS

NOTAS DE MATEMATICA

A. TINEO R. MANASEVICH

ON THE FIRST CONJUGATE POINT OF A QUASIDIFFERENTIAL EQUATION OF ORDER N.

DEPARTAMENTO DE MATEMATICA
MERIDA - VENEZUELA
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A B S T R A C T

In this paper we study some properties of the zeros of a quasidifferential equation of order n. We also study an associate equation of it. By establishing some properties - of the zeros of solutions of both equations we finally prove the first conjugate point of a quasidifferential equation - of order n exists if this equation is not disconjugate.

1.- INTRODUCTION. Let I be an interval. $C^r(I,n)$ denotes the space of functions $\underline{X} : I \to R^n$ of class C^r on I, $(r \geq 0)$. By short we write $C^r(I) = C^r(I,1)$. Also let $\{e_1,\ldots,e_n\}$ be the canonical basis of R^n.

Let $A = [a_{ij}]$ denote an $n \times n$ functional matrix where $a_{ij}: I \to R$, $1 \leq i, j \leq n$, are continuous functions such that $a_{ij} = 0$ if $j \geq i+2$, $a_{i,i+1}(t) \neq 0$ for $t \in I$ and $i = 1,\ldots,n-1$.

By definition we write:

$C^1_A(I,n) = \{ \underline{X} \in C^1(I,n) \mid < \underline{X}' - A\underline{X}, e_i > = 0,\ 1 \leq i \leq n-1\}$ where

$< \ >$ denotes the Euclidean scalar product.

Consider now the mapping $\rho : C^1_A(I,n) \to C^1(I)$ defined by $\rho(\underline{X}) = <\underline{X}, e_1>$.

We have ρ is a monomorphism and if $C^1_A(I)$ denotes the image of ρ then we have that $\rho : C^1_A(I,n) \to C^1_A(I)$ is an isomorphism. Define now the operator $L_A : C^1_A(I) \to C^{\circ}(I)$ by $L_A(\rho(\underline{X})) = < \underline{X}' - A\underline{X}, e_n >$. We have L_A is a linear operator which we call a quasi-differential operator of n^{th} order associate with A. We call the equation $L_A(x) = 0$ a quasi-differential equation of n^{th} order. Note if $\underline{X}(t) = (x_1(t),\ldots,x_n(t))$ is a solution of $\underline{X}' = A X$ then $x_1(t)$ satisfies $L_A(x) = 0$. Conversely if $u(t)$ is a solution of $L_A(x) = 0$ then there exist a unique $\underline{\overline{X}}=(x_1,\ldots,x_n)$ which satisfies $\underline{\overline{X}}'=A\underline{\overline{X}}$, with $x_1(t)=u(t)$. From this viewpoint we observe the equations $\underline{\overline{X}}'= A\underline{\overline{X}}$ and $L_A(x) = 0$ as being equivalent. Quasi-differential equations of the type we are dealing here have been considered by others see, [2] , [4] and [6] .

In section 2 we give some results concerning the zeros of a solution of a quasi-differential equation. In section 3 we define an associate system of $\underline{\overline{X}}' = A\,\underline{\overline{X}}$ and we establish - some properties of it. Finally in section 4 we prove our main theorem which briefly says: If L_A is not disconjugate in $[a, \infty)$ then the first conjugate point $\eta_1(a)$ of a does - exist.

2.- PRELIMINARY RESULTS. Let $\underline{\overline{X}} = (x_1,\ldots,x_n)$ be a solution of $\underline{\overline{X}}'=A\,\underline{\overline{X}}$. From now on, a solution means a non trivial solution. We write ord. $(\underline{\overline{X}}, t_\circ)= p$ to mean $\underline{\overline{X}}$ has a zero of order p at $t_\circ \in I$, i.e., $x_1(t_\circ) = \ldots = x_p(t_\circ) = 0$ and $x_{p+1}(t_\circ) \neq 0$, $1\leq p\leq n-1$. We denote

$$x_k^{(i)}(t_\circ) = \lim_{t \to t_\circ} \frac{x_k(t)-x_k(t_\circ)}{(t-t_\circ)^i}, \qquad (1)$$

when this limit exist.

PROPOSITION 1.- If $\underline{\overline{X}} = (x_1,\ldots,x_n)$ is a solution of $\underline{\overline{X}}'=A\,\underline{\overline{X}}$ such that ord. $(\underline{\overline{X}}, t_\circ) \geq p$ then unique $\underline{\overline{X}}=(x_1,\ldots,x_n)$ which satisfies $\underline{\overline{X}}'=A\underline{\overline{X}}$, with $x_1(t)=u(t)$. From this viewpoint we observe the equations $\underline{\overline{X}}'= A\underline{\overline{X}}$ and $L_A(x) = 0$ as being equivalent. Quasi-differential equations of the type we are dealing here have been considered by others see, [2] , [4] and [6] .

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$$x_k^{(i)}(t_\circ) = 0, \quad i=0,1,\ldots,p-1 \ , \quad k=1,\ldots,p-1. \qquad (2)$$

PROOF. (By induction on i). We first prove for $i=1$. The equation $\underline{X}' = A X$ can be written as

$$\dot{x}_k = \sum_{j=1}^{k+1} a_{kj}\, x_j \qquad k = 1,\ldots,n-1 \qquad (3)$$

$$\dot{x}_n = \sum_{j=1}^{n} a_{nj}\, x_j$$

We have

$$x_k^{(1)}(t_\circ) = \dot{x}_k(t_\circ) = \sum_{j=1}^{k+1} a_{kj}(t_\circ)\, x_j(t_\circ). \qquad (4)$$

Then since $x_j(t_\circ) = 0$, $1 \leq j \leq p$, from (4) we get $x_k^{(1)}(t_\circ) = 0$ if $k \leq p-1$. Suppose next our result is true for some i, $1 \leq i \leq p-2$, i.e., $x_k^{(i)}(t_\circ) = 0$, $k = 1,\ldots,p-i$. From (3), for $1 \leq k \leq p-1$ we can write,

$$\frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^i} = \sum_{j=1}^{k+1} a_{kj}(t)\, \frac{x_j(t)-x_j(t_\circ)}{(t-t_\circ)^i}. \qquad (5)$$

Taking the limit $t \to t_\circ$ in (5) and using the induction hypothesis it follows

$$\lim_{t \to t_\circ} \frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^i} = 0 \quad \text{if} \quad k+1 \leq p-i. \qquad (6)$$

From (6) and L'Hopital rule we then obtain

$$\lim_{t \to t_\circ} \frac{x_k(t)-x_k(t_\circ)}{(t-t_\circ)^{i+1}} = \frac{1}{i+1} \lim_{t \to t_\circ} \frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^{i}} = 0 \qquad (7)$$

for $k \leq p-i-1$. This finishes the induction.

PROPOSITION 2.- If $\underline{\overline{X}}$ is a solution of $\underline{\overline{X}}' = A\, \underline{\overline{X}}$ and ord. $(\underline{\overline{X}},t_\circ) \geq p$, then $x_k^{(p-k+1)}(t_\circ)$ does exist, $k=1,\ldots,p$, and

$$x_k^{(p-k+1)}(t_\circ) = \frac{1}{(p-k+1)!} a_{k,k+1}(t_\circ)\ldots a_{p,p+1}(t_\circ)x_{p+1}(t_\circ). \qquad (8)$$

PROOF. (By induction on k). We note firstly,

$$x_p^{(1)}(t_\circ) = \dot{x}_p(t_\circ) = \sum_{j=1}^{p+1} a_{pj}(t_\circ)x_j(t_\circ) = a_{p,p+1}(t_\circ)x_{p+1}(t_\circ).$$

This expression tell us our result is true for k=p. Assume - next the result is true for $k+1$, where $1<k+1\leq p$. We want to prove for k. From (3) we obtain,

$$\frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^{p-k}} = \sum_{j=1}^{k+1} a_{kj}(t) \frac{x_j(t)-x_j(t_\circ)}{(t-t_\circ)^{p-k}} \quad . \qquad (9)$$

From Proposition 1 we have

$$x_j^{(p-k)}(t_\circ) = 0, \quad j = 1,\ldots,k\ ,$$

then from (9) it follows,

$$\lim_{t \to t_\circ} \frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^{p-k}} = a_{k,k+1}(t_\circ) \quad x_{k+1}^{(p-k)}(t_\circ), \tag{10}$$

where, by the induction hypothesis,

$$x_{k+1}^{(p-k)}(t_\circ) = \frac{1}{(p-k)!} a_{k+1,k+2}(t_\circ)\ldots a_{p,p+1}(t_\circ)x_{p+1}(t_\circ). \tag{11}$$

Now by L'Hopital rule,

$$\lim_{t \to t_\circ} \frac{x_k(t)-x_k(t_\circ)}{(t-t_\circ)^{p-k+1}} = \frac{1}{p-k+1} \lim_{t \to t_\circ} \frac{\dot{x}_k(t)-\dot{x}_k(t_\circ)}{(t-t_\circ)^{p-k}} .$$

Then this limit exist and from (10) and (11) we finally obtain,

$$x_k^{(p-k+1)}(t_\circ) = \frac{1}{(p-k+1)!} a_{k,k+1}(t_\circ)\ldots a_{p,p+1}(t_\circ)x_{p+1}(t_\circ) \quad .$$

COROLLARY 1.- Let $\underline{\overline{X}} = (x_1,\ldots,x_n)$ be a non trivial solution of $\underline{\overline{X}}' = A\,X$, then the zeros of x_1 are isolated.

PROOF. Suppose there exist a $t_\circ$ and a sequence $\{t_n\}$, such that $t_n \neq t_\circ$, $\lim_{n\to\infty} t_n = t_\circ$ and $x_1(t_n)=0$. Then $x_1(t_\circ)=0$. Assume also $x_1(t_\circ) = \ldots. = x_p(t_\circ) = 0$, for some $1 \le p \le n-1$. From Proposition 2 it follows

$$\lim_{t \to t_\circ} \frac{x_1(t)-x_1(t_\circ)}{(t-t_\circ)^p} = \frac{1}{p!} a_{12}(t_\circ) \ldots. a_{p,p+1}(t_\circ) x_{p+1}(t_\circ) , \tag{12}$$

where $a_{12}(t_\circ) \ldots. a_{p,p+1}(t_\circ) \neq 0$. But

$$\lim_{t \to t_\circ} \frac{x_1(t)-x_1(t_\circ)}{(t-t_\circ)^p} = \lim_{n \to \infty} \frac{x_1(t_n)-x_1(t_\circ)}{(t_n-t_\circ)^p} = 0. \tag{13}$$

From (13) and (12) we get,

$$x_{p+1}(t_\circ) = 0. \tag{14}$$

By induction we then obtain, $x_1(t_\circ)=x_2(t_\circ)=\ldots.=x_n(t_\circ)$, which implies $\underline{X}$ is trivial. Contradiction.

3.- In this section we introduce a system of differential equations which we denote as the associate system of (3), and we give some results relating both systems.

Define the matrix $B = [b_{ij}]$ by

$$b_{ij} = (-1)^{i+j-1} a_{n-j+1,n-i+1}, \quad i \neq j, \; i,j=1,\ldots\ldots,n \tag{15}$$

$$b_{ii} = \sum_{\substack{j=1 \\ j \neq n-i+1}}^{n} a_{jj}.$$

We denote the system,

$$Y' = B\,Y \;,$$

where $Y = (y_1,\ldots\ldots,y_n)$, the associate system of (3). Note the matrix B has the same properties as the matrix A, i.e., $b_{ij} \equiv 0$ if $j \geq i+2$ and $b_{i,i+1}(t) \neq 0$, for $t \in I$ and $i=1,\ldots\ldots,n-1$

We have the following Propositions:

PROPOSITION 3.- Let $\underline{\overline{X}}_1,\ldots\ldots,\underline{\overline{X}}_{n-1}$ be (n-1) vector solutions of (3). Denote by M(t) the n x (n-1) matrix whose column - are the vectors $\underline{\overline{X}}_i(t)$, $(i=1,\ldots\ldots,n-1)$, and by $M_k(t)$ the (n-1) x (n-1) matrix obtained by eliminating the (n-k+1) row in - M(t), k=1, n. Let us define $y_k(t) = \det.\ (M_k(t))$, then - the vector $Y = (y_1,\ldots\ldots,y_n)$ is a solution of (16).

PROOF. By direct evaluation.

PROPOSITION 4.- Let $\underline{\overline{X}}_1,\ldots\ldots,\underline{\overline{X}}_n$ be n vector solutions of (3)

Define y_{ij} as the cofactor corresponding to the (n-i+1, n-j+1) element of the matrix whose column vectors are $(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_n)$. Then if $Y_\ell = (y_{1\ell},\ldots,y_{n\ell})$, $\ell=1, n$, the following is true: i) $Y_1,\ldots,Y_n$ are solutions of (16). ii) $Y_1,\ldots,Y_n$ are linearly independent solutions of (16) $\Longleftrightarrow \underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_n$ are linearly independent solutions of (3). iii) Let $W(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_k)$, $W(Y_1,\ldots,Y_k)$ be equal to $\det.(x_{\ell m} \mid 1\leq \ell,m\leq k)$ and $\det.(y_{\ell m} \mid 1\leq \ell,m\leq k)$ respectively, $1\leq k\leq n$, then if $W(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_n) \neq 0$ we have

$$W(Y_1,\ldots,Y_i)=W(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_n)^{i-1}W(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_{n-i}), \qquad (17)$$

$i = 1,\ldots,n$. For $i=n$, (17) is interpreted as

$$W(Y_1,\ldots,Y_n) = [W(\underline{\overline{X}}_1,\ldots,\underline{\overline{X}}_n)]^{n-1} \qquad (18)$$

PROOF:

i) follows from Proposition 3); ii) and iii) follow - from general properties of determinants. See, for example, [3, p.p. 168], for a proof of iii).

PROPOSITION 5.- Let $\underline{\overline{X}}_1(t),\ldots,\underline{\overline{X}}_n(t)$, be n vector solutions of 3), such that $\underline{\overline{X}}_i(t_\circ)=e_{n-i+1}$, $i=1,\ldots,n$, $t_\circ \in I$. Then the zeros of $w_k(t) = W(\underline{\overline{X}}_1(t),\ldots,\underline{\overline{X}}_k(t))$, $1\leq k<n$, do not cluster at $t_\circ$.

PROOF. We assume first that $2k\leq n+1$; then because of the conditions at $t_\circ$, we have $w_k(t_\circ) = 0$. Now from the identity:

$$w_k(t)=(t-t_\circ)^{k(n-k)}\det.(f_{ij}(t)), \quad t\neq t_\circ, \tag{19}$$

where

$$f_{ij}(t) = \frac{x_{ij}(t)}{(t-t_\circ)^{n-i-j+1}}, \quad i,\ j=1,\ldots,k, \tag{20}$$

it follows,

$$\lim_{t\to t_\circ} \frac{w_k(t)}{(t-t_\circ)^{k(n-k)}} = \det.\ (f_{ij}(t_\circ)), \tag{21}$$

where with the convention of (1)

$$f_{ij}(t_\circ) = x_{ij}^{(n-i-j+1)}(t_\circ). \tag{22}$$

We recall here if $\underline{X}_j(t) = (x_{1j},\ldots,x_{nj})$ is a solution of (3) with a zero of order $n-j$ at $t_\circ$, then

$$x_{ij}^{(n-j-i+1)}(t_\circ) = \frac{1}{(n-j-i+1)!}\, a_{i,i+1}(t_\circ)\ldots a_{n-j,n-j+1}(t_\circ).$$

$$\cdot\ x_{n-j+1}(t_\circ)\ .$$

Now on evaluating det. $(f_{ij}(t_\circ))$ we obtain,

$$\det.(f_{ij}(t_\circ))=a_{12}a_{23}^2\ldots a_{k-1,k}^{k-1}(a_{k,k+1}\ldots a_{n-k,n-k+1})^k \;\cdot$$

$$\cdot\; a_{n-1,n}\, a_{n-2,n-1}^2 \ldots a_{n-k+1,n-k+2}^{k-1} \cdot \begin{vmatrix} \frac{1}{(n-1)!} & \cdots & \frac{1}{(n-k)!} \\ \vdots & & \vdots \\ \frac{1}{(n-k)!} & \cdots & \frac{1}{(n-2k+1)!} \end{vmatrix} \quad ,(23)$$

where the coefficients $a_{i,i+1}$ are evaluated at $t_\circ$. We note that by hypothesis $a_{i,i+1}(t) \neq 0$, $t \in I$, $i=I,\ldots,n-1$, then from (23) and (21) it follows

$$\lim_{t \to t_\circ} \frac{w_k(t)}{(t-t_\circ)^{k(n-k)}} \neq 0. \qquad (24)$$

It is now clear that the zeros of $w_k(t)$, $t \in I$, can not cluster at $t_\circ$ because if there exists a sequence $\{t_i\}$, $t_i \neq t_\circ$, $\lim_{i\to\infty} t_i = t_\circ$ and such that $w_k(t_i)=0$, then we would have

$$\lim_{t \to t_\circ} \frac{w_k(t)}{(t-t_\circ)^{k(n-k)}} = \lim_{t_i \to t_\circ} \frac{w_k(t_i)}{(t_i-t_\circ)^{k(n-k)}} = 0 \;, \qquad (25)$$

which would contradict (24). Thus we have proved our result for $2k \leq n+1$. For $\frac{n+1}{2} < k < n$, we consider the solutions $Y_j(t)$ of (16), constructed as in Proposition 4 from the solutions $\underline{\overline{X}}_j(t)$, $j=1,\ldots,n$. Since $\underline{\overline{X}}_j(t_o)=e_{n-j+1}$ then $Y_j(t_o) = (-1)^{\frac{n(n-1)}{2}} e_{n-j+1}$, $j=1,\ldots,n$. Thus just as for the solutions $\underline{\overline{X}}_j(t)$ we obtain now: the zeros of $w_i^*(t) = W(Y_1(t),\ldots,Y_i(t))$ do not cluster at t_o, $2i \leq n+1$. From (17) with $i=n-k$, it then follows, the zeros of $W(\underline{\overline{X}}_1(t),\ldots,\underline{\overline{X}}_k(t))$ do not cluster at t_o, for $\frac{n-1}{2}<k<n$, which of course implies our result.

4.- In this section we consider the interval I of section 1) as $I= [a,\infty)$. We say the equation $\underline{\overline{X}}' = A\,\underline{\overline{X}}$ is not disconjugate in $[a,\infty)$, (equivalently, the equation $L_A(x)= 0$ is not disconjugate in $[a,\infty)$), if there exist a nontrivial solution of $\underline{\overline{X}}' = A\,\underline{\overline{X}}$ which has n zeros in $[a,\infty)$. (The zeros are counted with their multiplicities). We denote by $\eta_1(a)=\inf.\{b>a \mid$ a non trivial solution of $\underline{\overline{X}}' = A\,\underline{\overline{X}}$ with n zeros in $[a,b]\}$. If $\eta_1(a) > a$, then we say $\eta_1(a)$ is the first conjugate point of a. If $\eta_1(a) = a$, we say the first conjugate point of a does exist.

We now prove our main theorem:

THEOREM 1.- If the equation $\overline{\underline{X}}' = A\,\overline{\underline{X}}$ is not disconjugate in $[a, \infty)$ then there exists $\eta_1(a)$.

PROOF.- We first quote the following result due to Sherman [6].

LEMMA 1.- If the equation $\overline{\underline{X}}' = A\,\overline{\underline{X}}$ is not disconjugate in $[a,\infty)$ and $\eta_1(a) = a$, then there exists a sequence of points $\{b_i\}$ such that

i) $b_i > b_{i+1} > a$ $(i=1,2,\ldots.)$ and $\lim_{i\to\infty} b_i = a$,

ii) there exists a sequence of non-trivial solutions $\{Z_i\}$ of (3) and a number k $(0<k<n)$ such that Z_i has a zero of order n-k at a and a zero of order k at b_i.

Let us consider next the vector solutions $\underline{\overline{X}}_1(t),\ldots,\underline{\overline{X}}_n(t)$ of (3) which satisfy $\underline{\overline{X}}_j(a) = e_{n-j+1}$, $j=1,\ldots,n$.

Then we can write

$$Z_i = \sum_{j=1}^{k} \lambda_{ij}\underline{\overline{X}}_j \quad \text{with} \quad \sum_{j=1}^{k} \lambda_{ij}^{2} \neq 0. \tag{26}$$

Since $Z_i=(z_{1i},\ldots,z_{ni})$ has a zero of order k at b_i we have

$$z_{1i}(b_i) = z_{2i}(b_i) = \ldots\ldots\ldots = z_{ki}(b_i) = 0. \qquad (27)$$

(27) is seen as an homogeneous system of k algebraic equations which admits the non -trivial solution $(\lambda_{i1},\ldots,\lambda_{ik})$, it then follows the determinant of the coefficients of (27) satisfies:

$$W(\underline{\overline{X}}_1(b_i), \ldots\ldots\ldots, \underline{\overline{X}}_k(b_i)) = 0. \qquad (28)$$

But (28) contradicts Proposition 5. Thus if the equation $\underline{\overline{X}} = A\,\underline{\overline{X}}$ is not disconjugate in $[a,\infty)$ we have $\eta_1(a)$ exist.

R E F E R E N C E S

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